Problem 1

We are to show four properties of the vector commutator,

\[ [X, Y]f = X(Y(f)) - Y(X(f)), \]

namely

(a) linearity,
(b) the Leibnitz rule,
(c) the component formula,
(d) transformation as a vector field.

Let \( a, b \in \mathbb{R} \) be constants and let \( f, g \) be \( C^\infty \) functions from \( M \to \mathbb{R} \). Here we go:

(a) To show \([X, Y] \) is linear, we evaluate the commutator below using linearity of \( X \) and \( Y \) twice, and then regrouping terms:

\[
[X, Y](af + bg) = X\left(Y(af + bg)\right) - Y\left(X(af + bg)\right)
\]

\[
= X\left(aY(f) + bY(g)\right) - Y\left(aX(f) + bX(g)\right)
\]

\[
= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bY(X(g))
\]

\[
= a\left(X(Y(f)) - Y(X(f))\right) + b\left(X(Y(g)) - Y(X(g))\right)
\]

\[
= a[X, Y](f) + b[X, Y](g).
\]

(b) The Leibnitz property of \( X \) is \( X(fg) = fX(g) + gX(f) \). To show \([X, Y] \) satisfies the Leibnitz property, evaluate the commutator below. We use the Leibnitz property of \( X \) and \( Y \) twice, then cancel the underlined terms below, and finally regroup terms:

\[
[X, Y](fg) = X\left(Y(fg)\right) - Y\left(X(fg)\right)
\]

\[
= X\left(fY(g) + gY(f)\right) - Y\left(fX(g) + gX(f)\right)
\]

\[
= fX(Y(g)) + X(f)Y(g) + gX(Y(f)) + X(g)Y(f)
\]

\[
- fY(X(g)) - Y(f)X(g) - gY(X(f)) - Y(g)X(f)
\]

\[
= f\left(X(Y(g)) - Y(X(g))\right) + g\left(X(Y(f)) - Y(X(f))\right)
\]

\[
= f[X, Y](g) + g[X, Y](f).
\]

(c) We next derive the component formula for the commutator. Here we use Carroll’s notation: a vector \( V \) is written in terms of its components \( V^\mu \) as

\[
V = V^\mu \partial_\mu,
\]

and when \( V \) acts on a function \( f \) we have

\[
V(f) = V^\mu (\partial_\mu f). \tag{1}
\]
With this we can write the commutator as
\[
[X, Y](f) = [X^\mu \partial_\mu, Y^\nu \partial_\nu](f)
\]
\[
= X^\mu \partial_\mu(Y^\nu)(\partial_\nu f) + X^\mu Y^\nu(\partial_\mu \partial_\nu f) - Y^\nu \partial_\nu(X^\mu)(\partial_\mu f) - Y^\nu X^\mu(\partial_\nu \partial_\mu f)
\]
\[
= \left( X^\lambda \partial_\lambda(Y^\mu) - Y^\lambda \partial_\lambda(X^\mu) \right)(\partial_\mu f).
\]

To get to the third line I used the fact that partial derivatives commute (as do components, for that matter, since they are just numbers) and relabeled some summed indices. Comparing this expression to (1) we see that the components of \([X, Y]\) are
\[
[X, Y]^\mu = X^\lambda \partial_\lambda(Y^\mu) - Y^\lambda \partial_\lambda(X^\mu).
\]

(d) In (c.) we showed that \([X, Y]\) has components along basis vectors, but we did not properly show that it is a vector. We do this now by showing that the components of \([X, Y]\) transform as a vector.

Under a coordinate transformation vector components and basis vectors transform as follows:
\[
X^\mu' = \frac{\partial x^\mu'}{\partial x^\mu} X^\mu, \quad \partial_\mu' = \frac{\partial x_\mu'}{\partial x_\mu} \partial_\mu.
\]

