1 D-branes

1.1 The bosonic string

So far, we have dealt with the open string as a simpler place to do computations.

However, we should also consider the possibility of having D-branes: a special locus where the strings can end. Around these locations, the string is required to end in a special locus, and one can check that this locus leads to Dirichlet boundary conditions on the end of the string (if we choose the locus to be a hyperplane).

Let us consider two $d + 1$ dimensional D-branes that are parallel. One located at $\vec{a}$ and another located at $\vec{b}$, so that $|\vec{a} - \vec{b}|$ is the separation of the branes.

We say this as follows on the worldsheet. We require that the following boundary conditions be put at $\sigma = 0$ and $\sigma = \pi$:

\[ X_\perp(0, \tau) = \vec{a} \]  
\[ X_\perp(\pi, \tau) = \vec{b} \]  

For the ground state of this string, the boundary conditions at both ends force us to have a gradient. The contribution to the energy in the Polyakov string action is proportional to

\[ \nabla X^2_\perp \]  

and this is minimized if the gradient is constant (and time independent).

Thus classically, we get an extra contribution to $L_0$ that is proportional to

\[ |\vec{a} - \vec{b}|^2 \]
If we have a vertex operator language, we would have that the ordinary derivatives are related to the oscillators of $X$, while we would also have a term that includes the zero modes of $X$ and that we write as $\exp(ikX)$. One can easily check that this only gives contributions to the dimension of the operator from the direction with Neumann boundary conditions (those in which the ends are free to move).

This can be easily checked by doing standard OPE’s, with Green’s functions associated to the different types of boundary conditions:

$$G_N \sim \log |z - w|^2 + \log |z - \bar{w}|^2$$  

and

$$G_D \sim \log |z - w|^2 - \log |z - \bar{w}|^2$$

where the inside of the open string is mapped to the upper half plane. The point $\bar{w}$ is the image of $w$ under reflection, as is required to get the right boundary conditions. Then, we match the coefficient of the Green’s function $\log |z - w|^2$ to the standard normalization.

The quantization proceeds as usual. We get a contribution to $L_0$ from momentum parallel to the branes that is proportional to

$$L_0 \sim 2p_{\parallel}^2 + N + \alpha |\vec{a} - \vec{b}|^2 - 1$$

The term $(-1)$ is the usual zero point energy.

Now, we want to compute the one loop amplitude in the cylinder topology.

From the spacetime point of view, we need to do the following integrals:

$$\int dp_{\parallel} \sum_m \log(p^2 + m^2)$$

where the sum is over the species of particles of the open string channel.

These logarithms are captured by a Schwinger proper time parametrization

$$\log(p^2 + m^2) = \int_0^\infty \frac{dt}{t} \exp(-t(p^2 + m^2))$$

For the closed string, the parameter $t$ ended up being proportional to $\Im m(\tau)$ on the worldsheet, and the sum ended up being a character of Virasoro: after all, $m^2$ depends linearly $L_0$, so the sum is a sum over oscillators with an $\exp(-tL_0)$ type dependence.
For the closed string, the region near $\Im m(\tau) = 0$ was removed by modular invariance. There is no obvious analogous concept for the open string, as the shape of the cylinder for $t$ small is very different than the shape of the cylinder for $t$ large.

In the end, the answer we get is very similar to the answer for the closed string, with a few modifications.

First, the time parameter $t$ is real and not complex. Secondly, the zero point energy has an extra contribution from the separation of the D-branes. Third, we only integrate over the parallel momentum to the string.

The oscillator sum can be done readily. There exists a parameter $s$ linear in $t$ that makes it look essentially identical to the closed string calculation, but with the understanding that it is only half of the oscillators (let us say, like picking all the left movers only). The oscillator sum is

$$\eta(is)^{-24}$$

For $s$ chosen appropriately. In this equation $s$ is real, so $\eta$ is real and we don’t need absolute values.

The extra pieces that are needed are

$$Z_{cyl} = \int dp_\parallel \int_0^\infty ds \frac{ds}{s} \eta(is)^{-24} \exp(-\alpha s p_\parallel^2 - \beta s |\vec{a} - \vec{b}|^2)$$

where $\alpha, \beta$ are constants that can be calculated by careful normalization of the integral.

The potential UV divergence from this integral comes from the region $s \sim 0$. This region corresponds to a cylinder, where the open string has width $\pi$ and the time is order $s$.

By rescaling the volume of the worldsheet (a symmetry of the string), we can arrange it so that the the time is of size $2\pi$ and the width is of order $1/s$ instead. In the region $s \to 0$, this can be interpreted as a very long tube, and a long tube corresponds in the closed string channel to the standard propagation of physical string states when the particles are going on-shell. However, this is an Euclidean integral, so the effective time of propagation is imaginary.

The closed string interpretation is that we are exchanging closed strings between branes $A$ and $B$, and that one brane is serving as a source of closed strings, while the other one is serving as a sink. The propagation is over a distance $|\vec{a} - \vec{b}|$. 

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how do we estimate the contribution from \( s = 0 \)?, we estimate it by going to the closed string channel, and expanding near \( s' = 1/s \) being very long.

Notice that if we take \( \tau = is \), then the modular transform

\[
is = \tau \rightarrow -\frac{1}{\tau} = is' = \frac{i}{s}
\]

takes our parameter \( s \) for open string propagation, to the time parameter in the closed string channel.

