String Theory 230A Homework # 1 Solutions

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Problem 2 - Geodesic Equations

Consider a relativisitic point particle in $d+1$ dimensions, with action

$$S_\tau = -m \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} d\tau = -m \int ds$$

(a)

Under the reparametrization

$$\tau' = \tau'(\tau)$$

the action becomes

$$\begin{aligned} S_\tau &= -m \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}} \left(\frac{d\tau'}{d\tau}\right) d\tau \\ &= -m \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}} d\tau' \\ &= S_{\tau'} \end{aligned}$$

where we have repeatedly used the chain rule. So the action is invariant under reparametrization.

(b)

If we choose

$$\frac{ds}{d\tau} = \left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2}$$

then

$$\left(-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = 1$$

by part (a).

(c)

We compute the following variations

$$\begin{aligned} \frac{\partial S}{\partial \dot{x}^\alpha} &= g_{\alpha\mu}\dot{x}^\mu \\ \frac{\partial S}{\partial x^\alpha} &= \frac{1}{2}(\partial_\alpha g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu \\ \frac{d}{ds}\left(\frac{\partial S}{\partial \dot{x}^\alpha}\right) &= (\partial_\nu g_{\alpha\mu})\dot{x}^\mu \dot{x}^\nu + g_{\alpha\mu}\ddot{x}^\mu \end{aligned}$$

where we have set

$$\left(-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = 1$$

after computing the variation. After symmetrizing

$$(\partial_\nu g_{\alpha\mu})\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu}\right)\dot{x}^\mu \dot{x}^\nu$$

the Euler-Lagrange equation yields

$$\ddot{x}_\mu + \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0.$$

Finally we raise the index on $\ddot{x}_\mu$ by contracting both sides with $g^{\alpha\beta}$ yielding the geodesic equation

$$\boxed{\ddot{x}^\beta + \Gamma^\beta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0.}$$

String Theory 229A Homework Solution Set #1

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January 24, 2006

1. (15 Points)

   (a) The Lagrangian for the point particle in an EM field is, from the lecture notes:

$$\mathcal{L} = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu} \pm eA_\mu \dot{x}^\mu \tag{1}$$

   Therefore the Euler-Lagrange equations give the equations of motion:

$$\frac{\partial}{\partial \tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{2}$$

$$\frac{\partial}{\partial \tau}\left(\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} \pm eA_\mu\right) = \pm e\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu \tag{3}$$

$$m\dot{u}_\mu \pm e\frac{\partial x^\nu}{\partial \tau}\frac{\partial}{\partial x^\nu}A_\mu = \pm e\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu \tag{4}$$

$$\tag{5}$$

$$m\dot{u}_\mu \pm e(\partial_\nu A_\mu - \partial_\mu A_\nu)\dot{x}^\nu = 0 \tag{6}$$

$$m\dot{u}_\mu = \pm eF_{\mu\nu}\dot{x}^\nu \tag{7}$$

   (b) Instead of the hint in the problem, we can obtain the identical system by choosing a particular parameterization, namely we choose τ such that $-\dot{x}^\mu \dot{x}_\mu = m^2$. Taking this constraint, we see that the equation of motion derived in part a becomes simply:

$$\ddot{x}_\mu = \pm eF_{\mu\nu}\dot{x}^\nu \tag{8}$$

$$m^2 = -\dot{x}^\mu \dot{x}_\mu \tag{9}$$

   Choose the positive sign for e so we stop carrying around those pesky $\pm$'s. Plugging in for the explicit form of $F_{\mu\nu}$, we get

$$\ddot{x}_1 = \ddot{x}_2 = 0 \tag{10}$$

$$\ddot{x}_3 = -eB\dot{x}_4 \tag{11}$$

$$\ddot{x}_4 = eB\dot{x}_3 \tag{12}$$

   plus the constraint above. These are a standard set of coupled differential equations, their solutions are well know, and easily derived:

$$x_1 = a_1 + a_2\tau \tag{13}$$
$$x_2 = b_1 + b_2\tau \tag{14}$$
$$x_3 = c_1 + \frac{c_2}{eB}\sin(eB\tau) + \frac{c_3}{e^2B^2}\cos(eB\tau) \tag{15}$$
$$x_4 = c_4 - \frac{c_2}{eB}\cos(eB\tau) + \frac{c_3}{e^2B^2}\sin(eB\tau) \tag{16}$$
$$\tag{17}$$