Substituting this into (2) gives
\[
[X, Y]^\mu' = X^\lambda' \partial_\lambda(Y^\mu') - Y^\lambda' \partial_\lambda(X^\mu')
\]
\[
= \frac{\partial x^\lambda'}{\partial x^\lambda} X^\lambda \partial_\sigma \left( \frac{\partial x^\mu'}{\partial x^\mu} Y^\mu \right) - \frac{\partial x^\lambda'}{\partial x^\lambda} Y^\lambda \partial_\sigma \left( \frac{\partial x^\mu'}{\partial x^\mu} X^\mu \right)
\]
\[
= \delta^\mu_\lambda X^\sigma \partial_\sigma \left( \frac{\partial x^\mu'}{\partial x^\mu} Y^\mu \right) - \delta^\mu_\lambda Y^\sigma \partial_\sigma \left( \frac{\partial x^\mu'}{\partial x^\mu} X^\mu \right)
\]
\[
= X^\lambda' \partial_\lambda \left( \frac{\partial x^\mu'}{\partial x^\mu} Y^\mu \right) - Y^\lambda' \partial_\lambda \left( \frac{\partial x^\mu'}{\partial x^\mu} X^\mu \right)
\]
\[
= X^\lambda Y^\mu \frac{\partial x^\mu'}{\partial x^\lambda \partial x^\mu} + \frac{\partial x^\mu'}{\partial x^\mu} X^\lambda \partial_\lambda(Y^\mu) - X^\lambda Y^\mu \frac{\partial x^\mu'}{\partial x^\lambda \partial x^\mu} - \frac{\partial x^\mu'}{\partial x^\mu} Y^\lambda \partial_\lambda(X^\mu)
\]
\[
= \frac{\partial x^\mu'}{\partial x^\mu} [X, Y]^\mu.
\]

Indeed, the components of \([X, Y]\) transform like a vector, so \([X, Y]\) is a vector.

(e) Although a slight aside, we now have the opportunity to prove the Jacobi identity for vector fields. Using the definition of the commutator twice and the linearity of \(Z\), we find
\[
[[X, Y], Z]f = [X, Y](Z(f)) - Z([X, Y]f)
\]
\[
= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(X(f))
\]
\[
= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(X(f))).
\]

Similarly,
\[
[[Y, Z], X]f = Y(Z(X(f))) - Z(Y(X(f))) - X(Y(Z(f))) + X(Z(Y(f)),
\]
\[
[[Z, X], Y]f = Z(X(Y(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Y(Z(X(f)).
\]

When we add the last three lines, all the terms on the right-hand sides cancel out in pairs. Since \(f\) is arbitrary, this proves the Jacobi identity,
\[
[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.
\]
Problem 2

In components,

\[
(d(\omega \wedge \eta))_{\mu_1...\mu_{p+q+1}} = (p + q + 1) \frac{(p + q)!}{p!q!} \partial_{[\mu_1} (\omega)_{\mu_2...\mu_{p+1}} \eta_{\mu_{p+2}...\mu_{p+q+1}]} \\
= \frac{(p + q + 1)!}{p!q!} \left( (\partial_{[\mu_1} \omega_{\mu_2...\mu_{p+1}}) \eta_{\mu_{p+2}...\mu_{p+q+1}] + \omega_{[\mu_{p+2}...\mu_{p+1]} (\partial_{\mu_1} \eta_{\mu_{p+2}...\mu_{p+q+1}}) \right)
\]

We drop the inner set of antisymmetrization brackets in the second line because the terms are antisymmetrized in all of their indices. Note also that we do not pick up a factor of \((-1)^p\) when we place the \(\partial_{\mu_1}\) derivative on \(\eta\). If you like, you can think of differentiating in \(x^{\mu_1}\) first, and then antisymmetrizing. But now let’s proceed to move the \(\mu_1\) index in the second term all the way to the left, picking up the \((-1)^p\):

\[
(d(\omega \wedge \eta))_{[\mu_1...\mu_{p+q+1} = \frac{(p + q + 1)!}{p!q!} \left( (\partial_{[\mu_1} \omega_{\mu_2...\mu_{p+1}}) \eta_{\mu_{p+2}...\mu_{p+q+1}] + (-1)^p \omega_{[\mu_1...\mu_{p+1} (\partial_{\mu_{p+1}} \eta_{\mu_{p+2}...\mu_{p+q+1}}) \right)
\]

We explicitly restore a subset of antisymmetrized indices for cosmetic reasons. We also multiply the first term by \((p + 1)/(p + 1)\), and the second by \((q + 1)/(q + 1)\):

\[
(d(\omega \wedge \eta))_{\mu_1...\mu_{p+q+1}} = (p + 1) \frac{(p + q + 1)!}{(p + 1)!q!} \left( (\partial_{[\mu_1} \omega_{\mu_2...\mu_{p+1}}) \eta_{\mu_{p+2}...\mu_{p+q+1}] + (q + 1) \frac{(p + q + 1)!}{p!(q + 1)!} \left( (-1)^p \omega_{[\mu_1...\mu_{p+1} (\partial_{\mu_{p+1}} \eta_{\mu_{p+2}...\mu_{p+q+1}}) \right)
\]