We can use the modular transformation of \( \eta \), to write it in terms of \( \eta(is') \) instead.

When \( s' \) is large, the \( q \) expansion of \( \eta(is') \) converges rapidly (after all \( q \sim \exp(-s') \) is exponentially small in this region.) The \( q \) expansion of \( \eta^{-24} \) is schematically given by a series with coefficients that includes the following powers of \( q \):

\[
a_{-1}q^{-1} + a_0q^0 + a_1q^1 + \ldots
\]

When we rescale the circle to have size \( 2\pi \), this affects the definition of \( q \) in terms of \( s' \).

In any case, the series converges rapidly, and the first few terms should suffice. Also, the integral over \( p_\parallel \) can be done readily: it is Gaussian with width \( s \). The full integral we need to calculate is a sum of terms that look like

\[
Z_{cyl} \sim \sum_m \int \frac{d's'}{s} (s')^{2d+1-(d+1)} \exp \left( -s'm^2 - s' \frac{|\vec{a} - \vec{b}|^2}{(s')^2} \right)
\]

where \( m \) is a mass label and various normalization coefficients are omitted. The integral is an integral of the (classical) action for particles of mass \( m \) to propagate between \( a \) and \( b \) in time \( s' \). This is a semiclassical interpretation of what a Green’s function computes. The factors of \( s' \) come from the modular transform of \( \eta \) and the integral over \( p_\parallel \). The term \( |\vec{a} - \vec{b}|/s' \) is the classical velocity along a linear trajectory starting at \( \vec{a} \) and ending at \( \vec{b} \). (This follows from the covariant point particle action in the first set of lectures of the first quarter of the course).

The masses \( m62 \) that show up are exactly the closed string states. The amplitudes for the coefficients are interpreted as the amplitude to produce the corresponding string on brane \( A \) multiplied by the amplitude to absorb it in \( B \). Also, physical polarizations are included in the calculation implicitly.

The leading term is the mass of the tachyon, which is negative. That gives us an exponentialy divergent integral. It can be done by analytic continuation.
in \( m^2 \) from a region where the corresponding integral converges. This is a divergence that states that if you have an amplitude to produce a tachyon, then you will force the system to roll down the hill and the tachyon will grow exponentially in time. It is a nuisance of bosonic string, and we know that we will get problems like this repeatedly.

The second term is from the propagation of massless modes. This is the most interesting contribution (especially because we can do the integral in closed form).

We can rescale \(|\vec{a} - \vec{b}|\), and do the integral in \( s = 1/s' \) instead. The full answer scales as

\[
\frac{1}{|\vec{a} - \vec{b}|^{24-(d+1)}} = \frac{1}{|\vec{a} - \vec{b}|^{26-2-(d+1)}}
\]

This is the Green’s function of a massless particle in \( 26 - (d + 1) \) dimensions associated to

\[
\frac{1}{\nabla^2}
\]

This is, a Green’s function for exchange of massless particles in the transverse directions to the brane.

The massless closed strings are the graviton and the dilaton and an antisymmetric two form. One can show that only the graviton and the dilaton contribute.

The result is the gravitational potential between the two D-branes. It scales like

\[
\frac{G_NT_1T_2}{r_{\perp}^{26-2-(d+1)}}
\]

where the \( T \) are the tensions of the two branes, and \( G \) is the newton gravitational constant in higher dimensions. Since this is a one loop integral with no external particles, it has no powers of the coupling constant. \( G_N \) scales like \( g_s^2 \), as each three point vertex of closed strings requires a power of \( g_s \), and an exchange diagram with a graviton in the middle goes like \( g_s^2 \). We conclude that the tension of the D-branes is of order \( 1/g_s \).

The fact that their tension scales like an inverse power of the coupling constant means that these objects are non-perturbative in nature. The fact that we are able to find an absolute normalization for the tension for these objects from the open string channel is a great achievement. When we introduced these objects, there was no argument for how heavy or light they would be. The consistency between the open and closed string calculations determines this.
This is the closest analog of modular invariance that we can have in the presence of D-branes: there should be consistency between the open string and closed string channel, when we view the worldsheet time between different times on the worldsheet.

1.2 NS-R superstring

We have already seen in various instances that we can also consider the problems of string theory in the presence of worldsheet boundaries. So far, we have studied these possibilities as both a generalization of closed strings, and as a simplification for calculations.

Our purpose is to study these possibilities more systematically in the following, together with the inclusion of Chan-Paton factors, etc.

The idea of a D-brane is simple: we require the string ends to lie on a special submanifold of spacetime, where by definition, the D-brane is located. For simplicity, we will consider only systems of flat D-branes (although in principle it is possible to generalize calculations for curved D-branes, in practice this proves to be extremely difficult).

Let us start with a single D-brane, such that it is extended in $p$ dimensions and time. This will be called a D$p$-brane.

We choose a coordinate system so that $x^0, \ldots, x^p$ are along the D-brane, and $x^{p+1}, \ldots, x^9$ are orthogonal to the D$p$-brane. We will choose the D-brane to be fixed at locations $a^{p+1}, \ldots, a^9$ on the $x^{p+1}, \ldots, x^9$ plane.

