1 Homework set 6, due Feb 22nd

1. Consider a free (left-moving) boson field, which is characterized by the holomorphic operator of weight (1, 0), $\partial X$ and by the correlator

$$\langle \partial X(z)\partial X(w) \rangle = \frac{1}{(z - w)^2}$$  \hspace{1cm} (1)

Define the mode expansion of $\partial X$ by the following Laurent series

$$i\partial X = \sum_n \alpha_n z^{-n-1}$$  \hspace{1cm} (2)

(a) Write $\alpha_n$ as a contour integral around the origin of $\partial X$.

(b) Calculate the commutator $[\alpha_n, \alpha_m]$ by performing contour integrals in complex variables.

(c) Consider an antiperiodic expansion for $\partial X$ around the origin, characterized by the label $n$ of the oscillators above being a half integer. This is, we have inserted a twist operator $\sigma(0)$ such that

$$\partial X(z)\sigma(0) \sim \frac{1}{z^{1/2}}\mu(0)$$  \hspace{1cm} (3)

What is the difference in conformal dimensions $h(\sigma) - h(\mu)$?

(d) Calculate the Greens function

$$\langle \partial X(z)\partial X(w) \rangle_A$$  \hspace{1cm} (4)

in the presence of antiperiodic boundary conditions by using the commutation relations of the $\alpha_n$. If you prefer, you can calculate this correlator using complex analysis techniques as used in class.

(e) Use the definition of the stress tensor

$$T(z) = \lim_{w \to z} \frac{1}{2} \partial X(w)\partial X(z) - \frac{1}{2(z - w)^2}$$  \hspace{1cm} (5)

To calculate $\langle T(z) \rangle_A$ from the previous result. Extract the conformal dimension of $\sigma$ from this result.
2. **Even more commutators** Consider an abstract OPE given by

\[ J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{abc} J^c(w)}{z-w} \]  

(6)

for \( J_a \) conformal field of weight one, labeled by some semi-simple Lie algebra with generators \( T^a \), \( k \) a constant, and \( f_{abc} \) the structure constants of the Lie algebra. (For example, work with \( SU(2) \) with \( tr(T^aT^b) = \frac{1}{2}\delta^{ab} \), \( [T^a, T^b] = i f^{abc} T^c \) and \( f^{abc} = \epsilon^{abc} \), where \( T^a \) are realized by Pauli matrices).

Define \( J_n \) by

\[ J^a_n = \oint \frac{dz}{2\pi i} J^a(z) z^n \]  

(7)

(a) Calculate

\[ [J_m, J_n] \]  

(8)

The relations encoded in these commutators are called *Kac-Moody algebras*.

(b) The condition of unitarity for this algebra is that \( (J^a_n)^\dagger = J_{-n}^a \). In the case of the \( SU(2) \) algebra, show that the \( J_0^a \) satisfy the \( SU(2) \) Lie algebra generator relations with \( J_0^a \) self-adjoint matrices (they give angular momentum commutation relations).

(c) Define for the \( SU(2) \) current algebra \( J_0^\pm = J_1^1 \pm iJ_2^2 \). Write the \( SU(2) \) OPE algebra in terms of \( J^\pm(z), J^3(z) \). What is the unitarity relation in terms of the \( J^\pm \)?

(d) Show that \( J_1^+, J_{-1}^-, J_0^3 + a \) satisfy the \( SU(2) \) Lie algebra relations for \( a \) some constant determined from \( k \).

(e) A highest weight state representation of the \( SU(2)_k \) Kac-Moody algebra is defined by

\[ J_n|s\rangle = 0 \]

for all \( n > 0 \), and \( J_0^+|s\rangle = 0 \), while \( J_0^-|s\rangle = s|s\rangle \). Show by using the \( J_0 \) algebra, that unitarity of the \( SU(2) \) Kac-Moody algebra implies that \( s \) is a positive half integer, by considering the norm of the states \( (J_0^-)^n|s\rangle \)

(f) Show that for the previous setup, unitarity of the representation determined by \( |s\rangle \) implies that \( k \) is quantized and that \( s \) is bounded by \( k \), by considering the norm of the states \( (J_{-1}^+)^n|s\rangle \).
3. $c = 1/2$ Minimal model

By conformal invariance, the four point function of the twist fields is given by

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle = \left( \frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}} \right)^{1/8} F(x) \quad (9)$$

where $x = z_{12}z_{34}/z_{13}z_{24}$ is the conformally invariant cross ratio and $z_{ij} = z_i - z_j$.

Show by using the level two null state/operator descendant from $[\sigma]$, given by a linear combination $aL_{-2}\sigma + bL_{-1}^2\sigma$ that $F$ satisfies the following differential equation

$$\left( x(1 - x)\partial_x^2 + (1/2 - x)\partial_x + 1/16 \right) F(x) = 0 \quad (10)$$

Remember that $L_{-1}$ and $L_{-2}$ are given by contour integrals of the stress tensor around the operator. You need to use the Ward identity for the stress tensor in order to derived this result. This is equation 5.7 is Ginsparg’s notes. Note also that extra factors of $x$ have been absorbed in the prefactor. These simplify the form of the differential equation.