1) Consider a massive scalar field satisfying the usual Klein-Gordon equation:

\[ \nabla^2 \phi - m^2 \phi = 0 \]

in the Poincare patch of \( AdS_{d+1} \).

a) Consider a mode \( \phi(t, \vec{x}, r) = e^{-i\omega t + ik_i x^i} \phi(r) \). Rewrite the equation for \( \phi(r) \) as a standard Schrödinger equation

\[ [-\partial_z^2 + V(z)] \psi(z) = \omega^2 \psi(z) \]

where \( z \) is a new radial coordinate and \( \psi(z) = A(z)\phi(z) \) with \( A(z) \) chosen to remove first derivative terms.

b) Show that solutions with finite norm in the usual Schrödinger sense (i.e., square integrable in \( z \)) have finite Klein-Gordon norm in \( AdS_{d+1} \).

c) Setting \( k_i = 0 \), show that if \( m^2 < m_{BF}^2 \equiv -d^2/4L^2 \), the Schrödinger equation has a normalizable state with negative energy. (It suffices to find a test function with \( \langle H \rangle < 0 \).) This corresponds to a purely imaginary frequency and hence an exponentially growing mode in spacetime.

d) Show that if \( m^2 \geq m_{BF}^2 \), \( \omega^2 \) is always positive so there is no instability. This is the Breitenlohner-Freedman bound for stability of anti-de Sitter spacetime. Note that a small negative mass squared does not cause an instability.

2) Look over the list of proposed topics for presentations on the course website. Email me your top two preferences by 5pm April 20.