Problem 1 (Carroll Chapter 3, problem 12)

(a) Consider:
\[ [\nabla_{\rho}, \nabla_{\mu}]K_{\sigma} = \nabla_{\rho} \nabla_{\mu} K_{\sigma} - \nabla_{\mu} \nabla_{\rho} K_{\sigma} = -R^{\nu}_{\sigma \mu \rho} K_{\nu}. \]

Using the Killing equation we can also write this as:
\[ \nabla_{\rho} \nabla_{\mu} K_{\sigma} + \nabla_{\mu} \nabla_{\sigma} K_{\rho} = -R^{\nu}_{\sigma \rho \mu} K_{\nu}. \tag{1} \]

Now, we will write two more versions of equation (1) with indices permuted:
\[ \nabla_{\mu} \nabla_{\sigma} K_{\rho} + \nabla_{\sigma} \nabla_{\rho} K_{\mu} = -R^{\nu}_{\rho \mu \sigma} K_{\nu}. \tag{2} \]
\[ \nabla_{\sigma} \nabla_{\rho} K_{\mu} + \nabla_{\rho} \nabla_{\mu} K_{\sigma} = -R^{\nu}_{\mu \sigma \rho} K_{\nu}. \tag{3} \]

Now consider (1) + (2) − (3). We get
\[ 2\nabla_{\rho} \nabla_{\sigma} K_{\rho} = -(R^{\nu}_{\sigma \rho \mu} + R^{\nu}_{\rho \mu \sigma} - R^{\nu}_{\mu \sigma \rho}) K_{\nu} = 2R^{\nu}_{\mu \sigma \rho} K_{\nu} = 2R_{\mu \sigma \rho \nu} K^{\nu}. \]

Raising the index \( \rho \) gives the desired result,
\[ \nabla_{\mu} \nabla_{\sigma} K^{\rho} = R^{\rho}_{\sigma \mu \nu} K^{\nu}. \]

(b) We are asked to show that \( K^{\alpha} \nabla_{\alpha} R = 0 \), which says the Ricci scalar does not change as we move along a Killing vector field. The short way to do this is to use the result that the Killing parameter \( \lambda \) can be used as a coordinate in a coordinate system in which \( \frac{\partial}{\partial \lambda} = K^{\alpha} \partial_{\alpha} \) and the metric is independent of \( \lambda \). Then since \( R \) is a scalar, \( K^{\alpha} \nabla_{\alpha} R = K^{\alpha} \partial_{\alpha} R = \frac{\partial R}{\partial \lambda} \). But since the metric is independent of \( \lambda \), the Ricci scalar \( R \) must be as well (since it is built from the metric) so \( \frac{\partial R}{\partial \lambda} = 0 \) hence \( K^{\alpha} \nabla_{\alpha} R = 0 \). We can also derive this from the result in part (a) by contracting \( \rho \) with \( \mu \):
\[ R_{\sigma \nu} K^{\nu} = \nabla_{\mu} \nabla_{\sigma} K^{\mu} \]

Now apply \( \nabla^{\sigma} \) on both sides:
\[ \nabla^{\sigma}(R_{\sigma \nu} K^{\nu}) = \nabla^{\sigma} \nabla_{\mu} \nabla_{\sigma} K^{\mu} \tag{4} \]

The righthand side is:
\[ \nabla^{\sigma} \nabla_{\mu} \nabla_{\sigma} K^{\mu} = \nabla_{\sigma} \nabla_{\mu} \nabla^{\sigma} K^{\mu} - \nabla_{\sigma} \nabla_{\mu} \nabla^{\sigma} K^{\mu} = -\nabla_{\sigma} \nabla_{\mu} \nabla^{\sigma} K^{\mu} + \nabla_{\sigma} \nabla_{\mu} \nabla^{\lambda} K^{\sigma} - \nabla_{\sigma} \nabla_{\mu} \nabla^{\mu} K^{\lambda} \]
\[ = -\nabla_{\mu} \nabla_{\sigma} \nabla^{\mu} K^{\sigma} - R^{\mu \lambda \sigma} \nabla^{\lambda} K^{\sigma} - R^{\mu \lambda} \nabla^{\sigma} \nabla^{\lambda} K^{\sigma} \]
\[ = -\nabla_{\mu} \nabla_{\sigma} \nabla^{\mu} K^{\sigma} - R^{\mu \lambda \sigma} \nabla^{\lambda} K^{\sigma} + R_{\lambda} \nabla^{\lambda} K^{\sigma} \]
\[ = -\nabla_{\mu} \nabla_{\sigma} \nabla^{\mu} K^{\sigma} = 0 \]

In the first line we used the Killing equation. In the second line we used the relation between commutators of covariant derivatives and Riemann tensors. In the fourth line we renamed dummy indices. To obtain line 5 we used the symmetry property of Ricci
tensor, and if we rename the dummy indices we see the expression is the negative of the middle quantity in first line, so it vanishes. Thus, the lefthand side of (4) vanishes:

\[ 0 = \nabla^\sigma (R_{\sigma\nu} K^\nu) = \nabla^\sigma R_{\nu\sigma} K^\nu + R_{\sigma\nu} \nabla^\sigma K^\nu = \nabla^\sigma R_{\nu\sigma} K^\nu = \frac{1}{2} K^\nu \nabla_\nu R \]

which is the desired result. In the second line we used the Killing equation and the symmetry of \( R_{\alpha\beta} \). In the last line the Bianchi identity has been used (Carroll 3.150).

