Week 4

Reading material from the books

- *Polchinski, Chapter 2*
- *Becker, Becker, Schwartz, Chapter 3*

1 Conformal field theory

So far, we have seen that the string theory can be described as a quantum field theory coupled to gravity in two dimensions. In order for the coupling to be consistent, one finds that the field theory must be invariant under local rescalings of the metric: a theory with this property is called a conformal field theory.

For this to be true, the particles have to be all massless: a mass would set a preferred scale in the system and this forbids the theory from being insensitive to scale variations.

So in essence, all 'particles' in a CFT are massless. For free theories, this is not a problem, but in interacting theories we have a hard time calculating standard scattering quantities like an S-matrix. They end up having severe infrared singularities from emission of 'soft radiation'.

Thus, one should look for other observables. A typical observable in a field theory is a local operator insertion, described by some operator that depends on the coordinates

$$\mathcal{O}(x) \tag{1}$$

General observables are going to be correlators of various of these operator insertions. These general correlations are also called Green's functions for a field theory.

When we do rescalings by some factor $g_{\mu\nu}(x) \to \exp(2\sigma(x))g_{\mu\nu}(x)$ it would be nice if the operator transforms in a convenient way:

$$\mathcal{O}(x) \to \exp(-\Delta\sigma(x))\mathcal{O}(x) \tag{2}$$

Apart from this, $\mathcal{O}$ could also be a tensor under general transformations of variables. This is, we would like to find operators that transform covariantly. These special operators are going to be the basic building blocks of conformal field theories.

The quantity Δ is going to be called their conformal dimension.

Before we get deep into conformal field theory, we would like a simple example of such a theory: the free field massless boson in two dimensions. The idea is to explore a little bit how general correlations of fields look like, and then go from there to establish the foundations of conformal field theory more seriously.

Before we return to the problem of quantizing strings and studying the string theory scattering amplitudes, we need to get the Conformal Field theory techniques straightened out.

2 Normal Ordering

Most of the notation we will follow is in the first few pages of chapter 2 of Polchinski

The *really important equations* are given by

$$\langle \mathcal{F}[X] \rangle = \int \mathcal{D}X \exp(-S)\mathcal{F}[X] \tag{3}$$

which defines the path integral formula. This is completely equivalent to canonical quantization in the case of free fields, and it's a more subtle object that defines the quantum field theory in the interacting case.

We will be dealing in the meantime with a free field theory for a single boson. Thus

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_1 X \partial_1 X + \partial_2 X \partial_2 X) \tag{4}$$

The second most important issue, is that everything will be written in terms of complex coordinates. This is the result from the analytic continuation of light cone coordinates to Euclidean signature by Wick rotation.

This is, we will choose

$$\sigma^{\pm} \sim \sigma^1 \pm \sigma^0 \to \sigma^1 \mp i\sigma^2 \tag{5}$$

where σ^2 is the Euclidean Wick rotation of σ^0. These two coordinates are called $z, \bar{z}$. It is hard to get a straight convention between all the books etc. We choose it so that left movers correspond to z and right-movers correspond to $\bar{z}$

The coordinates $z, \bar{z}$ are to be treated as completely independent coordinates that happen to be complex conjugate to each other in the physical setup of Euclidean geometry. Apart from that, they are unrelated to each other.

Moreover, in the Lorentzian analytic continuation they are independent real lightcone coordinates.

We also introduce the holomorphic and antiholomorphic derivatives, with the convention

$$\begin{aligned} \partial_z z &= \partial z = 1 && (6)\\ \partial_z \bar{z} &= \partial \bar{z} = 0 && (7)\\ \partial_{\bar{z}} z &= \bar{\partial} z = 0 && (8)\\ \partial_{\bar{z}} \bar{z} &= \bar{\partial} \bar{z} = 1 && (9) \end{aligned}$$

The change of variables also gives us

$$dz d\bar{z} = d^2 z = 2 d\sigma^1 d\sigma^2 \tag{10}$$

and the action takes the form

$$S = \frac{1}{2\pi\alpha'} \int d^2 z (\partial X \bar{\partial} X) \tag{11}$$

The equations of motion are given by

$$\bar{\partial}(\partial X) = 0 = \partial(\bar{\partial} X) \tag{12}$$

Thus, the operator $\partial X(z, \bar{z})$ is holomorphic (depends only on z). We write this as

$$\partial X(z) \tag{13}$$

similarly we do this for antiholomorphic fields.