Now we can rewrite the factors with derivatives as exterior derivatives, absorbing the coefficients of \(p + 1\) and \(q + 1\), and the inner antisymmetrization brackets:

\[
(d(\omega \wedge \eta))_{\mu_1...\mu_{p+q+1}} = \frac{(p + q + 1)!}{(p + 1)!q!} (d\omega)_{[\mu_1...\mu_{p+1}} \eta_{\mu_{p+2}...\mu_{p+q+1}]} + \frac{(p + q + 1)!}{p!(q + 1)!} (-1)^p \omega_{[\mu_1...\mu_{p+1} (d\eta)_{\mu_{p+1}...\mu_{p+q+1}]
\]

Finally, we can rewrite this as a sum of wedge products, absorbing the rest of the combinatorial factors:

\[
(d(\omega \wedge \eta))_{\mu_1...\mu_{p+q+1}} = ((d\omega) \wedge \eta)_{\mu_1...\mu_{p+q+1} + (-1)^p (\omega \wedge (d\eta))_{\mu_{p+1}...\mu_{p+q+1}}
\]
Problem 3

The “rotating” coordinates are defined by:

\[
\begin{align*}
t' &= t , \\
x' &= (x^2 + y^2)^{1/2} \cos(\phi - \omega t) , \\
y' &= (x^2 + y^2)^{1/2} \sin(\phi - \omega t) \quad \text{where} \quad \phi = \tan^{-1}(y/x) , \\
z' &= z .
\end{align*}
\]

To find \(x\) and \(y\) as functions of the rotating coordinates, note that

\[
\begin{align*}
x^2 + y^2 &= x'^2 + y'^2 , \\
y/x &= \tan(\phi' + \omega t') \quad \text{where} \quad \phi' = \tan^{-1}(y'/x') .
\end{align*}
\]

Relation (4) follows from using \(\tan(\phi - \omega t) = y'/x'\) and \(t = t'\) to rewrite \(y/x = \tan \phi\).

Substituting (4) into (3) and solving for \(x\) and \(y\) as desired gives:

\[
\begin{align*}
t &= t' , \\
x &= (x'^2 + y'^2)^{1/2} \cos(\phi' + \omega t') , \\
y &= (x'^2 + y'^2)^{1/2} \sin(\phi' + \omega t') \quad \text{where} \quad \phi' = \tan^{-1}(y'/x') , \\
z &= z' .
\end{align*}
\]

The metric in the original coordinates is \(g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\). The metric in the rotating coordinates can be found by directly computing \(g'_{\mu'\nu'} = (\partial x^\alpha / \partial x'^\mu)(\partial x^\beta / \partial x'^\nu) g_{\alpha\beta}\) or by evaluating

\[
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt'^2 + \left(\frac{\partial x}{\partial x'^\mu} dx^\mu\right)^2 + \left(\frac{\partial y}{\partial x'^\mu} dx^\mu\right)^2 + dz'^2
\]

and then identifying the metric from \(ds^2 = g_{\mu'\nu'} dx'^\mu dx'^\nu\). After differentiating and simplifying, the result is

\[
\left(\frac{\partial x}{\partial x'^\mu} dx^\mu\right)^2 + \left(\frac{\partial y}{\partial x'^\mu} dx^\mu\right)^2 = dx'^2 + dy'^2
\]

\[
+ 2\omega \left(x'dy' - y'dx'\right) dt' + \omega^2 \left(x'^2 + y'^2\right) dt'^2 .
\]

Hence,

\[
ds^2 = dx'^2 + dy'^2 + dz'^2
\]

\[
+ [1 + \omega^2(x'^2 + y'^2)] dt'^2 + 2\omega(x' dt' dy' - y' dt' dx') .
\]

Equivalently,

\[
g_{\mu'\nu'} = \begin{pmatrix}
-1 + \omega^2(x'^2 + y'^2) & -\omega y' & \omega x' & 0 \\
-\omega y' & 1 & 0 & 0 \\
\omega x' & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} .
\]