If the left end of a string ends on the D$p$-brane, we choose to represent the string worldsheet by a space coordinate $\sigma$ and a time coordinate $\tau$ on the worldsheet. Then we have the following boundary conditions on the string

$$\partial_\sigma X^i|_{\sigma=0} = 0 \quad \forall i = 0, \ldots, p$$

$$X^i|_{\sigma=0} = a^i \quad \forall i = p + 1, \ldots, 9$$

The last condition also reads $\partial_\tau X^i_{\sigma=0} = 0, \forall i = p + 1, \ldots, 9$. These are Neuman and Dirichlet boundary conditions respectively.

Similarly, we can consider the right end to be on the same D$p$-brane, or on a different D$q$-brane.

These conditions read $\partial_\tau X^i_L|_{\sigma=0} = \pm \partial_\tau X^i_R|_{\sigma=0}$ where the sign depends on whether we have Dirichlet or Neuman boundary conditions.

In the superstring, we can consider the Ramond sector. In the $R$ sector, fermions have the same periodicity conditions as the bosons. These translate
with the same sign conventions as the bosons (on both boundaries of the open string). This is required by worldsheet supersymmetry. Because the boundary conditions are the same, boson and fermion ground state zero-point energies cancel against each other, except for a possible classical contributions from the bosons, if the string is stretched between two locations. When we consider the same problem with the NS boundary conditions, we have to change the boundary conditions on the right boundary by a minus sign. This produces interesting zero-point energy contributions on the NS sector. Furthermore, there is a GSO projection (choice of minus signs in the partition function \((-1)^F\)) that might keep the possibility of a tachyon or not.

The quick rule of thumb is that the classical contribution to the ground state energy from stretching the string between two D-branes, is the minimal length between them times the string tension.

This actually follows from a BRST calculation. In the end, it boils to the statement that classically \(\partial X \neq 0\) if \(\vec{X}(0, \tau) = \vec{a}\), and \(\vec{X}(\pi, \tau) = \vec{b}\).

The most interesting question one can ask about the spectrum of open strings is whether the D-brane preserves some supersymmetry or not. We will start with the simple case where for each \(X^i\) the left and right boundary conditions are Dirichlet or Neumann.

There are four different types of boundary conditions: DD, DN, ND, NN. Of these four, the first and last have fermion zero modes in the R sector, while for the other two, the fermions have zero modes in the NS sector.

Thus, if the ND boundary conditions are present, a fermion zero mode will tell us that there always exists a state in the system whose energy is exactly the ground state energy of the NS sector.

Apart from the classical contribution, the ground state energy is

\[
-\frac{1}{2} + \frac{\#ND}{8}
\]  

(21)

This can be degenerate with the R ground state only if \(\#ND = 4\). This is a special case that can preserve some supersymmetry.

The other possible case, is when \(\#ND = 0\), because then there are no fermion zero modes, and the ground state can be projected out. Finally, for \(\#ND = 8\), one can also get degeneracies between bosons and fermions, but the R ground state would not be paired with a corresponding boson. (This
is ok, in 2-D SUSY, one can have left-moving supersymmetries, and a right moving fermion is allowed in the spectrum without requiring a superpartner—

We will see more of this structure when we study the heterotic string.

Now, we turn to the problem of whether a D-brane preserves some supersymmetry, or none at all. How would we test this possibility?

Notice that the boundary conditions tell us roughly that objects like \( \partial X \) are continuous across the boundary, if we identify them on the other side across the boundary with \( \bar{\partial} X \) with some sign. This is a symmetry \( \sigma \to -\sigma \), so that \( \partial \sigma \to \bar{\partial} \sigma \), and one also has an action \( \psi^i \to \pm \bar{\psi}^i \), that determines how the left and right movers are paired.

Notice that this identification of left movers into right movers is a morphism of the left moving conformal field theory into the right moving conformal field theory.

To satisfy the boundary conditions, we can calculate using the method of images. For example for a D-brane located at the origin, we require \( X(0, \tau) = 0 \) for some of the \( X \)

\[
\langle X(z, \bar{z}) X(w, \bar{w}) \rangle \sim \log |z - w|^2 \pm \log |z - \bar{w}|^2
\]  

(22)

Here we are choosing \( z = \tau + i\sigma \), so that \( \sigma > 0 \) is inside the worldsheet. The complex conjugation \( z \to \bar{z} \) produces an image point on the other side of the boundary.

For the case of Dirichlet boundary conditions when we take the limit \( \Im m(z) \to 0 \), we get that the correlation function above vanishes if we choose the minus sign.

In particular, objects like

\[
\oint \partial X^i
\]  

(23)

around an open string vertex operator determine the momentum of the vertex operator along the direction \( X^i \). For a vertex of the form \( \exp(ikX^i) \), we get a contribution from the right part of the contour coming from \( \partial X \), and on the left by \( \bar{\partial} X \), but it can be calculated using the correlation function above. For Dirchlet boundary conditions, the momentum is zero calculated this way. This is just right, because the string can not leave the brane without becoming infinitely long (this costs infinite energy) or breaking (this requires non-trivial string interactions).

It is easy to work all of this out for the elementary free fields \( X, \psi \), and the \( b, c, \beta, \gamma \) ghosts. However, we need to be very careful when we try to do
this for vertex operators in the Ramond sector (twist fields). This is because it is not obvious how the bosonized fields would transform between the left and the right movers.

