Problem 1

(a)
We need to compute the OPE of the vertex operator

\[ O_{\mu\nu} = f_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu \exp (ik_\alpha X^\alpha) \]

with the stress energy tensor

\[ T = -\frac{1}{\alpha'} : \partial X^\mu \bar{\partial} X^\mu : \]

We only need to Taylor expand the contraction rule

\[ : F : : G : = \exp \left( -\frac{\alpha'}{2} \int dz dw \ln (z - w) \frac{\delta}{\delta X_F(z)} \frac{\partial}{\delta X_G(w)} \right) : F G : \]

to second order since only two \( X(z) \) fields occur in \( T(z) \).

(b)

Problem 3

(a)
From world-sheet translation symmetry

\[ \delta z = \epsilon v(z) \]

we have the conserved current

\[ j(z) = iv(z)T(z). \]

The infinitesimal transformation of \( T(z) \) under world-sheet translation symmetry is

\[ \text{Res}_{z \rightarrow w} j(z) T(w) = \frac{1}{\imath \epsilon} \delta_{t} T(z) \]

Using the \( T(z)T(w) \) OPE,

\[ \text{Res}_{z \rightarrow w} j(z) T(w) = \text{Res}_{z \rightarrow w} iv(z)T(z)T(w) \]

\[ = iv(z) \left[ \frac{c}{2\imath \epsilon^2} + \frac{2}{w^2} T(w) + \frac{1}{\imath \epsilon} \partial_w T(w) \right] \]
Finally we must Taylor expand
\[ v(z) = v(w) + (z-w)\partial_w v(w) + \frac{(z-w)^2}{2}\partial_w^2 v(w) + \frac{(z-w)^3}{6}\partial_w^3 v(w) \]
to extract the residue
\[ i \left[ \frac{c}{12} \partial_w^3 v(w) + 2\partial_w v(w)T(w) + \partial_w T(w) \right] = -\frac{i}{\epsilon} \delta \epsilon T(w) \]
Factoring out the \( \epsilon \)'s
\[ \delta \epsilon T(w) = - \left[ \frac{c}{12} \partial_w^3 v(w) + 2\partial_w v(w)T(w) + \partial_w T(w) \right] \]
(b)
\[ \{g(f(z)), z\} = \frac{2 \left[ g''' f'^3 + 3 g'' f'' f' + g' f''' \right] g' f' - 3 \left[ g'' f'^2 + g' f'' \right]^2}{2 (g' f')^2} \]
\[ = \left[ \frac{2 g'''}{2g'^2} \right] f'^4 \frac{f'^3}{f'^2} + \left[ \frac{2 f'''}{2f'^2} \right] \left[ \frac{3 f''}{2f'^2} \right] \]
\[ = \{g(w), w\} f'^2 + \{f(z), z\} \]
since the cross terms \[ \frac{6g'' f'' f'^2 - 6g'' f'' f' f'^2}{2 (g' f')^2} \]
cancel. We’ve derived the composition rule
\[ \{g(f(z)), z\} = \{g(w), w\} f'^2 + \{f(z), z\} \]
(c)
World-sheet translation symmetry \( z \rightarrow z + \epsilon v(z) \) is the infinitesimal form of the transformation \( z \rightarrow f(z) \). As a warm-up to determining how the stress energy tensor transforms under this symmetry we first consider how the scalar field \( X^\mu(z) \) transforms.
\[ \delta X^\mu(z) = -\epsilon v(z) \partial_z X^\mu(z). \]
For future use we collect three useful facts
\[ T'(z + \epsilon v(z)) = T'(z) + \epsilon \partial v(z) T(z) \]
to first order in \( \epsilon \) since \( T(z) \) and \( T'(z) \) are equal to first order in \( \epsilon \).
\[ (\partial_z f)^{-2} = 1 - 2\epsilon \partial v(z) \]
to first order in \( \epsilon \), and
\[ \{f(z), z\} = \frac{2 \epsilon \partial^3 v(z)(1 + \epsilon \partial v(z)) - 3\epsilon^2(\partial^2 v(z))^2}{2(1 + \epsilon \partial v(z))^2} \]
\[ = \epsilon \partial^3 v(z). \]
Reducing the finite form of the transformation law
\[ (\partial_z f)^2 T'(f(z)) = T(z) - \frac{c}{12} \{f(z), z\} \]
yields
\[ T'(z + \epsilon v(z)) = (\partial_z f)^{-2} \left[ T(z) - \frac{c}{12} \{ f(z), z \} \right]. \]
Substituting in our previous results,
\[ T'(z) + \epsilon \partial_v v(z) T(z) = \left( 1 - 2\epsilon \partial_v v(z) \right) \left[ T(z) - \frac{c}{12} \epsilon \partial^3 v(z) \right]. \]
From the definition of variation
\[ \epsilon \delta_v T(z) = T'(z) - T(z) \]
and collecting terms,
\[ \delta T(z) = -\frac{c}{12} \partial^3 v(z) - 2 \partial v(z) T(z) - v(z) \partial T(z). \]