Also, we have the Green's function of the scalar field, that is given by

$$\langle X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) \rangle = -\alpha'/2 \log(|z - z'|^2) \langle 1 \rangle \sim G_F(z_1, \bar{z}_1; z_2, \bar{z}_2) \tag{14}$$

We calculate this in infinite flat space. The logarithm is the Green's function for the Laplacian in two dimensions.

Path integrals always compute Time ordered products of fields. These are the combinations that appear in studying perturbation theory anyhow, and they also reflect our notions of how measurements are performed. In Euclidean formulations this is implicit, as there is no good notion of time ordering.

In general we are interested in calculating vevs of the form

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_n(z_n, \bar{z}_n) \rangle \tag{15}$$

for arbitrary insertions. In free field theory these can all be reduced to computations that involve $G_F(z_1, \bar{z}_1; z_2, \bar{z}_2) = G_F(z_1, z_2) = G_F(z_{12})$. (We will abuse notation and drop extra coordinates whenever equations get too long, and hopefully there will be no confusion because of it.)

The other very important equation is the definition of derivatives of operators.

We define

$$\partial\mathcal{O} \tag{16}$$

by taking derivatives on vevs of $\mathcal{O}$ with other fields:

$$\langle \partial_z \mathcal{O}(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \partial_z \langle \mathcal{O}(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle \tag{17}$$

We will also abbreviate many times the presence of arbitrary insertions by the convention of inserting dots $(\dots)$. The equation above would look as follows

$$\langle \partial_z \mathcal{O}(z,\bar{z})\dots\rangle = \partial_z \langle \mathcal{O}(z,\bar{z})\dots\rangle \tag{18}$$

Finally, there are singularities when two or mode different X fields coincide in location. These are the usual singularities of field theory and they only reflect the fact that we have infinitely many degrees of freedom. In practice this is part of the problem of renormalization of the theory. In free field theory they can be eliminated by using expressions that have well defined limits, and this gives rise to the (so called) normal ordering procedure.

Roughly, it is defined by removing the contractions of fields inside expressions written between pairs of ::. For example

$$:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2): \;=\; X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) - \langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\rangle/\langle 1\rangle \tag{19}$$

$$:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2): \;=\; X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) + \alpha'/2\log(|z_{12}|^2) \tag{20}$$

We will not assume that the path integral is normalized, so $\langle 1\rangle = Z$ is the partition function itself.

There is a useful formula that defines normal ordering for arbitrary functionals of the fields

$$:\mathcal{F}[X]: := \exp\left(\frac{\alpha'}{4}\int d^2z d^2z' \log|z-z'|^2 \frac{\delta}{\delta X(z_1)}\frac{\delta}{\delta X(z_2)}\right)\mathcal{F}[X] \tag{21}$$

This is a combinatorial formula. The exponential is a formal sum of terms, and one is supposed to evaluate these terms order by order in the expansion

of the exponential. terms of order n involve n contractions between $2n$ fields. The $1/n!$ from the exponential avoids double counting. Also, each integral appearing in the expansion has it's own variable.

We will use this formula heavily. Normal ordered expressions are free of singularities from colliding operators. Also, so long as we have normal ordered formulas we can use equations of motion between the normal ordered pieces.

Delta function singularities (and derivatives of delta function singularities) are zero everywhere, except when two fields coincide (this property is sometimes called ultralocal). These are also called contact singularities or contact terms.

Normal ordering removes the "unwanted" infinities in polynomial expressions (formal power series) in the fields. The procedure is well suited to define smooth limits of composite operators.

Many times we will use the convention $\alpha' = 2$. This is very important to keep in mind.

2.1 Some examples of normal ordering

Consider an operator of the form

$$\mathcal{O}_k(z,\bar{z}) = \exp(ikX(z,\bar{z})) \tag{22}$$

It is easy to show that the normal ordered expression differs from the "bare operator" just in the normalization

$$:\mathcal{O}_k(z,\bar{z}):\sim \exp(k^2\log(0))\mathcal{O}_k(z,\bar{z}) \tag{23}$$

This property is usually called *wave-function renormalization*, which means that up to an (usually infinite) multiplicative factor we know how to calculate expressions that contain it.

