Week 5
Reading material from the books

- *Polchinski, Chapter 15*

- *Infinite conformal symmetry in two-dimensional quantum field theory*, by Belavin, Polyakov and Zamolodchikov, 

1 The Conformal Ward identity

On discussing the operator state correspondence, we can ask what is the meaning of a primary field.

The definition of being primary of weight $h$ is that

$$T(z)\mathcal{O}(0) \simeq \frac{h}{z^2} \mathcal{O}(0) + \frac{1}{z} \partial \mathcal{O}(0)$$  \hspace{1cm} (1)

By taking contour integrals, this is equivalent to $[L_0, \mathcal{O}(0)] = h\mathcal{O}(0)$ and that $[L_n, \mathcal{O}(0)] = 0$ for $n \geq 1$. That is, the corresponding state satisfies

$$L_n|\mathcal{O}\rangle = 0$$  \hspace{1cm} (2)

for $n \geq 1$ and $L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle$, so the state is a lowest weight state of Virasoro. Since any state can be obtained by acting with (possibly many) $L$ on a lowest weight state, we find that at the level of operators, the primary ones are special, and the rest of them are related to them by Virasoro. These are called descendants. If conformal symmetry is thought of as a group symmetry, then the Wigner-Eckart theorem suggest that once we know the correlation functions of the primaries, the rest should be group theory.

To do that, we want to understand how to insert descendants on a correlation function that is already known. The idea is that descendants are obtained by acting with commutators with the $L$ operators, and that these are represented by contour integrals of insertions of $T(z)$. So if we know how to insert various $T$ into a correlation function, we are in shape, as then we can integrate the results over our favorite contours.

The main idea to do so is that $T(z)$ is holomorphic, except at the insertion of the operators $\mathcal{O}$, where it becomes meromorphic, with poles of first and second order.
Now, we need to understand the asymptotic behavior of $T(z)$ at infinity. Using the change of variables $w = 1/z$, we find that $T(w)dw^2 = T(z)dz^2$. Since $w = 0$ is the point at infinity and it is a regular point (no insertions at infinity), we find that therefore

$$\lim_{w \to 0} T(w) \simeq \lim_{z \to \infty} T(z) z^4$$

exists.

This implies that $T(z)$ decays like $1/z^4$ at infinity.

Because of this, we can use theorems of complex analysis to tell us what happens when we introduce $T(z)$ in a correlation function. The expression

$$\langle T(z) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle \simeq G(z, z_1, \ldots, z_n)$$

must be a meromorphic function of $z$ that decays at infinity as $1/z^4$, and has single and double poles at the insertions $z_i$ (these arise from the OPE of $T$ with the $\mathcal{O}$). For functions that vanish at infinity, they are determined by their polar part alone (this is usually proved by contradiction and the maximum principle: the maximum and minimum in norm of a holomorphic function on a domain occurs at the boundary of the domain). Using this we find that

$$\sum_i \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = 0 \quad (8)$$

This shows that insertions of $T$ can be computed by applying a differential operator to Green’s functions of primary fields.

Now, using the regularity at $z \to \infty$, we find that the Green’s functions must satisfy three equations, one each from the terms that decay as $z^{-1}$, $z^{-2}$ and $z^{-3}$. The first one is

$$\sum_i \partial_i \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = 0 \quad (8)$$

this indicates translation invariance of the Green’s functions: they are only functions of $z_{ij} = z_i - z_j$. The next one is

$$\sum_i (h_i + z_i \partial_i) \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = 0 \quad (9)$$
This is equivalent to saying that the Green’s function is scaling, with weight $-\sum h_i$. The operator $\sum z_i \partial_i$ is also sometimes also known as the Euler operator.

The last one is related to special conformal transformations and reads

$$\sum_i (2z_i h_i + z_i^2 \partial_i) \langle O_1(z_1) \ldots O_n(z_n) \rangle = 0$$  \hspace{1cm} (10)

Let us apply these to the study of correlation functions of few operators. We find that for one point functions

$$\partial \langle O(z) \rangle = 0$$  \hspace{1cm} (11)

so a vacuum expectation value of a single operator is constant. Secondly

$$(h + z \partial) \langle O(z) \rangle = h \langle O(0) \rangle = 0$$  \hspace{1cm} (12)

So either $h = 0$ or the one point function vanishes. We will assume that (for unitary) conformal field theories, that there is only one operator of dimension 0, the identity operator that does nothing, so that

$$\langle 1 \rangle = 1$$  \hspace{1cm} (13)

All other operators will have $h > 0$.

