Problem 1 - Lorentz Symmetry

(a) Noether Currents
Consider a global symmetry
\[ \phi \rightarrow \phi + \delta \epsilon \phi \]
with \( \epsilon \) an infinitesimal parameter. Imagine variation with \( \epsilon \) x dependent. The Lagrangian transforms as,
\[ \mathcal{L} \rightarrow \mathcal{L} + \epsilon \partial_{\alpha} J^\alpha \]
If the variation is constant then the change in the Lagrangian must be a total derivative implying
\[ \partial_{\alpha} J^\alpha = 0. \]
The infinitesimal variation corresponding to a Lorentz transformation is
\[ \delta X^\mu = a_{\mu} \nu X^\nu \]
where \( a_{\mu\nu} = \eta_{\mu\beta} a_{\nu}^\beta \) is antisymmetric. Recall the Polyakov action
\[ S = \frac{-T}{2} \int d^2 \sigma \eta^{\alpha\beta} \partial_{\alpha} X \cdot \partial_{\beta} X \]
where \( \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( T = (2\pi\alpha')^{-1} \).
\[ J^\mu_{\alpha} = T (X^\mu \partial_{\alpha} X^\nu - X^\nu \partial_{\alpha} X^\mu) \]

(b) Problem 2 - Folded String Solution

(a)
Consider the motion of a string of the form
\[ x^0 = Et \]
\[ x^1 = C \sin \sigma \cos t \]
\[ x^2 = D \sin \sigma \sin t \]
The Euler-Lagrange equations
\[ \Box X^\mu = \left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 \]
are easily seen to be satisfied. However we must also check the Virasoro constraints
\[ T_{10} = T_{01} = \dot{X} \cdot X' = 0 \]
\[ T_{00} = T_{11} = \frac{1}{2} (\dot{X}^2 + X'^2) = 0. \]
The first constraint yields
\[ (\cos \sigma \sin \sigma \cos t \sin t)(-C^2 + D^2) = 0 \]
so we find the relation
\[ C = \pm D \]
The second constraint yields
\[ E^2 = C^2 + D^2 \]
after using \((\cos^2 \sigma + \sin^2 \sigma) = 1\) repeatedly.

(b)
The linear momentum of the string is
\[ P^\mu = T \int_0^\pi d\sigma d\tau \frac{dX^\mu}{d\tau} \]
The angular momentum is
\[ J^{\mu \nu} = T \int_0^\pi d\sigma \left( X^\mu \frac{dX^\nu}{d\tau} - X^\nu \frac{dX^\mu}{d\tau} \right) \]

(c)

(d)

**Problem 3 - pp-Waves**

(a)
We work in conformal gauge \(\eta_{\alpha \beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) on the worldsheet. The action
\[ S = -\frac{T}{2} \int d\sigma d\tau \left( -\dot{X}^\mu \dot{X}_\mu + X'^\mu X^\mu \right) \]
\[ = -\frac{T}{2} \int d\sigma d\tau \left( \dot{X}^+ \dot{X}^- + X_+^2 (\dot{X}^+)^2 - (\dot{X}^+)^2 - X'^+ X'^- - X_+^2 (X'^+)^2 + (X'^-)^2 \right) \]
The canonical conjugate momentum
\[ p^- = \frac{\partial S}{\partial \dot{X}^-} = \frac{T}{2} \dot{X}^+ = \frac{AT}{2} \]
Euler-Lagrange equations yield

\[ \frac{d}{dt}p_- = \frac{\partial S}{\partial X^-} = 0 \]

So

\[ p_- = \text{Const.} \]

\[ p_+ = \frac{\partial S}{\partial X^+} = \dot{X}^- + 2ATX^2 \]

where \( X^+ = A\tau \).

\[ S = -\frac{T}{2} \int d\sigma d\tau = A\dot{X}^- + A^2X^2 + \eta^{\alpha\beta}(\partial_\alpha X^\perp \partial_\beta X^\perp) \]

which shows the transverse oscillators are free massive scalar fields with respect to a flat metric on the worldsheet.

(b)

To set up lightcone coordinates on the worldsheet we require that

\[ \sqrt{-g^{\alpha\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

following the class notes.

(c)

In conformal gauge the equations of motion

\[ \partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta X_\mu \right) = \frac{\partial S}{\partial X^\mu} \]

simplify to

\[ \partial_+ \partial_- X^\pm = 0 \]

and

\[ \partial_+ \partial_- X^\perp = -2X^\perp \left( \frac{dX^+}{d\tau} \right)^2 \]

This is easily seen to be satisfied by

\[ X^+ = C(\sigma^+ - \sigma^-) \]

(d)

Simplifying the Euler-Lagrange equations from part (c) using \( X^+ = C(\sigma^+ - \sigma^-) \), the scalar fields \( X^\perp \) satisfy the wave equation

\[ \partial_+ \partial_- X^\perp = -2X^\perp C^2 \]

which is that of a free massive scalar field with respect to a flat metric on the worldsheet.
The most general solution for $X^-(\sigma^+, \sigma^-)$ is

$$X^-(\sigma^+, \sigma^-) = X_L^-(\sigma^+) + X_R^-(\sigma^-).$$

To find the full solution we must solve the Virasoro constraints:

$$g_{++} = G_{\mu\nu} \frac{dX^\mu}{d\sigma^+} \frac{dX^\nu}{d\sigma^+} = 0$$
$$g_{--} = G_{\mu\nu} \frac{dX^\mu}{d\sigma^-} \frac{dX^\nu}{d\sigma^-} = 0$$

$$g_{++} = -\partial_+ X^+ \partial_+ X^- + \partial_+ X^+ \partial_+ X^\perp - (X^\perp)^2 (\partial_+ X^+)^2$$
$$g_{--} = -\partial_- X^+ \partial_- X^- + \partial_- X^\perp \partial_- X^\perp$$

Using $X^+ = C(\sigma^+ - \sigma^-)$, the Virasoro constraints simplify to

$$g_{++} = -C \partial_+ X^- + (\partial_+ X^+)^2 - C^2 (X^\perp)^2$$
$$g_{--} = C \partial_- X^- + (\partial_- X^\perp)^2$$

which determine $X_L^-(\sigma^+)$ and $X_R^-(\sigma^-)$ respectively.