Another useful formula is equation (2.2.10) in the book. It tells us how to "concatenate" normal ordered formulae.

$$:\mathcal{F}::\mathcal{G}:=:\mathcal{F}\mathcal{G}:+\sum \text{cross contractions}$$

One can show easily that with the definition above

$$\begin{aligned} :\mathcal{O}_k(z,\bar{z})::\mathcal{O}_{k'}(z',\bar{z}'): &= |z-z'|^{\alpha' kk'}:\mathcal{O}_k(z,\bar{z})\mathcal{O}_{k'}(z',\bar{z}'): &(24)\\ &= (z-z')^{kk'}(\bar{z}-\bar{z}')^{kk'}:\mathcal{O}_k(z,\bar{z})\mathcal{O}_{k'}(z',\bar{z}') &(25)\end{aligned}$$

We will usually use Taylor series expansions:

$X(z) = X(0) + z\partial_z X(0) + \dots$ (this also includes derivatives with respect to $\bar{z}$).

Between normal ordered expressions, X satisfies the equations of motion, so there are no "cross terms" between z and $\bar{z}$. Expressions that contain $\partial^n X$ in the exponential are to be expanded as a formal power series.

From here we obtain the following

$$: \mathcal{O}_k(z,\bar{z}) :: \mathcal{O}_{k'}(0,\bar{0}) := |z|^{\alpha' kk'} : \mathcal{O}_{k+k'}(0,\bar{0}) + O(z,\bar{z}) : \tag{26}$$

that describes the leading "singularity" when the operators collide. Clearly the prefactor diverges if $\alpha' kk' < 0$, and this just says that the extra terms come with extra powers of $z, \bar{z}$ and are therefore less singular terms.

In fact, we get a full power series of terms that are less and less singular as powers of z. This result generalizes to arbitrary local expressions, and defines what is called the OPE (operator product expansion). The idea is that anytime we have operators coming close to each other, there will be a singular behavior that can be written as a series expansion of the form

$$: A_1(z,\bar{z}) :: A_2(0,\bar{0}) := \sum (z)^{a_3} (\bar{z})^{\tilde{a}_3} c_{12}^3 : A_3(0,\bar{0}) : \tag{27}$$

over all possible local operators, with constant coefficients c_{12}^3, that are independent of z, but depend on the choices of operators. Usually only finitely many of the terms on the right hand side are actually divergent expressions when $z \to 0$.

Many times we use the symbol

$$: A_1(z,\bar{z}) :: A_2(0,\bar{0}) :\sim \sum (z)^{a_3} (\bar{z})^{\tilde{a}_3} c_{12}^3 : A_3(0,\bar{0}) : \tag{28}$$

where we *sum only over the singular terms* in the OPE.

This is usually an asymptotic expansion. however, in CFT's one can argue that the result is convergent. The reason for this is that asymptotic series in z are usually corrected at order $\exp(-a/z)$ (this is the usual way in which perturbation expansions are corrected by instantons or other classical solitons). However, this would require that the theory have a natural scale (a mass, etc). However for conformal field theories all fields are massless, and the theory has no natural scale. This means that these formal expansions can not be corrected non-perturbatively in the usual form. For physics applications

this means that these corrections should be zero, and therefore the expression on the right hand side is convergent for some finite radius of convergence.

Finally the definition of the quantum stress energy tensor is given by

$$T(z) = \frac{-1}{\alpha'} : \partial X(z) \partial X(z) : \tag{29}$$

This is a choice of normalization. It is many times useful to take $\alpha' = 2$ and then a lot of equations simplify.

For example

$$T(z)T(0) \sim \frac{1}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(z) \tag{30}$$

These are the only singular terms in the OPE of the stress-energy tensor.

Similarly, one can do the double contractions between T and other operators $\mathcal{O}_k$ to find

$$T(z) : \mathcal{O}_k(0,\bar{0}) :\sim \frac{k^2}{2(z)^2} : \mathcal{O}_k(0,\bar{0}) : + \frac{1}{z}\partial : \mathcal{O}_k(0,\bar{0}) : \tag{31}$$

IN general, we can expect that we will get a formula of the following form

$$T(z) : A(0) :\sim \sum_{i<0} z^i A_i(0) \tag{32}$$

It's easy to show (although really tedious) that $A_{-1}(0) = \partial A(0)$

For any formal power series in X and it's derivatives.

There are some nice operators for which $A_{-2} = hA$. These are so called *scaling operators* of conformal weight h.

Finally, there are even nicer scaling operators for which $A_{-k} = 0$ for $k > 2$. These are called primary operators of conformal weight h.(Also called tensors). These operators play a special role in CFT.

Notation: Holomorphic operators only depend on z, and not on $\bar{z}$. Similarly, anti-holomorphic operators depend only on $\bar{z}s$ and not on z. For these operators we only write one coordinate z or $\bar{z}$, as opposed to the double coordinate $(z, \bar{z})$.