Instead, we would like to require that the remnant of Lorentz invariance dictate the final answer.

For example, we would want the OPE expansion in a correlation function like

\[ \psi^\mu(z) \Xi^\alpha(w) \]  

(24)
to be analytic when continued past the boundary of the worldsheet.

This suggests that

\[ \bar{\Xi}^\beta(\bar{z})|_{\sigma=0} \simeq M^\beta_\alpha \Xi^\alpha(z)|_{\sigma=0} \]  

(25)

Similarly, the OPE of \( \Xi \Xi \) should be analytic when continued past the boundary. These continuations require that \( M \) is a unitary transformation.

In general, \( M \) can always be written as a linear combination of products of Gamma matrices.

The OPE’s \( \psi \Xi \) should be compatible with this expression.

This implies that

\[ \gamma^i M = M \gamma^i \quad \forall i = 0, \ldots, p \]  

(26)

\[ \gamma^j M = -M \gamma^j \quad \forall j = p + 1, \ldots, 9 \]  

(27)

There are two possibilities. If \( p \) is even, then we choose \( M \sim \gamma^0 \ldots \gamma^p \).

This is an odd number of \( \Gamma \) matrices, so they anti-commute with \( \gamma^j \) for \( j > p \).

If \( p \) is odd, we choose \( M \sim \gamma^{p+1} \ldots \gamma^9 \).

These two choices differ by multiplication by \( \gamma^{10} = \prod_i \gamma^i \), which is a matrix that anti-commutes with all the \( \gamma^i \).

Notice that the GSO projection on the left movers, that involves \((-1)^F\) projects onto a given chirality of the spin fields. The right movers will be the same or of a different chirality than the left movers. The chirality is exactly the eigenvalue of the spin field under \( \gamma^{10} \).

The big question is now whether supersymmetry is preserved or not: basically, whether the matrix \( M \) keeps the chirality, or whether it changes it. The idea is that the match between left and right movers allows for some supersymmetry, if \( \oint \exp(-\phi/2) \Xi^\alpha \) goes to \( \oint \exp(-\bar{\phi}/2) \bar{X}^\beta \) for massless physical polarizations at zero momentum (these are states that obey the GSO projection condition, and they have fixed chirality)
If $M$ has an even number of $\gamma$ matrices, then it preserves the chirality. Whereas if $M$ has an odd number of $\gamma$ matrices it flips the chirality. The parity of the number of $\gamma$ matrices in $M$ depends on whether $p$ is even or odd.

This distinguishes type IIA string theory from type IIB string theory.

For type IIA string theory, D0-branes, D2-branes, \ldots, D2p-branes are supersymmetric (preserve some supersymmetries).

For type IIB string theory, D1-branes, D3-branes, \ldots, D(2p+1)-branes preserve some supersymmetry.

The supersymmetries that are preserved are associated to linear combinations of the form

\[ Q \sim \oint \frac{1}{\sqrt{2}} (\bar{\Xi}_\alpha + \Gamma \Xi_\beta) \]  

(28)

The factor of $\sqrt{2}$ is there to guarantee that $\{Q, Q\} \sim P$ has the usual normalization.

Notice that this results from a projection operator on the set of all supersymmetries. Only half of them survive, and they mix left and right moving supersymmetries.

The projection operator is roughly of the form

\[ P_\Gamma = \frac{1 + \Gamma}{2} \]  

(29)

where $\Gamma$ takes a left mover and turns it into a right mover, and vice-versa, so that $\Gamma^2 = 1$.

To check whether two D-branes leave some supersymmetry unbroken, we need to check that $P_\Gamma$ and $P_{\Gamma'}$ have some common eigenspace of eigenvalue one.

These will cut the supersymmetry further. For D-branes at angles, this depends on the angles between the D-branes. An easy test is whether the fermions and bosons associated to strings stretching between the two D-branes have a lot of degeneracies in energies or not.

D-brane configurations that preserve some amount of supersymmetry are called BPS.

2 T-duality

In the study of bosonization, we saw that the free fermions corresponded to a periodic boson with a particular periodicity.
Moreover, we understood how the boson partition function was built by including winding states.

A winding state is one where
\[ X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + w2\pi R \] (30)

here \( R \) is the radius of compactification of the circle for the boson \( X \).

One finds from here that
\[ P_L \sim n/R + wR/2, \quad P_R \sim n/R - wR/2 \] (31)

A quick calculation shows that the spectrum of masses of a string looks as follows
\[ m^2 \sim p_L^2/2 + P_R^2/2 + (N + \tilde{N} - 2) = n^2/R^2 + w^2R^2/4 + (N + \tilde{N} - 2) \] (32)

while
\[ P_LP'_L - P_RP'_R \sim nw' - n'w \in \mathbb{Z} \] (33)

The last equation means that all operators \( \exp(iP_LX_L) \exp(iP_RX_R) \) are mutually local to each other, so it is suitable to define a string theory (otherwise the integrated correlation functions would not make much sense).

The surprise is that the spectrum looks very symmetrical in the way winding and momentum appears. Indeed, If we change \( n \leftrightarrow w \) and \( R \to 2/R \) (remember we are using units where \( \alpha' = 2 \)), the the spectrum of strings does not change.

