String Theory 230A Homework # 4 Solutions

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**Problem 1 (Polchinski 2.1)**

\[
\partial \bar{\partial} \ln |z|^2 = \partial \bar{\partial} (\ln z + \ln \bar{z}) \\
= \partial \left( \frac{1}{\bar{z}} \right)
\]

Away from \((z, \bar{z}) = (0, 0)\), \(\ln |z|^2\) is \(C^\infty\) and we can freely apply equality of mixed paritals to conclude

\[
\partial \bar{\partial} \ln |z|^2 = \partial \bar{\partial} (\ln z + \ln \bar{z}) \\
= \bar{\partial} \partial \left( \frac{1}{z} \right)
\]

Applying the divergence theorem

\[
\int_R d^2z \left( \partial_z v^z + \partial_{\bar{z}} v^{\bar{z}} \right) = i \oint v^z d\bar{z} - v^{\bar{z}} dz
\]

with \(v_z = 1/\bar{z}\)

\[
\int_R \partial \bar{\partial} \ln |z|^2 = \int_R d^2z \partial \bar{\partial} \left( \frac{1}{z} \right) \\
= i \oint \frac{dz}{\bar{z}}
\]

Evaluating over the contour \(z = Re^{i\theta}\), we have \(\bar{z} = Re^{-i\theta}, d\bar{z} = -i\bar{z} d\theta\) which lets us evaluate the contour integral

\[
i \oint \frac{d\bar{z}}{z} = i \oint (-i) d\theta = 2\pi
\]

Therefore

\[
\int_R \partial \bar{\partial} \ln |z|^2 = 2\pi
\]

and we conclude that

\[
\partial \bar{\partial} \ln |z|^2 = 2\pi \delta^2(z, \bar{z}).
\]
Problem 2 - Linear Dilaton CFT

(a)
For a free scalar field with (modified) stress energy tensor

\[ T(z) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) : + \beta \partial^2 X \]

the OPE : \( T(z)T(0) \) : will contain the usual contribution

\[ T_0(z)T_0(0) \sim \frac{1}{2z^4} + \frac{2T_0(0)}{z^2} + \frac{\partial T_0(0)}{z} \]

where \( T_0(z) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \) but will have additional contributions from

\[ : \beta \partial^2 X(z) : \beta \partial^2 X(w) : \sim -\frac{\alpha'^2}{2} \partial^2 \partial^2 \ln |z-w|^2 \]

\[ \sim \frac{6\alpha' \beta^2}{2(z-w)^4} \]

There are cross contractions of the form

\[ -\frac{1}{\alpha'} : \beta \partial^2 X(z) : \partial' X(w) \partial' X(w) : + (z \leftrightarrow w) = \frac{-2\beta \partial' X(w)}{(z-w)^3} + \frac{2\beta \partial X(z)}{(z-w)^3} \]

Taylor expanding

\[ \partial X(z) = \partial' X(w) + (z-w) \partial^2 X(w) + \frac{(z-w)^2}{2} \partial^3 X(w) + \ldots \]

we have

\[ \frac{-2\beta \partial' X(w)}{(z-w)^3} + \frac{2\beta \partial X(z)}{(z-w)^3} = \left[ \frac{-2\beta \partial' X(w)}{(z-w)^3} + \frac{2\beta \partial X(w)}{(z-w)^3} \right] + \frac{2\beta}{(z-w)^2} \partial^2 X(w) + \frac{\beta}{(z-w)} \partial' \partial^2 X(w). \]

The term in [ ]'s cancels and we are left with the appropriate modification to preserve the OPE

\[ T(z)T(0) \sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} \]

where we have determined

\[ c = 1 + 6\alpha' \beta^2 \]

(b)

(c)

Problem 3

(a)
Using the two-point functions

\[ : X_L(z)X_L(z') : = X_L(z)X_L(z') + \frac{\alpha'}{2} \ln(z-z') \]

\[ : X_R(\bar{z})X_R(\bar{z}') : = X_R(\bar{z})X_R(\bar{z}') + \frac{\alpha'}{2} \ln(\bar{z}-\bar{z}') \]

\[ : X_L(z)X_R(\bar{z}') : = 0 \]
We derive the more general contraction rule
\[ F := \exp \left( -\frac{\alpha'}{2} \int dz dz' \ln(z-z') \frac{\delta}{\delta x_F} \frac{\partial}{\partial X_{GL}} - \frac{\alpha'}{2} \int d\bar{z} d\bar{z}' \frac{\delta}{\delta x_{FR}} \frac{\delta}{\delta x_{GR}} \right) : F G : \]

using this,
\[ O_{k,\bar{k}}(z,\bar{z}) := \exp \left( \frac{1}{4} k \bar{k} \ln(z) + \frac{1}{4} \bar{k} k \ln(\bar{z}) \right) : O_{k',\bar{k}'}(0,\bar{0}) : \]

See also Polchinski 8.2.19.