This is true for the stress energy tensor.

Similarly, $\partial X(z, \bar{z}) = \partial X(z)$, because the equations of motion can be used for X.

Via this definition, we can always state that between normal ordered products

$$X(z,\bar{z}) = X_L(z) + X_R(\bar{z}) \tag{33}$$

, and we can define the operators

$$\mathcal{O}_{k,\tilde{k}}(z,\bar{z}) =: \exp(ikX_L(z)) \exp(ik' X_R(\bar{z})) : \tag{34}$$

With this split, we split the Green's function of X into a holomorphic and an antiholomorphic piece

$$: X_L(z)X_L(z') : \;=\; X_L(z)X_L(z') + \log(z-z') \tag{35}$$
$$: X_R(\bar{z})X_R(\bar{z'}) : \;=\; X_R(\bar{z})X_R(\bar{z}') + \log(z-z') \tag{36}$$

Notice that the normal ordering procedure that defines the OPE's are not single valued on the complex plane. This motivates the following definition:

Two operators $: A_1(z,\bar{z}) :$ and $: A_2(z',\bar{z}') :$ are said to be *mutually local* if their OPE is single valued on the complex plane, namely, if in the OPE expansion

$$: A_1(z,\bar{z}) :: A_2(0,\bar{0}) := \sum (z)^{a_3} (\bar{z})^{\tilde{a}_3} c_{12}^3 : A_3(0,\bar{0}) : \tag{37}$$

the only terms with $c_{12}^3 \neq 0$ are those for which $a_3 - \tilde{a}_3$ is an integer. Namely, the combination that appears is $|z\bar{z}|^{a_3} \bar{z}^{\tilde{a}_3 - a_3}$, and this expression is single valued on the complex plane when $\bar{z}$ is the complex conjugate variable to z.

A system of operators A_j, $j \in U$ over some labeling set is said to be a set of mutually local operators if any pair of operators in U (they could be repeated) is mutually local.

We will also be concerned with systems of operators that are closed under the OPE expansion, e,g., any operator that appear in the OPE of two operators in the set, is also in the set. We also include linear combinations of operators in these systems.

For example, in the free field theory of X, the system of operators defined by the OPE's of $T(z)$ with itself and their linear combinations is such a set, and one can show that this set is a set of mutually local operators: these are given by all polynomials in ∂X and it's derivatives, with only an even number of X fields.

3 Radial quantization and the Virasoro algebra

Let us consider a 2D CFT on the Euclidean plane. Let us write the metric of the plane in polar coordinates. We find that

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{38}$$

Let us introduce a new variable $\tau = \log(r)$, or equivalently $r = \exp(\tau)$.

In the coordinates θ, τ, the metric above is given by

$$ds^2 = \exp(2\tau)(d\tau^2 + d\theta^2) \tag{39}$$

and this is conformally equivalent to a flat metric

$$\tilde{ds}^2 = \exp(-2\tau) ds^2 = d\tau^2 + d\theta^2 \tag{40}$$

however, we have to remember that θ is a periodic variable with period 2π, so the geometry of the plane is conformally equivalent to the geometry of the cylinder.

If we analytically continue the metric to Lorentzian signature by $\tau \to it$, we find that we can relate the cylinder time τ, to a Lorentzian time t.

In regular path integrals, we end up computing time ordered expectation values of the fields. Time ordering refers to ordering in the t time coordinate. We can 'extend' this notion to Euclidean coordinates, so that a path integral on the cylinder computes τ-ordered correlation functions. This is what we would want for a natural relation between time orderings in a Lorentzian cylinder to be represented faithfully in the natural Wick rotated coordinate system.

When we go back to the original coordinates r, θ, we end up finding that $\tau' > \tau$ (Euclidean time ordering) implies that $r > r'$. This is, a natural notion of ordering in the Euclidean plane is Radial ordering. We will assume in what follows that the Euclidean path integral is computing Radial ordered correlation functions by the above procedure.

Having this notion lets us also think in terms that are related clearly to Canonical quantization as well, where one would have the operator commutation relations defining a field theory.

3.1 Conserved charges

If we consider a general quantum mechanical system, usually we describe it in terms of some Hamiltonian evolution, described by (a time independent Hamiltonian) H. A physical quantity Q is a conserved charge if

$$i[H, Q] = 0 = \dot{Q} \tag{41}$$

The left hand side above is the Heisenberg equation for the time evolution of the operator Q. Sometimes Q depends explicitly in time, and one has to generalize the equation above.

