String theory Final exam
Phys 230 A, Winter 2008

March 17, 2008, 11 am
Due date: Thursday March 20, 5 pm
Turn in your exams in Prof. Berenstein's office, Broida 6.133

This is an open book exam. You are allowed to use any internet resource or book available to you. You are also allowed to use software like Mathematica or Maple to help you in your computations. If you use this software, turn in a copy of the input and output.

You are not allowed to receive help from anyone to present this exam. You are not allowed to talk (or correspond) about this exam to anyone other than the professor for the course, or your T.A. The exam is written so that it should take less than 24 hours to solve it. The extra time will just give you some flexibility to schedule your time.

This exam is worth 40% of your grade. There are no allowances to turn the exam late, unless you communicate with the professor ahead of schedule and present a valid excuse.

1. **Bosonization**

   Consider a conformal field theory with two left moving fermions ψ^1, ψ^2, for a total central charge $c = 1$. One can also consider a conformal field theory for a single left moving scalar ϕ (the holomorphic half of an ordinary boson $X(z, \bar{z})$), which also has central charge one. One can imagine that the two conformal field theories might be related (identical). This relation is called bosonization. This problem will explore such a concept.

   (a) Consider the complex combination

$$\psi^{\pm}(z) = \frac{\psi^1 \pm i\psi^2}{\sqrt{2}} \tag{1}$$

Write the OPE's of $\psi^{\pm}(z)$ with each other, staring from the OPE's of $\psi^{1,2}$.

(b) Consider the following definition of the charge current

$$j =: \psi^+ \psi^- : (z) \tag{2}$$

Show that j is a conformal primary of weight one and calculate the OPE expansion of j with $\psi^{\pm}$.

(c) The bosonic field theory also has a conserved current, represented by $-i\partial\phi$. We would like to identify this current with j as defined above. For this to be correct, the OPE of the currents with themselves should be identical. Check whether the OPE

$$j(z)j(w) \sim \frac{A}{(z-w)^2} \tag{3}$$

for both currents have the same normalization. If they don't, normalize the currents so that they do.

(d) The charge associated to $-i\partial\phi$ measures the exponent of vertex operators of the form $\exp(ik\phi)$ giving k. What should be the correct value of k that would permit us to identify $\exp(ik\phi)$ with $\psi^{\pm}$? You should also show that the vertex operator has the same dimension as $\psi^{\pm}$

(e) If ψ^+ is identified with $\exp(ik_0\phi)$, we can consider the operator $V_n \sim \exp(ink_0\phi)$, which would be the vertex operator of the minimal object with charge n (this is n times the charge of the fundamental object). Show that one should identify this object with the operator given by

$$V_n(z) \simeq \psi^+(z)\partial\psi^+(z)\ldots\partial^{n-1}\psi^+(z) \tag{4}$$

This is, show that the operator on the right hand side is primary and has the same conformal weight and charge as V_n (Hint: you can argue using oscillator methods that this is the state of charge n with the smallest energy and that the Virasoro generators do not change the charge of a state. Otherwise, just compute the OPE with T and j). For V_{-n} we change the plus signs for minus signs and the same argument holds.

(f) Calculate the most singular terms in the OPE of $V_n(w)V_m(z)$, where m, n can be either positive or negative integers.

2. **BRST cohomology**

Consider the holomorphic half of the closed bosonic string at level 2 (this is roughly the same as the open string at level 2, except for some factors of 2). This is, consider states that have two holomorphic derivatives and some momentum. Calculate the BRST cohomology of the level 2 states that are of the form $c(z)V(z)$ where V is a matter vertex operator. Compare your results with the (open string) lightcone quantization of the bosonic string at level two, and show that the two results agree.

3. **Superconformal algebra**

In the superstring theory the set of Virasoro constraints that result from setting the stress energy tensor to zero $T = 0$, is enlarged to also include $G = 0$, where T, G form a superconformal algebra. This algebra has the following OPE relations

$$T(z)G(w) \sim \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \tag{5}$$

$$G(z)G(w) \sim \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w} \tag{6}$$

$$T(z)T(w) \sim \frac{(3\hat{c}/2)}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{7}$$

where $\hat{c}$ is a rescaled version of the central charge.

To each of the constraints we associate a ghost system. For T, it is the usual b, c fermionic ghosts of the string, of conformal weight $2, -1$, while for G we get a β, γ bosonic ghost system of conformal weights $3/2, -1/2$.

(a) Show that for a combined b, c system and β, γ system of conformal weights

$$h_b = \lambda \quad , \quad h_c = 1 - \lambda$$
$$h_\beta = \lambda - 1/2 \quad , \quad h_\gamma = \frac{3}{2} - \lambda$$

the following definitions of T, G satisfy the superconformal algebra relations

$$
\begin{aligned}
T &= :(\partial b)c: - \lambda\partial(:bc:) + :(\partial\beta)\gamma: - \frac{1}{2}(2\lambda-1)\partial(:\beta\gamma:) \quad (8)\\
G &= -\frac{1}{2}(\partial\beta)c + \frac{(2\lambda-1)}{2}\partial(\beta c) - 2b\gamma \quad (9)
\end{aligned}
$$

These are equations (10.1.16–10.1.18) in the second volume of Polchinski. For this, remember that b, c are conjugate fermions and β, γ are conjugate bosons, with OPE's given by

$$
\begin{aligned}
b(z)c(w) \sim c(z)b(w) &\sim \frac{1}{z-w} \quad (10)\\
\beta(z)\gamma(w) \sim -\gamma(z)\beta(w) &\sim \frac{-1}{z-w} \quad (11)
\end{aligned}
$$

(b) From your results, what is the value of $\hat{c}$ for the ghost system of the string?