If we assume simplifying initial conditions such that $a_1 = 0$ an $a_2 = 1$, then $\tau = x_1 = t$, and we can rewrite the solution in a more standard classical form:

$$\vec{x}(t) = (b_1 + b_2 t)\hat{x}_2 + \left(c_1 + \frac{c_2}{eB}\sin(eBt) + \frac{c_3}{e^2B^2}\cos(eBt)\right)\hat{x}_3 \tag{18}$$
$$+ \left(c_4 - \frac{c_2}{eB}\cos(eBt) + \frac{c_3}{e^2B^2}\sin(eBt)\right)\hat{x}_4 \tag{19}$$

This motion is now immediately identified as a helix in the 3-4 plane, as we would expect for the motion of a charged particle in a constant magnetic field.

(c) Starting from the covariant form of the lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\dot{x}^\mu \dot{x}_\mu - m^2) \pm eA_\mu \dot{x}^\mu \tag{20}$$

(note there is a typo in the lecture notes with the sign of m^2) The conjugate momenta of the x^μ is trivially found:

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -\dot{x}_\mu \pm eA_\mu \tag{21}$$

So the Hamiltonian is:

$$H \equiv p^\mu \dot{x}_\mu - \mathcal{L} \tag{22}$$
$$= -p^\mu(p_\mu \mp eA_\mu) + \frac{1}{2}(\dot{x}^\mu \dot{x}_\mu - m^2) \mp eA^\mu \dot{x}_\mu \tag{23}$$
$$= -p^\mu(p_\mu \mp eA_\mu) + \frac{1}{2}((p_\mu \mp eA_\mu)(p^\mu \mp eA^\mu) - m^2) + eA^\mu(p_\mu \mp eA_\mu) \tag{24}$$
$$= -\frac{1}{2}(p_\mu \mp eA_\mu)(p^\mu \mp eA^\mu) - \frac{1}{2}m^2 \tag{25}$$

Because the original action was manifestly covariant, we know that the constraint will just be $H = 0$ or $H|\psi\rangle = 0$ on quantum states. Let's verify this. We know that the constraint with $\eta = 1$ from the η equations of motion imply that $m^2 = -\dot{x}^\mu \dot{x}_\mu$. Therefore, in terms of the momentum variables, $m^2 + (p_\mu \mp eA_\mu)(p^\mu \mp eA^\mu) = 0$, which is exactly $H = 0$ as we derived above.

So, defining $D_\mu = \partial_\mu \mp ieA_\mu$, if we replace the momentum in the normal way $p_\mu \to -i\partial_\mu$, the quantum constraint is:

$$H|\psi\rangle = 0 \tag{26}$$
$$(-i\partial_\mu \mp eA_\mu)(-i\partial^\mu \mp eA^\mu) + m^2)|\psi\rangle = 0 \tag{27}$$
$$(\partial_\mu \mp ieA_\mu)(\partial^\mu \mp ieA^\mu) - m^2)|\psi\rangle = 0 \tag{28}$$
$$(D_\mu D^\mu - m^2)|\psi\rangle = 0 \tag{29}$$

That is the Klein-Gordon equation.

2. 15 Points

(a) The action of the point particle coupled to a background of the form given in the problem becomes:

$$S = -m \int d\tau \sqrt{\dot{x}^+ \dot{x}^- + \beta x_\perp^2 \dot{x}^{+2} - \dot{x}_\perp^2} \tag{30}$$

Using reparameterization invarience to choose $x^+ = \tau$, it becomes:

$$S = -m \int d\tau \sqrt{\dot{x}^- + \beta x_\perp^2 - \dot{x}_\perp^2} \tag{31}$$