If $Q$is Hermitian, we can generate a Unitary group of transformations given by

$$\exp(i\xi Q) \tag{42}$$

These are physical rotations in the Hilbert space that do not modify the energy. However, they can modify other observables. The Q are generators of infinitesimal transformations.

It is commonly the case that observables transform in a nice way under these rotations. For example, in radially symmetric problem in two dimensions, Q would be the angular momentum: the canonical conjugate variable to the angel rotations. The coordinates x^1, x^2 would transform into each other, and it is easy to show that

$$i[Q, x^1] = -x^2 \tag{43}$$

$$i[Q, x^2] = x^1 \tag{44}$$

If we take the combinations $x^1 \pm ix^2 = x^{\pm}$, we would find that

$$[Q, x^{\pm}] = \pm x^{\pm} \tag{45}$$

so that the operators $x^{\pm}$ are eigenvectors of Q acting on the space of operators. Their corresponding eigenvalues are called the charges of $x^{\pm}$ (The angular momentum carried by the corresponding operator).

If we have perturbations proportional to $x^{\pm}$ (let us say from some time varying circularly polarized electric field), then one has selection rules for single photon absorption/emission: namely, that $\Delta Q = \pm 1$.

The action of Q on the set of operators is important, and the eigenvalues of Q are the charges of the corresponding operators.

When we generalize to field theory, usually we have that the conserved charges arise from a Nother current:

$$Q = \int d^d x j^0 \tag{46}$$

where we have in mind a theory in d dimensions. Again, we are interested in computing the charges of local operators

$$[Q, \mathcal{O}(x)] \sim \lambda_{\mathcal{O}} \mathcal{O}(x) \tag{47}$$

If we have

$$[\int d^d y j^0(y), \mathcal{O}(x)] \sim \lambda_{\mathcal{O}} \mathcal{O}(x) \tag{48}$$

and j^0 is a local operator, we have that so long as $y \neq x$, then

$$[j^0(y), \mathcal{O}(x)] = 0 \tag{49}$$

It must be the case that

$$[j^0(y), \mathcal{O}(x)] \sim \delta^d(x-y)\mathcal{O}(x) \tag{50}$$

However, since the conserved charge equation is valid between integral signs, we can have c-numbers on the right hand side that integrate to zero, and have singular support at y. For example

$$[j^0(y), \mathcal{O}(x)] \sim \delta^d(x-y)\mathcal{O}(x) + \nabla_x \delta^d(x-y) + \nabla_x^2 \delta^d(x-y) + \dots \tag{51}$$

These extra terms are called Schwinger terms, and on certain conditions they are also called an anomaly. The anomaly vanishes if the Schwinger terms vanish.

If we have more than one charge, we can get a Lie algebra of charges, by taking multiple commutators

$$[Q^a, Q^b] = i f^{ab}{}_c Q^c \tag{52}$$

and we want to compute these structure constants f_c^{ab}. They are required to satisfy the consistency conditions of a Lie algebra (the Jacobi identity).

We want to calculate these in the case of conformal field theories, where the currents that are conserved are related to the stress energy tensor.

The simplest problem is how to make commutators in a path integral formulation of the theory.

The idea is to define the equal time commutator as a limit

$$[A(t), B(t)] = \lim \epsilon \to 0^+ \left(A(t+\epsilon)B(t) - B(t)A(t-\epsilon)\right) \tag{53}$$

The right hand side is clearly time ordered, and therefore can be computed in a path integral.

Moreover, If A is conserved, then so long as $\epsilon > 0$, we have that $A(t+\epsilon) = A(t+0^+)$ as operators acting on the Hilbert space of states after time t. Hence, in this case we can be less careful in how we take the limit.

Also, current conservation (a form of Green's theorem) usually lets us deform the 'contour of integration' to be an arbitrary spatial hypersurface with reasonable boundary conditions.

3.2 The Virasoro algebra

The first thing we have to worry about are the allowed *global* symmetries of conformal invariant theories. So far we are studying a conformal field theory coupled to a fixed geometric shape.

As we have argued so far, rescalings of the background metric are not physically meaningful, because conformally invariant theories are not sensitive to them. However, diffeomorphisms are meaningful: if you change coordinate systems by translations, you get new states: the translation of the old state. Rotations, or Lorentz transformations are also physical. These are the isometries of a background metric. So we find that the transformations that preserve the background metric are associated to physical operations on the Hilberst space of states.