(b)
Operators \( O_{k,\bar{k}} \) and \( O_{k',\bar{k}'} \) are mutually local if and only if their OPE is single-valued on the complex plane which requires \( z^{k} \bar{k} \) to be single-valued. Therefore
\[ \langle k, k' \rangle = k \bar{k}' - \bar{k} k' \]

must be an integer.

(c)
Given a set \( S \) of mutually local operators closed under the OPE and two operators \( O_{n(k,\bar{k})}, O_{m(k',\bar{k}')}, \in S \) then
\[ O_{n(k,\bar{k})} \in S \]
since it occurs in the OPE of \( O_{k,\bar{k}} \) with itself \( n \) times. Similarly we can take the OPE of \( O_{n(k,\bar{k})} \) and \( O_{m(k',\bar{k}')} \) to conclude \( O_{n(k,\bar{k})} + m(k',\bar{k}') \in S \).

(c) Bonus
This part was unclear, but additionally we can show that if \( O_{k,\bar{k}}, O_{k',\bar{k}'} \in S \) are mutually local operators then \( O_{n(k,\bar{k})} + m(k',\bar{k}') \) is mutually local with all other operators in \( S \). For example with given three mutually local operators \( O_{k,\bar{k}}, O_{k',\bar{k}'}, O_{l,\bar{l}}, \in S \) then
\[ O_{n(k,\bar{k})} + m(k',\bar{k}') \]
is mutually local to \( O_{l,\bar{l}} \) if
\[ n(lk + m\bar{k}') - \bar{l}(nk + m\bar{k}) \]
is an integer. A simple rearrangement of terms shows this expression equals
\[ n(lk - \bar{l}k) + m(lk' - \bar{l}k') \]
which is a sum of integer terms since \( O_{k,\bar{k}} \) and \( O_{k',\bar{k}'} \) are mutually local.

Problem 4
From the Ward identities we can always show that the OPE of the stress energy tensor \( T(z) \) with an operator \( O(0,\bar{0}) \) is of the form
\[ T(z) O(0,\bar{0}) = \frac{\partial O}{\partial z} + \sum_{j \geq 2} a_{-j} O_j(a,\bar{0}). \]
However in the spirit of this problem we directly show this for operators of the form
\[ \mathcal{O} = \partial^{a_1} X \partial^{a_2} X \ldots \partial^{a_k} X. \]

First we verify the claim for the special case
\[ \mathcal{O}^{(n)}(w, \bar{w}) = \partial^n X(w, \bar{w}) \]

\[ : T(z) :: \mathcal{O}^{(n)}(w, \bar{w}) : = \partial X(z) \partial^n \left( \frac{1}{z - w} \right) \]
\[ = \frac{n!}{(z - w)^n} \left[ \partial X(w) + \cdots + \left( \frac{z - w}{n} \right)^n \partial^{n+1} X(w) + \cdots \right] \]
\[ = \frac{\partial \mathcal{O}^n(w, \bar{w})}{z - w} + \sum_{j \geq 2} a_j \mathcal{O}_j(o, \bar{0}). \]

where we have Taylor expanded \( X(z) \) about \( w \). Now consider a more general operator
\[ \mathcal{O}(w, \bar{w}) = \mathcal{O}^{(a_1)} \mathcal{O}^{(a_2)} \ldots \mathcal{O}^{(a_k)}(w, \bar{w}) \]

The contributions to the \( 1/(z - w) \) pole in the \( : T(z) :: \mathcal{O}(w, \bar{w}) :_1 \) contraction OPE with a single contraction are
\[ \partial \mathcal{O}^{(a_1)} \mathcal{O}^{(a_2)} \ldots \mathcal{O}^{(a_n)} + \mathcal{O}^{(a_1)} \partial \mathcal{O}^{(a_2)} \ldots \mathcal{O}^{(a_n)} + \cdots + \mathcal{O}^{(a_1)} \mathcal{O}^{(a_2)} \ldots \partial \mathcal{O}^{(a_n)} \]

which is simply \( \partial \mathcal{O}(w, \bar{w}) \). The terms with two contractions have all poles with order at least two and all operators in the OPE are expanded about \( w \) so there is no way to get a contribution to the \( 1/(z - w) \) pole.

Alternatively, the generating functional for normal ordering (Polchinski 2.2.10) can be used to give a short proof of this result.