(d)
Recall (or look up on Wikipedia) that a conformal transformation can be decomposed as follows
\[ \frac{az + b}{cz + d} = f_4 \circ f_3 \circ f_2 \circ f_1(z) \]
where
\[ f_1(z) = z + d/c \quad \text{Translation} \]
\[ f_2(z) = 1/z \quad \text{Inversion and reflection} \]
\[ f_3(z) = (- (ad - bc)/c^2)z \quad \text{Dilation and rotation} \]
\[ f_4(z) = z + a/c \quad \text{Translation} \]
Since \( S(f_j, z) = 0 \) for \( j = 1, 2, 3, 4 \) it follows from the composition rule that \( S(f, z) = 0 \) for \( f(z) = \frac{az + b}{cz + d} \). Alternatively when can use Mathematica to compute this directly
\[ -\frac{3}{2} \left( \frac{2z^2(a^2z^2 + 2acz + d^2)}{(daz + c)^2} - \frac{2ac}{(daz + c)^2} \right)^2 + \left( \frac{6a^2}{(daz + c)^2} + \frac{6ac^2}{(daz + c)^2} \right) \left( \frac{-c(b+az)}{(daz + c)^2} + \frac{a}{(daz + c)^2} \right) \]
which simplifies to 0.

(e)
For \( f(z) = \log z \),
\[ S(f(z), z) = \frac{4z^{-4} - 3z^{-4}}{2z^{-2}} = \frac{1}{2z^2} \]

Problem 2

(a)
We first simplify the formula for the conformal weights,
\[ h_{r,s} = \frac{c - 1}{24} + \frac{1}{4} (r\alpha_+ + s\alpha_-)^2 \]
Using Mathematica to compute and factor the determinant:

$$\text{det}(\text{L}_3)$$

The expected form from the Kac formula is

$$\langle h|L_1^3L_2^3L_3^3|h\rangle / \langle h|L_1^2L_2L_3^2|0\rangle = K$$

where

$$\alpha_\pm = (24)^{-1/2} \left[ (1 - c)^{1/2} \pm (25 - c)^{1/2} \right]$$

$$h_{r,s} = \frac{c - 1}{24} + \frac{1}{(4)(24)} \left[ r^2\alpha_+^2 + 2rs\alpha_+\alpha_- + s^2\alpha_-^2 \right]$$

$$= \frac{c - 1}{24} + \frac{1}{(4)(24)} \left[ (26 - 2c)(r^2 + s^2) + 2rs(-24) + 2(r^2 - s^2)\sqrt{(1 - c)(25 - c)} \right]$$

$$= \frac{c}{24} (1 - \frac{r^2 + s^2}{2}) + \left[ \frac{(r - s)^2}{4} + \frac{r^2 + s^2 - 2}{48} \right] + \frac{1}{48}(r^2 - s^2)\sqrt{(1 - c)(25 - c)}$$

for \((r, s) = (3, 1)\)

$$h_{3,1} = \frac{7 - c + \sqrt{(1 - c)(25 - c)}}{6}$$

Similarly

$$h_{2,1} = \frac{5 - c + \sqrt{(1 - c)(25 - c)}}{16}$$

The Kac determinant at level three is the determinant of:

$$\mathcal{M}^3(c, h) = \begin{pmatrix} 
\langle h|L_1^3L_2^3L_3^3|h\rangle & \langle h|L_1L_2L_3^2|1\rangle & \langle h|L_3L_1^2L_2^2|1\rangle \\
\langle h|L_1^3L_2L_3|0\rangle & \langle h|L_1^2L_2L_3|1\rangle & \langle h|L_3L_2^2L_1^2|1\rangle \\
\langle h|L_1^3L_3|0\rangle & \langle h|L_1^2L_3|1\rangle & \langle h|L_3L_1^2|0\rangle 
\end{pmatrix}$$

$$= \begin{pmatrix} 
24h(1 + h)(1 + 2h) & 12h(1 + 3h) & 24h \\
12h(1 + 3h) & h(8 + c + 8h) & 10h \\
24h & 10h & 2(c + 3h) 
\end{pmatrix}$$

Using Mathematica to compute and factor the determinant:

$$\text{det}(\mathcal{M}^3(c, h)) = 48h^2(2 + c - 7h + ch + 3h^2)(c - 10h + 2ch + 16h^2).$$

