1 Homework set 5, due Feb 15th

1. Consider the vertex operator

\[ f_{\mu \nu} : \partial X^\mu \bar{\partial} X^\nu \exp(ikX) : \]  \hspace{1cm} (1)

(a) Using the OPE with the stress energy tensor of many free scalar fields, what are the constraints on \( f_{\mu \nu} \) that need to be satisfied in order for this vertex operator to be a primary field?

(b) What is the scaling dimension of this operator? (Calculate \( h_L + h_R \)).

2. Consider the set of descendants of a primary state \( |v\rangle \) that is normalized to one, at level three. These are given by

\[ |v, \alpha\rangle \sim L_{-3}|v\rangle, L_{-2}L_{-1}|v\rangle, L_{-1}^3|v\rangle \]  \hspace{1cm} (2)

(a) Calculate the 3 \( \times 3 \) Kac matrix of inner products

\[ (M_3)_{\alpha, \beta} = \langle v, \alpha | v, \beta \rangle \]  \hspace{1cm} (3)

(b) Evaluate the determinant of \( M_3 \) and verify the Kac determinant formula in Ginsparg’s lectures. (You can use a computer algebra manipulation program to do this, so long as you show or print the output of the program)

(c) For the minimal model \( m = 3 \) with \( c = 1 - 6/(3 \cdot 4) \), consider the two primary fields \( \phi_{1,2} \) and \( \phi_{2,1} \) of conformal weights \( h_{1,2} \) and \( h_{2,1} \). Each of them has a null descendant at level 2, of the form \( aL_{-2} \cdot \phi + bL_{-1}^2 \cdot \phi \). Calculate the ratio of \( a/b \) for each of these two primary fields.

3. The stress tensor OPE shows that the tensor \( T(z) \) is not a primary field. Instead the infinitesimal transformation of \( T(z) \) induced by the OPE is given by

\[ \delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial \epsilon(z)T(z) + \frac{c}{12} \partial^3 \epsilon \]  \hspace{1cm} (4)

when the transformation is generated by the contour around \( z \)

\[ \oint \frac{dw}{2\pi i} T(w)\epsilon(w) \]  \hspace{1cm} (5)
(a) Show that equation 4 is true by explicit evaluation.

(b) This infinitesimal transformation equation can be integrated to a finite transformation $z \rightarrow f(z)$

$$T(z) \rightarrow (\partial f)^2 T(f(z)) + \frac{c}{12} S(f, z) \quad (6)$$

where $S(f, z)$ is called the Schwarzian, and it is given by

$$S(f, z) = \frac{\partial_z f \partial_z^3 f - \frac{3}{2} (\partial_z^2 f)^2}{(\partial_z f)^2} \quad (7)$$

Show that this transformation given by equation (6) satisfies the composition rule from $z \rightarrow f(z) \rightarrow g(f(z))$.

(c) Show that the transformation in equation (6) reduces to 4 for $f$ infinitesimally close to $z$.

(d) Calculate $S(f, z)$ for $f(z) = \frac{az+b}{cz+d}$.

(e) Calculate $S(f, z)$ for $f(z) = \log(z)$