More is true: all OPE’s don’t change, and these are the ones that determine the string interactions. What we see is that this is a symmetry of the string theory: all observables are the same, once we make the proper identifications.

Thus, as far as string theory is concerned, radius \( R \) and \( 2/R \) are identical.

We can also ask if something special happens at \( R = 2/R \).

Indeed, for these values of \( R \), one can show that there are states with \( \tilde{N} = 1, N = 0 \) and such that they are massless (similar for left movers and right movers with roles reversed) . These are extra massless vector fields in \( D - 1 \) dimensions. For the bosonic string, this is an enhanced \( SU(2)_L \times SU(2)_R \) current algebra.

The vertex operators associated to these states looks like
\[ \partial X^\mu \exp(ipRX_R) \] (34)
where $P_R^2 = 2$.

These have $SU(2)$ gauge theory interactions.

The compactification radius of the circle is a parameter of the theory, and it acts like by giving a mass to the massless vectors via the Higgs mechanism with respect to this $SU(2)$ symmetry about $R^2 = 2$.

What we see is that the enhancement of the symmetry is very unexpected from the point of view of spacetime physics. Gravitationally there does not seem to be anything special about these points.

Note also, that in the superstring, these states are absent because they are projected out by the $(1 + (-1)^F L,R)$ sums (GSO projection) that determine the physical spectrum of the theory.

Now we can generalize to compactifications on tori $T^n$. We expect to have a $2n$ dimensional lattice, where we can define two $n$-vectors $P_L$ and $P_R$. We will require that

$$P_L P'_L - P_R P'_R \sim P \cdot P' \in \mathbb{Z}$$

so that the spectrum of operators be mutually local.

This defines a map from the lattice to its dual (the dual lattice is the set of linear maps of a lattice to the integers). Thus it becomes important to ask if the lattice $\Lambda$ is self-dual or not. This is, if any integer linear map $M$ is of the form $P_M \cdot P$ for some $P_M$ in $\Lambda$. If this is true, then $\Lambda = \Lambda^*$. This can be shown to follow from modular invariance of the partition function.

**Insert $\theta$ series argument and modular inversion**

The mass operator is going to be proportional to

$$\frac{P_L^2}{2} + \frac{P_R^2}{2} + (N + \tilde{N} - 2)$$

The integer product above defines a Lorentzian metric of in a space of signature $(k, k)$, while we have a second positive definite quadratic form that determines the spectrum.

Given a solution of the first equation, since we are in a space of signature $(k, k)$, we can always Lorentz rotate it and obtain another solution to this lattice constraint. This will change the quadratic form $P_L^2 + P_R^2$ to get different values. However, if we make rotations of the $P_L$ alone (a $k$ dimensional subspace), or of $P_R$, we don’t change the spectrum of operators.

Thus the set of continuously connected possible choices to describe the physics is given by

$$O(k, k)/O(k)O(k)$$

(37)
We can show that on a k-torus, this has the same dimension as the number of components of constant values of $g_{ij}$ and $B_{ij}$. Which is the set of parameters that one would think of using in the gravitational compactification on $T^k$ to determine the physics.

3 Self-dual even lattices

We have already seen that the continuous part of the even self-dual Lorentzian lattices is up to equivalence given by

$$\frac{O(k,k)}{O(k) \times O(k)}$$ (38)

If the Lorentzian lattice has signature $(k, m)$, then the space of lattices above is given by

$$\frac{O(k,m)}{O(k) \times O(m)}$$ (39)

Now, some of these rotations might take the lattice exactly into itself, and so they would count as an automorphism of the lattice.

These just correspond to transformations in $O(k, m)$ that act by integer linear transformations of the lattice into itself. This subgroup is thus called $O(k, m, \mathbb{Z})$.

Thus the correct moduli space of different configurations is given by

$$\frac{O(k,m)}{O(k) \times O(m) \times O(k,m,\mathbb{Z})}$$ (40)

Actually, we might worry that the description above does not make sense because one would imagine that that the actions of $O(k, m, \mathbb{Z})$ and $O(k) \times O(m)$ would not commute and thus ruin the picture. Indeed, the proper way to describe the coset space is as

$$O(k) \times O(m) \backslash O(k, m) / O(k, m, \mathbb{Z})$$ (41)

The idea is that the set of linear transformations that take one lattice to another can be interpreted as left action on the group $O(k, m)$, while the discrete transformations amount to right action on the group. These two commute.
One of them is acting as an active transformation (change of geometry), while the other is acting as a passive transformation (change of basis).

In any case, the discrete group of identifications $O(k, m, \mathbb{Z})$ is called the T-duality group.

This generalizes what we have already seen for the circle. In the circle, we have that the full set of lattices is $O(1, 1)$. This is roughly like $O(2)$, but a non-compact version of it (it is just a boost in dimension $(1, 1)$.

$O(1) \times O(1)$ is trivial in that case. The only transformation that keeps the structure of the lattice is just sending the two generators $(\alpha, \beta) \rightarrow (-\alpha, -\beta)$.

This is the only integer linear transformation inside $O(1, 1)$ that is allowed. This is just a $\mathbb{Z}_2$ symmetry. This particular $\mathbb{Z}_2$ group is exactly the T-duality we already saw.