The expected form from the Kac formula is

$$\text{det}(\mathcal{M}^3(c, h)) = K \prod_{1 \leq r < s \leq 3} \prod (h - h_{r,s})^{P(3-rs)}$$

$$= K(h - h_{1,1})^{P(2)}(h - h_{1,2})^{P(1)}(h - h_{2,1})^{P(1)}(h - h_{1,3})^{P(0)}(h - h_{3,1})^{P(0)}$$

$$= Kh^2(h - h_{1,2})(h - h_{2,1})(h - h_{1,3})(h - h_{3,1})$$

where we have used the values of the partition function \(P(2) = 2, P(1) = 1, P(0) = 1\) and \(K\) is a constant. Using the quadratic formula to find the roots of the Kac determinant we see that they agree with Kac’s formula. We now show how the Kac matrix can be computed by hand. To simplify the computation of \(\langle h|L_1^3L_2^3L_3^3|h\rangle\) we note

$$L_0L_{-1}^n|h\rangle = (h + n)L_{-1}^n|h\rangle$$

and

$$L_1L_{-1}^1|h\rangle = 2h|h\rangle$$

$$L_1L_{-1}^2|h\rangle = 2(1 + 2h)L_{-1}^1|h\rangle$$

$$L_1L_{-1}^3|h\rangle = 6(1 + h)L_{-1}^2|h\rangle$$
Applying the standard commutation relations

\[
\langle h| L^3_1 L^3_{-1} | h \rangle = \langle h| L^2_1 2 L^2_0 L^2_{-1}|h \rangle + \langle h| L^2_1 L_{-1} L^2_{1} | h \rangle \\
= [(2(h+2) + 2(1+2h)] \langle h| L^2_1 L^2_{-1} | h \rangle \\
= 6(h+1) \langle h| L^2_1 L^2_{-1} | h \rangle \\
= 24h(1 + h)(1 + 2h)
\]

We see that we can easily compute \( \langle h| L^n_1 L^n_{-1} | h \rangle \) recursively in this fashion.

| n  | \( L_0 L^n_{-1}| h \rangle \) | \( L_1 L^n_{-1}| h \rangle \) |
|-----|---------------------------------|---------------------------------|
| 1   | \((h+1)\)                        | \(2h\)                           |
| 2   | \((h+2)\)                        | \(4h+2\)                         |
| 3   | \((h+3)\)                        | \(6h+6\)                         |
| 4   | \((h+4)\)                        | \(8h+12\)                        |
| 5   | \((h+5)\)                        | \(10h+20\)                       |

By induction

\[
\langle h| L^n_1 L^n_{-1} | h \rangle = \prod_{j=1}^{n} 2jh + \frac{j(j-1)}{2}.
\]

This is just a statement about the \( SU(2) \) subalgebra of the Virasoro algebra spanned by \( L_{\pm 1}, L_0 \). Now for a more involved term,

\[
\langle h| L_1 L_2 L^3_{-1} | h \rangle = \langle h| L_2 L_1 L^3_{-1} | h \rangle - \langle h| L_3 L^3_{-1} | h \rangle \\
= 6(1 + h)(6h) - 24h \\
= 12h(3h+1)
\]

where we have used the results for the Kac determinant at level 2.

(b)

For the minimal model \( m = 3 \) with \( c = 1 - 6/(m)(m+1) = 1/2 \), we compute the conformal weights using the formula from the previous problem, \( h_{1,2} = 1/16 \) and \( h_{2,1} = 1/2 \). To determine the null descendant at level 2, we look for states of the form \( (aL_{-2} + bL_{-1})|\phi_{r,s}⟩ \) of 0 norm. Such a state is then an eigenvector of the \( 2 \times 2 \) Kac determinant with eigenvalue zero. Recall the Kac determinant at level 2 is

\[
\begin{pmatrix}
4h + c/2 & 6h \\
6h & 4h(1+2h)
\end{pmatrix}
\]

For \( \phi_{2,1} \) with \( h_{2,1} = 1/2 \) this simplifies to

\[
\begin{pmatrix}
9/4 & 3 \\
3 & 4
\end{pmatrix}
\]

So the eigenvector corresponding to the 0 eigenvalue is \((4, -3)\). So \( a/b = -4/3 \).

For \( \phi_{1,2} \) with \( h_{1,2} = 1/16 \) the Kac determinant matrix is

\[
\begin{pmatrix}
1/2 & 3/8 \\
3/8 & 9/32
\end{pmatrix}
\]

So the eigenvector corresponding to the 0 eigenvalue is \((3, -4)\). So \( a/b = -3/4 \).