More complicated diffeomorphisms would usually change the form of the metric, so they would take you to a 'different theory'. The quantities are related by the transformation, but we can not identify the end state with an action that has taken you back to the initial field theory.

For conformal field theories we can relax the need to stick to isometries. We can relax this assumption if after the diffeomorphism we get into a metric that is Weyl-equivalent to the original one: a rescaling of the coordinate metric.

If we start with the Lorentzian metric in lightcone coordinates

$$ds^2 \sim d\sigma^+ d\sigma^- \tag{54}$$

and we do a transformation

$$\tilde{\sigma}^{\pm} - \delta\sigma^{\pm} = \sigma^{\pm} \tag{55}$$

we find that a modified metric is conformally equivalent to the original one if it is of the form

$$ds^2 = g(\tilde{\sigma}^+, \tilde{\sigma}^-) d\tilde{\sigma}^+ d\tilde{\sigma}^- \tag{56}$$

In order that $\tilde{g}_{++} = 0$, we find that

$$\frac{\partial \tilde{f}^-}{\partial \tilde{\sigma}^+} = 0 \tag{57}$$

this is, we find that $\tilde{\sigma}^+$ is a function of σ^+ alone, and similarly for $\tilde{\sigma}^-$.

If we are on a cylinder and we want to keep the periodicities of the identifications, we find that $\delta\sigma^+$ is given by a Fourier series

$$\delta\sigma^+ = \sum_{n=-\infty}^{\infty} \tilde{v}_n \exp(in\sigma^+) = \tilde{v}(\sigma^+) \tag{58}$$

$\tilde{v}_0$ corresponds to moving to the right and translating in time: a combination of 'rotations' on the circle, and time translations: $H - J$. Similarly, we use the notation v_n for the corresponding left movers.

Going to the Euclidean setup T_{--} becomes $T_{\xi\xi}$, and the conformal group is the group of infinitesimal changes of complex variables, where ξ is a periodic variable on the cylinder.

Since the diffeomorphisms that we are considering are infinitesimal symmetry transformations, there should be associated generators of these symmetries that are described by Operators. Moreover, the transformations act locally, so they should be associated to some Noether current that is conserved. Unfortunately, since the coordinates also move in time, the generators do not commute with the Hamiltonian, and they have explicit time dependence.

In general field theories, the local infinitesimal generators of diffeomorphisms (changes of coordinates) are give by the stress energy tensor. For us, if we want to generate a transformation given by some coefficients v as above, we would consider the expression

$$\int_{t=constant} d\sigma T_{--} v(\sigma^-) \tag{59}$$

If we consider that T should be a tensor, then it should transform under changes of coordinates in a nice way. The coordinate ξ is not too useful because of the periodicity. It is better to consider a variable $z = \exp(-i\xi)$ which covers the complex plane once, with a puncture at the origin. Under these changes of variables T should transform as a tensor, so

$$T_{zz}dzdz \sim T_{\xi\xi}d\xi d\xi \tag{60}$$

Also, we have to be careful with the integral: we can write it as a contour integral in ξ over the real axis between 0 and 2π, and this becomes an integral over the complex plane $\oint \frac{dz}{z}$.

It is convenient to choose normalizations that are used in complex integrals, so that we find that

$$Q = \frac{1}{2\pi i}\oint \left(dzT(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right) \tag{61}$$

The mode coefficients of $T_{\xi\xi}$ are going to be called L_n. When we write it to $T(z)$, the Fourier series becomes a Laurent series. however, we have to shift by two the coefficients because of the tensor transformations. Our final definition is

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \tag{62}$$

And we can recover L_n from T by doing the obvious residue integral

$$L_m = \oint \frac{z^{m+1}}{2\pi i}T(z) = \oint \frac{1}{2\pi i z}(z^{m+2}T(z)) \tag{63}$$

the complex contour is counterclockwise around the origin at fixed radius.

Now, we can put this together with our prescription for radial ordering to find an expression for the commutators of the generators of conformal trasnformations.

In the case of the free boson, a straightforward but tedious application of the residue theorem shows that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{64}$$

This is called the Virasoro algebra. For the free boson, we find that $c = 1$. If we have k free bosons, then we find that $c = k$. The operator c commutes

with everything. This is a consequence of the Jacobi identity. because it commutes with everything, it can be understood to be proportional to the identity, and takes a fixed value in any example.

This number c is called the 'central charge' of the Virasoro algebra. The coefficient c is a Schwinger term for the stress-energy current equal time commutators.