There are general theorems about these even-self dual lattices:

The first thing, is that $k - m$ must be a multiple of 8. (Since so far we have been dealing with $k = m$, this is trivially satisfied). The second statement is that if one has a basis of $k + m$ vectors $\alpha_i$ that generate the lattice, then the matrix

$$| \det(M_{ij}) | = | \det(\alpha_i \cdot \alpha_j) | = 1$$

has to have determinant one. This amounts to describing the dual basis inside the lattice with integer coefficients. (This is, the fundamental cell has volume one).

Because of this extra property, these lattices are also called even unimodular lattices.

The third important statement is that so long as both $k, m > 0$, all even self-dual lattices are equivalent to each other. The matrix $M$ can be put in the following form

$$M \sim \begin{pmatrix} \sigma & 0 & \ldots & \ldots \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & E_8 & \vdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where there are $|m - k|/8$ copies of the $E_8$ root lattice Dynkin diagram intersection form, and $\min(m, k)$ copies of the fundamental lattice in dimension $(1, 1)$, which we can choose to have the form

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
In the case \( m = 0 \), the problem becomes more interesting. As we can see from above, the dimension of the moduli space of lattices is zero. This means that the solution of the problem is discrete.

For us, we are interested in the case where \( k = 8 \) and \( k = 16 \). For \( k = 8 \) there is a unique solution. This is the root lattice of \( E_8 \) itself.

For \( k = 16 \) there are only two even self-dual lattices. In this case, they are the root lattice of \( E_8 \times E_8 \) and the root lattice of \( SO(32) \).

For \( k = 24 \) there are five different lattices. One of them is very famous. It is called the Leech lattice, and it was a very important part of the Monstrous Moonshine program in mathematics. This program finished the classification of simple groups by finding the monster group as the automorphisms of the Leech lattice, and the corresponding modular form associated to the lattice as containing a list of the dimensions of the representations of the Monster group.

## 4 T-duality action on spacetime fields

We have already seen that by counting dimensions, the dimension of the space of inequivalent set of lattices is equal to the dimension of the space of metric components and vevs for the antisymmetric tensor field \( B_{mn} \).

So how does the field \( B_{mn} \) couple to the worldsheet?

The answer is as follows: the \( A_\mu \sim B_{m\mu} \) field is a vector particle in \( d \) non-compact dimensions (associated to the spacetime index \( \mu \)). We have already seen that winding strings carry this charge. Thus in \( d \) dimensions there should be a coupling of the form

\[
\int A_\mu dx^\mu
\]

(45)

for a point particle, that measures the winding of the string. This can be converted to a two dimensional integral:

\[
\int B_{\mu\nu} \frac{dx^\mu}{d\sigma^\alpha} \frac{dx^\nu}{d\sigma^\beta} d^2\sigma
\]

(46)

in mathematics this is also noted as

\[
\int_\Sigma B^s
\]

(47)
the integral of the pull-back of the two form field $B$.

We are interested in $B_{\mu\nu}$ constant, so that $B$ is strictly a total derivative in the action.

However, just like for the electromagnetic gauge field $A_{\mu}$, just because $A$ is a locally a total derivative, does not mean that $A$ has no effect at all in quantum mechanics.

The main effect is to change the definition of the conjugate momentum variables, to $P \sim \dot{x} + A$. So that if $p$ is quantized, then the quantum Hamiltonian is

$$ (p - A)^2 $$

and the spectrum gets shifted by $A$.

The same thing happens in this case. In the presence of winding, if we try to decompose the modes of the string, we notice that the above coupling only modifies the oscillators term for the constant mode. This means it just modifies the values of $p_L$ and $p_R$ associated to the zero modes, but not the true oscillators.

Thus we can do the calculation just with the zero modes. Remember also that we have two different zero modes, one for the left movers and one for the right movers.

This is, $\partial X$ and $\bar{\partial} X$, when Fourier transformed on the cylinder, have a zero mode each, and they are not equal. This is because for winding strings, the vev of $p_L \sim \oint \partial X$ and $p_R \sim \oint \bar{\partial} X$ are different.

The effect of $X$ being periodic is to introduce winding sectors of the string.

If we compactify on a d-torus, we can use the components of the constant metric $g_{ij}$ and the $B$ field $B_{ij}$. We normalize the metric so that it is $g_{ij} = R^2 \hat{g}_{ij}$, where $\hat{g}$ has determinant equal to one, and the coordinates $x^i$ have periodicity one. We also have the components $B_{ij}$.

The worldsheet action for the zero modes (in conformal gauge), in the presence of windings $w^i$, namely

$$ X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) + w^i $$

is given roughly by

$$ \int \frac{1}{2} g_{ij} \dot{X}^i \dot{X}^j - \frac{1}{2} g_{ij} w^i w^j + \frac{1}{2} B_{ij} (w^i \dot{X}^j - w^j \dot{X}^i) $$

(49)

(in the Euclidean action, B appears with an extra factor of $i$).
Factors of $2\pi$ and proper normalizations are omitted: we restore them by requiring consistency with our expressions for vertex operators when required.