Now we follow the lecture notes, and find the Hamiltonian corresponding to this system. First we get the conjugate momentum:

$$p_\perp = \frac{\partial \mathcal{L}}{\partial \dot{x}_\perp} = \frac{m\dot{x}_\perp}{\sqrt{A}} \tag{32}$$

$$p_- = \frac{\partial \mathcal{L}}{\partial \dot{x}^-} = -\frac{m}{2\sqrt{A}} \tag{33}$$

$$A = \dot{x}^- + \beta x_\perp^2 - \dot{x}_\perp^2 \tag{34}$$

Note that

$$\sqrt{A} = -\frac{m}{2p_-} \tag{35}$$

$$\dot{x}_\perp = -\frac{p_\perp}{2p_-} \tag{36}$$

$$\dot{x}^- = \frac{m^2 + p_\perp^2}{4p_-^2} - \beta x_\perp^2 \tag{37}$$

The Hamiltonian is then

$$H = p_- \dot{x}^- + p_\perp \dot{x}_\perp - \mathcal{L} \tag{38}$$

$$= \frac{m^2 + p_\perp^2}{4p_-} - p_- \beta x_\perp^2 - \frac{p_\perp^2}{2p_-} - \frac{m^2}{2p_-} \tag{39}$$

$$= -\frac{1}{4}\left(\frac{m^2 + p_\perp^2}{p_-}\right) - p_- \beta x_\perp^2 \tag{40}$$

We know that p_- is constant since it is related to p^+ by $p_- = -\frac{1}{2}p^+$, so in terms of p^+ we simply have

$$H = \frac{m^2}{2p^+} + \frac{p_\perp^2}{2p^+} + \frac{\beta p^+ x_\perp^2}{2} \tag{41}$$

We recognize this as simply the hamiltonian of the simple harmonic oscillator, with β being the frequency, and p^+ playing the role of mass.

(b) The equations of motion (given the hamiltonian above) are trivially:

$$\ddot{x}_\perp + \beta x_\perp = 0 \tag{42}$$

This is simply the standard harmonic oscillator solution for the directions perpendicular to the light cone.

$$x_\perp(\tau) = c_1 \cos(\sqrt{\beta}\tau) + c_2 \sin(\sqrt{\beta}\tau) \tag{43}$$

For the one extra direction x^-, we use the fact that p_- is a constant, therefore A is a constant as defined above, call it d:

$$A = d \tag{44}$$

$$\dot{x}^- = d + \dot{x}_\perp^2 - \beta x_\perp^2 \tag{45}$$

$$\dot{x}^- = d + (c_2^2 - c_1^2)\cos(2\sqrt{\beta}\tau) - 2c_1c_2 \sin(2\sqrt{\beta}\tau) \tag{46}$$

$$x^- = e + d\tau + \frac{(c_2^2 - c_1^2)}{2\sqrt{\beta}} \sin(2\sqrt{\beta}\tau) + \frac{c_1c_2}{\sqrt{\beta}} \cos(2\sqrt{\beta}\tau) \tag{47}$$

with a little bit of algebra, where c_1, c_2, d, and e are constants.

(c) With the Lagrange multiplier, the action becomes:

$$S = \frac{1}{2}\int d\tau \left(\eta^{-1}(\dot{x}^- + \beta x_\perp^2 - \dot{x}_\perp^2) + \eta m^2\right) \tag{48}$$

The η equation of motion is:

$$\eta^2 = \frac{\dot{x}^- + \beta x_\perp^2 - \dot{x}_\perp^2}{m^2} \tag{49}$$

so upon substitution back into the action we recover the original action.
The Euler Lagrange equation of motion for $x_\perp$ is:

$$\eta^{-1}\ddot{x}_\perp + \eta^{-1}\beta x_\perp = 0 \tag{50}$$

(d) Choosing $\eta = 1$, the equation of motion becomes:

$$\ddot{x}_\perp + \beta x_\perp = 0 \tag{51}$$

exactly as we had before (actually, we could have chosen η as anything except 0 and this equation of motion would have been the same, but the next equation would not be) Combined with the constraint

$$m^2 = \dot{x}^- + \beta x_\perp^2 - \dot{x}_\perp^2 \tag{52}$$

we have the exact same set of equations as we did in part b, and therefore the solutions will be identical as well.

(e) Let's see the hamiltonian again:

$$H = \frac{m^2}{2p^+} + \frac{p_\perp^2}{2p^+} + \frac{\beta p^+ x_\perp^2}{2} \tag{53}$$

As stated above, this is nothing but the harmonic oscillator hamiltonian in the $D-2$ directions perpendicular to the light cone. So upon quantization, you will get $D-2$ uncoupled harmonic oscillators, each contributing energy $\sqrt{\beta}(n+\frac{1}{2})$ for some integer quantum number n. Since you have $D-2$ oscillators, you have $D-2$ quantum numbers to describe the state.