Now, let us go to two point functions. Translation invariance implies that

$$\langle O_1(z_1)O_2(z_2) \rangle = f(z_{12})$$  \hspace{1cm} (14)

Now scale covariance implies that

$$z_{12} f'(z_{12}) = -h_1 - h_2$$  \hspace{1cm} (15)

so that

$$f(z_{12}) = C_{12} z_{12}^{-h_1-h_2}$$  \hspace{1cm} (16)

The third equation implies that $h_1 = h_2$ so that $C_{12}$ is a quadratic form. Via the operator-state-correspondence (being rather careful), this quadratic form becomes the inner product in the Hilbert space of states and should be positive definite. The reason that $C_{ij}$ vanishes for $h_i \neq h_j$ is that $L_0$ is hermitian, so that two states with different energies must be orthogonal.
Now let us consider the compatibility of the Ward identity with the OPE. We start with two primary fields $O_1(z_1)O_2(z_2)$ inside a correlation function. We assume that

$$\langle \ldots O_1(z_1)O_2(z_2)\ldots \rangle = \langle \ldots \sum C^{[\alpha]}_{12} z_1^\alpha A_{[\alpha]}(z_2)\ldots \rangle$$

and now we act with $L_0$, assuming that the $A_{[\alpha]}$ are scaling of weight $h_{\alpha}$. We can evaluate the Ward identity on the left or the right. We want to act with $L_0$ (centered at at 0, so we take $z_2 \to 0$ in the right hand side.) For the right hand side we get

$$\langle \ldots \sum C^{[\alpha]}_{12} z_1^\alpha A_{[\alpha]}(L_0)\ldots \rangle = \langle \ldots \sum C^{[\alpha]}_{12} z_1^\alpha h_{\alpha} A_{[\alpha]}(0)\ldots \rangle$$

now for the left side, we use the fact that $L_0 \propto \oint T(w)w$ and we choose a contour that encircles both $z_1, z_2$. We get that

$$\langle \ldots [L_0, O_1(z_1)O_2(z_2)]\ldots \rangle = \langle \ldots (h_1 + h_2 + z_1 \partial_1 + z_2 \partial_2)O_1(z_1)O_2(z_2)\ldots \rangle$$

Now we take the limit $z_2 \to 0$, so the term $z_2 \partial_2 = 0$. We get then that

$$\langle h_1 + h_2 + z_1 \partial_1 \ldots \rangle = \langle \ldots \sum C^{[\alpha]}_{12} z_1^\alpha A_{[\alpha]}(0)\ldots \rangle = \langle \ldots \sum C^{[\alpha]}_{12} z_1^\alpha h_{\alpha} A_{[\alpha]}(0)\ldots \rangle$$

the equality of these two expressions as formal power series implies that

$$h_{\alpha} = h_1 + h_2 + \alpha$$

This gives a notion of ‘naive scaling’. each operator of weight $h_{\alpha}$ has dimension units $h_{\alpha}$, while $z^\alpha$ has dimension $-\alpha$.

### 1.1 Three point functions

The next layer of complication is the set of three point functions of CPO’s (Conformal primary operators).

Translation invariance implies that

$$\langle O_1(z_1)O_2(z_2)O_3(z_3) \rangle = G(z_{31}, z_{32})$$

The three equations can be thought of as the symbolic linear algebra problem

$$\partial_1 + \partial_2 + \partial_3 = 0$$

$$z_1 \partial_1 + z_2 \partial_2 + z_3 \partial_3 = -h_1 - h_2 - h_3$$

$$z_1^2 \partial_1 + z_2^2 \partial_2 + z_3^2 \partial_3 = -2h_1 z_1 - 2h_2 z_2 - 2h_3 z_3$$
and it shows that $\partial_1, \partial_2$ can be eliminated in favor of $\partial_3$. For example, multiplying the first one by $z_2$ and subtracting from the second one we find that
\[(z_1 - z_2)\partial_1 + (z_3 - z_2)\partial_3 = -h_1 - h_2 - h_3 \tag{24}\]
Similarly, multiplying the second one by $z - 2$ and subtracting from the third, we find that
\[z_1(z_1 - z_2)\partial_1 + z_3(z_3 - z_2)\partial_3 = -2z_1h_1 - 2z_2h_2 - 2z_3h_3 + z_2(h_1 + h_2 + h_3) \tag{25}\]
And now we can substitute the expression (24) in the last line to find that
\[z_31z_32\partial_3G = (h_2 + h_3 - h_1)(-z_31) + (h_1 + h_3 - h_2)(-z_32) \tag{26}\]
We can separate variables to
\[\frac{\partial_3G}{G} = -(h_2 + h_3 - h_1)\partial_3 \log(z_32) - (h_1 + h_3 - h_2)\partial_3 \log(z_31) \tag{27}\]
from here, we find that
\[G = \frac{C(z_{12})}{(z_{31})^{h_1+h_3-h_2}(z_{32})^{h_2+h_3-h_1}} \tag{28}\]
using symmetry between 123 we find that
\[G = \frac{C_{123}}{(z_{31})^{h_1+h_3-h_2}(z_{32})^{h_2+h_3-h_1}(z_{12})^{h_1+h_2-h_3}} \tag{29}\]
so the three point functions are uniquely determined up to a constant!
From what we have so far we can not normalize these constant away. Only the two point function can be calibrated by rescaling the operators if necessary.