The shifting of the canonical momenta is

$$\pi_i \sim g_{ij} \dot{x}^j + B_{ij} w^j$$

Now, the Hamiltonian will be

$$H \sim \frac{1}{2} g_{ij} \dot{X}^i \dot{X}^j + \frac{1}{2} g_{ij} w^i w^j \sim \frac{1}{2} \left( g^{ij} (\pi_i - B_{ik} w^k)(\pi_j - B_{jm} w^m) + g_{ij} w^i w^j \right)$$

And with $\pi_i \sim \frac{2\pi}{2\pi}$ being quantized we obtain a quadratic form on the momenta and winding $n_i, w^j$.

Notice that the $n_i$ are contracted using $g^{ij}$, while the $w^j$ are contracted using $g_{ij}$ (when B is zero). This property makes the lattice of windings and momenta dual to each other. Notice that the effect of the $B$ field is to mix momenta and winding. (For normalization conventions in the book see the equations (8.4.8) and (8.4.9))

The interesting news is that the spectrum of strings is determined by the vevs associated to $g, B$. The action of T-duality is to leave the spectrum of $H$ invariant while permuting the lattice, and keeping the canonical pairing of winding and momenta invariant (this is what we get from requiring that $p_R P'_R - P_L P'_L \sim w^i n_i' - n_j w^j$ be invariant under relabelings of states). This is requiring that the OPE’s be the same between the two different families of conformal primaries.

In any case, we can compare two different such lattices.

The most general lattice transformation will give us

$$\begin{pmatrix} n^i \\ w^j \end{pmatrix}' \sim \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n^i \\ w^j \end{pmatrix}$$

Keeping the pairing $n w$ invariant places restrictions on the matrices of integers $A, B, C, D$.

The T-duality will relate various values of $g, B$ to each other. Some of the T-dualities are just shifts of $B$ periods by $2\pi$ (shift momentum quantization by one), leaving $g$ fixed. Some others just correspond to coordinate redefinitions of the original torus (the so called mapping-class group). These are the obvious part of gauge symmetry.
The other transformations mix $g$ and $B$, and are non-trivial aspects of string geometry. These generalize the simple T-duality we have already encountered.

5 T-duality action on D-branes

Well, let us begin with a circle compactification on type II theories. The T-duality acts by a $\mathbb{Z}_2$ symmetry and sends $\partial X \to \partial X$ and $\bar{\partial} X \to -\bar{\partial} X$.

Notice that this has the effect of exchanging Dirichlet and Neumann boundary conditions for the open strings along the compactified direction.

This means that T-duality sends type IIA strings theory to type IIB string theory and vice versa, because we are changing the dimensionality of the supersymmetric branes by one with this operation.

A D-brane wrapped on the circle is T-dual to a D-brane that does not wrap the circle. Because the two theories have different chirality, to obtain the IIB spinors from the type IIA spinors, we need to act with a $\Gamma$ MATRIX.

If we compactify $x^9$, it seems natural to use $\gamma^9$ to do this.

A new thing we encounter is that D-branes that are transverse to the compactified direction have a moduli space of positions where they can sit. What is the dual realization of this space of configurations?

Given a reference $D$ brane, we can ask how do we determine the relative position of this D-brane with respect to another one along the circle.

Looking at the open string spectrum is a good idea. The open string spectrum will see a winding charge plus a little bit from the difference in angle.

The T-dual of winding is momentum, so we should see momentum quantization up to a little bit.

But we have already discussed an effect that gives us momentum quantization plus a little bit: we need a gauge field to do that.

There is an obvious gauge field when we have a D-brane. This is the open string bosonic ground state. Just like we can turn on a non-trivial B field by modifying the zero mode piece for the string oscillators of the closed string, we can do the same for the open string.

We just insert

$$\int A_\mu dx^\mu$$

(53)
for each end of the string, where \( A_\mu \) is the corresponding gauge field of the given D-brane.

This can be converted into an integral over the worldsheet too

\[
\int_\Sigma \hat{B}_{\mu\nu} dx^\mu dx^\nu
\]  

\[(54)\]

where \( \hat{B} \) depends on the separation between the branes, and it is the electromagnetic gauge field associated to an (any) interpolating connection between \( A_{\text{left}} \) and \( A_{\text{right}} \) pulled back to the worldsheet.

This is just like a \( B \) field, and it is a total derivative. This gives the required shift in quantization of the zero modes.

The values of \( \oint A_\mu dx^\mu \) around the compactified circles are called Wilson lines. Thus the T-duality map sends D-brane positions to Wilson lines and viceversa.

Moreover, this gives us a recursion relation for the tensions of the D-branes. This is because T-duality should be compatible with how we determine the spectrum of massive objects. Thus, doing two T-dualities we can turn a D0-brane into a D2-brane etc.

L**esson:** The target space of string theory is not unique. The string spectrum and spectrum of D-brane tensions does not determine a unique geometry, but it does determine a unique point in the set of possible theories up to dualities.

This is very important, because later on we will generalize these dualities further.

We can also work with a single T-duality on a two torus, to see other interesting effects.

In particular, let us consider a D1-brane that winds in the (1, 1) direction. This can be considered as a bound state of a D-brane running in the (1, 0) direction and a \( D \) brane running in the (0, 1) direction. This is, a combination of these D-branes can be deformed into the first one.

We can ask, what is the T-dual of the diagonal brane, for a T-duality along (0, 1)?