The inverse metric is the inverse of this matrix,

\[
g^{\mu'\nu'} = \begin{pmatrix}
-1 & -\omega y' & \omega x' & 0 \\
-\omega y' & 1 - \omega^2 y'^2 & \omega^2 x'y' & 0 \\
\omega x' & \omega^2 x'y' & 1 - \omega^2 x'^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} .
\]
Problem 4

(a) Let $\xi^a$ be a vector field on $\mathbb{R}^2$ whose components in a cartesian basis are $\xi^x = xy$, $\xi^y = x^2 + y^2$. To find the components of $\xi^a$ in the basis defined by polar coordinates, we’ll use a coordinate basis, $\xi = \xi^r \frac{\partial}{\partial r} + \xi^\theta \frac{\partial}{\partial \theta}$, and the vector transformation property $\xi'^\mu = (\partial x'^\mu / \partial x^\nu) \xi^\nu$:

$$\xi^r = \frac{\partial r}{\partial x^\nu} \xi^\nu + \frac{\partial r}{\partial y^\nu} \xi^\nu, \quad \xi^\theta = \frac{\partial \theta}{\partial x^\nu} \xi^\nu + \frac{\partial \theta}{\partial y^\nu} \xi^\nu.$$ 

Using $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$ we obtain:

$$\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\sin \theta \frac{r}{r}, \quad \frac{\partial \theta}{\partial y} = \cos \theta \frac{r}{r}.$$ 

Thus

$$\xi^r = \cos \theta (r \cos \theta)(r \sin \theta) + \sin \theta (r^2) = r^2 \sin \theta (1 + \cos^2 \theta),$$

$$\xi^\theta = -\frac{\sin \theta}{r} (r \cos \theta)(r \sin \theta) + \frac{\cos \theta}{r} (r^2) = r \cos^3 \theta.$$ 

Note that in this coordinate basis, $\frac{\partial}{\partial \theta}$ is not normalized to unity.

(b) We are given the vector fields:

$$\xi = xy \frac{\partial}{\partial x} + (x^2 + y^2) \frac{\partial}{\partial y}, \quad \eta = y^3 \frac{\partial}{\partial x} + x^3 \frac{\partial}{\partial y}.$$ 

Let’s calculate the commutator operating on an arbitrary function $f$:

$$[\xi, \eta] f = (xy \frac{\partial}{\partial x} + (x^2 + y^2) \frac{\partial}{\partial y})(y^3 \frac{\partial}{\partial x} + x^3 \frac{\partial}{\partial y}) f - (y^3 \frac{\partial}{\partial x} + x^3 \frac{\partial}{\partial y})(xy \frac{\partial}{\partial x} + (x^2 + y^2) \frac{\partial}{\partial y}) f$$

Consider the very first terms from both lines:

$$xy \frac{\partial}{\partial x} \left( y^3 \frac{\partial}{\partial x} \right) f - y^3 \frac{\partial}{\partial x} \left( xy \frac{\partial}{\partial x} \right) f = -y^4 \frac{\partial f}{\partial x}.$$ 

Notice that the terms with $\frac{\partial^2 f}{\partial x^2}$ cancel – this will happen with all pairs of terms, so there won’t be any second derivatives. The terms that will survive are the ones in which the partial derivative in $\xi$ act on the components of $\eta$ and vice versa:

$$\frac{\partial f}{\partial y} + 3y \frac{\partial f}{\partial x} - y^2 \frac{\partial f}{\partial x} - 2x y \frac{\partial f}{\partial y} - 2 y x \frac{\partial f}{\partial x}.$$ 

Since $f$ is arbitrary,

$$[\xi, \eta] = (3x^2 y^2 + 2y^4 - x^4) \frac{\partial}{\partial x} + (x^3 y - 2x y^3) \frac{\partial}{\partial y}$$

or in component form,

$$[\xi, \eta]^x = 3x^2 y^2 + 2y^4 - x^4, \quad [\xi, \eta]^y = x^3 y - 2x y^3.$$ 

The above calculation shows that in a coordinate basis, the commutator has components $[\xi, \eta]^\mu = \sum_\nu (\xi^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu \xi^\mu)$. Note that since the given vector fields $\xi, \eta$ don’t commute, they cannot describe a coordinate basis.