1.2 Minimal models

The idea of minimal models is that for $c < 1$, there are special short representations that are unitary. A short representation is one where at least one descendant is null. These occur for
\[h_{p,q}(m) = \frac{[(m + 1)p - mq]^2 - 1}{4m(m + 1)} \tag{30}\]
where \( p, q \) are integers in the window \((m - 1) \geq p \geq q \geq 1 \) (other values of \( p, q \) with the same dimension as those above count as the same fields). Such a field has a null descendant at level \( pq \).

In these models, the central charge is given by

\[
c = 1 - \frac{6}{m(m+1)} > 0
\]  

and \( m \) is an integer. The simplest such model has \( m = 3 \), and that corresponds to

\[
c = \frac{1}{2}
\]  

There are three types of null fields allowed. The first one is \( h_{1,1} = 0 \) (this one always exists). Such an operator has a null descendant at level 1. The only such descendant is \( \partial \mathcal{O} \simeq 0 \), so that the operator is constant. It is a \( c \)-number. It is the identity operator and is in a sense trivial.

The others are for \( h_{1,2} = h_{2,2} = 1/16 \), and \( h_{2,1} = h_{1,3} = 1/2 \). Each of these has the property that they have a null descendant at level 2.

Such a descendant is of the form

\[
(\alpha L_{-1}^2 + \beta L_{-2})|\mathcal{O}\rangle = 0
\]  

This is identifying the null descendant with a state of norm zero being identically zero in the Hilbert space of states.

The operator of dimension 1/16 will be called \( \sigma \), and the other \( \epsilon \). This model represents the critical Ising model (the simplest second order phase transition).

Consider a correlation function with two of the \( \sigma \) fields. We want to understand the singularity when \( \sigma \) gets close to \( \sigma \). Let us apply \( L_{-2} \) and use usual tricks of complex analysis shifting contours. We find that we need to insert a \( \oint T(w)/2\pi i(w-z) \), but we want to change the contour to include all singularities except \( z \). At the singularity \( z_1 \), we find that one gets the following behavior

\[
\frac{2h_1}{(z-z_1)^2} + \frac{1}{z-z_1} \partial_1
\]

and all the other terms are regular at \( z_1 \). We are assuming that we are acting on the correlation function of just primaries. Now, \( L_{-1}^2 \) acts as \( \partial_z^2 \).

The differential equation that we need to satisfy is of the singular form

\[
\alpha \partial_z^2 + \beta \left[ \frac{2h_1}{(z-z_1)^2} + \frac{1}{z-z_1} \partial_1 \right] \simeq 0
\]  

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Now let us study the OPE. It is clear that

$$\sigma(z)\sigma(z_1) \simeq \frac{c_{\sigma\sigma}}{(z-z_1)^{1/8}} + \frac{c_{\sigma\sigma\sigma}}{(z-z_1)^{1/16}}\sigma(z_1) + c_{\sigma\sigma\epsilon}(z-z_1)^{3/8}\epsilon(z_1) + \ldots$$  (36)

We know that $c_{\sigma\sigma} \neq 0$ if the field is allowed. The most singular terms of the null field go like

$$[\alpha(-1/8)(-1-1/8) + \beta(2h_1+1/8)](z-z_1)^{1/8+2} = 0$$  (37)

so we get a relation between $\alpha, \beta$, because we know that $c_{\sigma\sigma} \neq 0$.

Expansing we also find that $c_{\sigma\sigma\sigma}$ is inconsistent with these values of $\alpha, \beta$, but not $c_{\sigma\sigma\epsilon}$. We can replace $\partial_1 \simeq -\partial_z$ to understand the singularity (the other variables are regular and far away).

We express this by saying that

$$\sigma \times \sigma \simeq [1] + [\epsilon]$$  (38)

the fact that $\epsilon$ shows up on the right hand side implies that $\epsilon \times \sigma \simeq [\sigma] + \ldots$.

If one considers more general minimal models, one can have $\psi_{1,2}$ and $\psi_{2,1}$. Quite a bit of work with differential equations as above goes to show that

$$\psi_{1,2} \times \psi_{p,q} \simeq [\psi_{p,q+1}] + [\psi_{p,q-1}]$$  (39)

and that

$$\psi_{2,1} \times \psi_{p,q} \simeq [\psi_{p+1,q}] + [\psi_{p-1,q}]$$  (40)

where none of the objects of the right hand side can take values out of the preferred box. This is shown by realizing that for each primary $\psi_{2,1}$ there is an infinite family of higher order descendants that appear again and again. These extra differential equations kill the other “unwanted” pieces in the OPE.