T-duality along (0, 1) acts on the first D-brane and makes it into a point. It acts on the second brane and makes it into a two torus.

Thus, we conclude that a D0 brane can be captured by a D2-brane. Moreover, one can argue that the D0 brane is delocalized: there is no preferred horizontal line in the configuration of the (1, 1) string.
How do the strings realize that there is D0-brane charge on the D2 brane?

Well, if the D2 brane is very large, adding a D0-brane to it will raise the local energy density of the D2 brane by very little. Thus, if the D0-brane delocalizes, it looks like a low energy excitation of the D2-brane.

A low energy excitation should be captured by massless fields. Since the D0 brane charge is conserved, there should be an associated integer-valued number for the corresponding configuration.

The answer is that the D-brane has a gauge field. Thus D0 brane inside a D2-brane can be captured as a topologically non-trivial configuration of the gauge field on the D2-brane. For a \( U(1) \) theory, there is such a number: the magnetic flux of the corresponding EM field through the two torus.

(For mathematicians: the first Chern class of the associated line bundle)

This might seem surprising at first, but it can be explained as follows:

If the circle along \((0,1)\) is small, and the circle along \((1,0)\) is huge, we can ignore the big circle for local questions. The D1-brane along the \((1,1)\) direction is almost parallel to a \(D1\) brane along the \((1,0)\) direction. We can think of an adiabatic approximation where we are varying slowly the position of the D-brane in the \((0,1)\) plane.

After T-duality, positions become Wilson lines. Thus we should get configurations where the Wilson line is varying slowly.

By a contour argument the Wilson line on two close contours along the \((1,0)\) direction bound a 2-surface. Moreover, we can use the ”integrated Maxwell equations” to show that the difference of the contours above measures the magnetic field inside the contour, and that it is non-zero.

The magnetic field has to be quantized by Dirac’s argument. We can bring a second D2 brane near by (let us call it \(D2'\)). Thus the open strings stretching between D2 and D2’ feel a magnetic field on the left and not on the right. Therefore they feel a net magnetic field.

Requiring that the path integral for the string is consistent when \(A\) is not globally well defined gives us quantization of this magnetic field.

Indeed, we have seen that \(A\) couples to the string by a boundary interaction of the form

\[
\int A_\mu \frac{dX^\mu}{d\tau} d\tau \sim \int A_\mu dx^\mu
\]

and that it looks similar to a \(B\) field, so there is some freedom to move things around.

For \(N\) D-branes on top of each other, we get in general a \(U(N)\) gauge group. This is because we can not distinguish the branes: the spectrum of
strings between branes $i$ and $i$, and branes $i$ and $j$ look identical. All of the particles present include massless vectors associated to these strings stretching between the branes. $U(N)$ gauge fields can have non-trivial topological numbers (Chern classes).

Similar reasoning with other pairs of ”diagonal” branes on high dimensional tori shows that a D0 brane bound to a $D4$ brane becomes an instanton configuration (associated to a non-trivial second Chern class), etc.

In general, classifying the set of different possible D-brane charges gives a very non-trivial invariant of a manifold: K-theory.

6 The heterotic string

We have all the ingredients we require now to define the heterotic string.

Let us imagine that instead of requiring supersymmetry in the left and right movers of the string, we require supersymmetry only on the left movers.

From the left mover point of view, we live in 10 dimensions, and having left movers in the R sector is enough to produce spin fields.

For the right movers, we seem to run into some problem. We should have 10 X directions so that we can match the zero modes, but the central charge is only 10, and we are short of 26, which is the critical dimension for the bosonic string. We must “fix” this by adding something.

Since we know very few things, we add a 16 dimensional right moving boson lattice.

Now, we impose modular invariance. For the left movers, we know that the worldsheet fermions can be assembled into a modular invariant chiral partition function. For the right movers, we have added this extra bosonic lattice. We need to make it into a modular invariant partition function as well.

Well, we have already seen the solution to that problem. There are two distinct even self-dual lattices that would give rise to a modular invariant partition function. These are the root lattice of $E_8 \times E_8$ and the root lattice of $SO(32)$. We can also think of thee as 16 + 16 left moving fermions with a chiral partition function for each 16, or as 32 free fermions with same boundary conditions for all fermions together.

The first partition tells us that $E_8$ is roughly $SO(16)$ plus a spinor of $SO(16)$, and similarly for the second one. The $E_8$ lattice is made of the standard NS sector for the spinors and the R sector. For 16 fields, the twist
fields give dimension $16/16$, as we have seen from various points of view, and it gives rise to a non-trivial enhanced symmetry current algebra.

From our analysis of the bosonic string, we know how to describe the massless spectrum of enhanced particles (in picture ($-1$)) for the right movers:

$$\exp(-\bar{\phi})\bar{\psi}^\mu j_\alpha \exp(ikX)$$

where $j_\alpha$ is one of the currents of dimension one defined above as part of the $E_8$ or $SO(32)$ current algebra.

Thus string theory predicts that there is another supersymmetric string in 10 D. It has only one spacetime SUSY, and it has a gauge group $SO(32)$ or $E_8 \times E_8$. Surprisingly, these cancel gravitational and gauge anomalies in 10 D! This discovery of cancellation of anomalies made Green and Schwarz famous and really got the "first string revolution started".