**Problem 2 (Carroll Chapter 3, problem 14)**

The Killing vectors are:

\[ R = \partial_\phi, \quad S = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \quad T = -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi. \]

We want to show:


In evaluating each commutator, you must not forget to use a test function \( f \). Also, all terms that are proportional to second derivatives of \( f \) are underlined below, and cancel out.

\[ [R, S] f = [\partial_\phi, \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi] f \]
\[ = \partial_\phi (\cos \phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi f) - (\cos \phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi f) \partial_\phi f \]
\[ = (-\sin \phi \partial_\theta f + \cot \theta \cos \phi \partial_\phi f + \cos \phi \partial_\phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi^2 f) \]
\[ - (\cos \phi \partial_\theta \partial_\phi f - \cot \theta \sin \phi \partial_\phi^2 f) \]
\[ = T f \]

Similarly,

\[ [T, R] f = [-\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi, \partial_\phi] f \]
\[ = (\sin \phi \partial_\theta f - \cot \theta \cos \phi \partial_\phi^2 f) \]
\[ - (\cos \phi \partial_\theta f + \cot \theta \sin \phi \partial_\phi f - \sin \phi \partial_\phi \partial_\theta f - \cot \theta \cos \phi \partial_\phi^2 f) \]
\[ = \cos \phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi \]
\[ = S f \]

Similarly,

\[ [S, T] f = [\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi] f \]
\[ = (\cos \phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi f)(-\sin \phi \partial_\theta f - \cot \theta \cos \phi \partial_\phi f) \]
\[ - (\sin \phi \partial_\theta f - \cot \theta \cos \phi \partial_\phi)(\cos \phi \partial_\theta f - \cot \theta \sin \phi \partial_\phi f) \]
\[
\begin{align*}
\cos^2 \phi \csc^2 \theta \partial_\phi f &+ \cot \theta \sin \phi \cos \phi \partial_\theta f - \cot^2 \theta \sin^2 \phi \partial_\phi f \\
- \cos \phi \sin \phi \partial_\phi^2 f - \cot \theta \cos^2 \phi \partial_\theta \partial_\phi f + \cot \theta \sin^2 \phi \partial_\phi \partial_\theta f + \cot^2 \theta \cos \phi \sin \phi \partial_\phi^2 f \\
-(\sin^2 \phi \csc^2 \theta \partial_\phi f + \cot \theta \cos \phi \sin \phi \partial_\theta f + \cot^2 \theta \cos^2 \phi \partial_\phi f) \\
- \cos \phi \sin \phi \partial_\phi^2 f - \cot \theta \cos^2 \phi \partial_\theta \partial_\phi f + \cot \theta \sin^2 \phi \partial_\phi \partial_\theta f + \cot^2 \theta \cos \phi \sin \phi \partial_\phi^2 f) \\
= \cos^2 \phi \csc^2 \theta \partial_\phi f + \cot \theta \sin \phi \cos \phi \partial_\theta f - \cot^2 \theta \sin^2 \phi \partial_\phi f \\
-(\sin^2 \phi \csc^2 \theta \partial_\phi f + \cot \theta \cos \phi \sin \phi \partial_\theta f + \cot^2 \theta \cos^2 \phi \partial_\phi f) \\
= \csc^2 \theta \partial_\phi f - \cot^2 \theta \partial_\phi f \\
= \partial_\phi f \\
= Rf
\end{align*}
\]

**Problem 3**

(a) Varying \( S \) with respect to the inverse metric gives

\[
\delta S = \int d^4x \delta g^{\alpha \beta} \left( \frac{\sqrt{-g}}{2} g_{\alpha \beta} \left( -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} \right) - \frac{\sqrt{-g}}{4} \left( F_{\rho \sigma} F_{\mu \nu} g_{\rho \mu} + F_{\alpha \beta} F_{\beta \nu} g_{\alpha \mu} \right) \right)
\]

\[
= \int d^4x \delta g^{\alpha \beta} \left( \frac{\sqrt{-g}}{8} F_{\mu \nu} F_{\mu \nu} g_{\alpha \beta} - \frac{\sqrt{-g}}{2} F_{\sigma \alpha} F_{\sigma \beta} \right)
\]

where the first term comes from varying the volume element and the second from varying the two metrics used to contract the indices of \( F_{\mu \nu} \). Then the stress tensor, using (4.75) of Carroll, is

\[
T_{\alpha \beta} = \frac{-2}{\sqrt{-g} \delta g^{\alpha \beta}} \delta S_M = F_{\sigma \alpha} F_{\sigma \beta} - \frac{1}{4} F_{\mu \nu} F_{\mu \nu} g_{\alpha \beta}
\]

(b) The overall factor of 1/2 in the action was a typo. The action should read

\[
S = -\int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right]
\]

Varying \( S \) with respect to the inverse metric gives

\[
\delta S = -\int d^4x \sqrt{-g} \delta g^{\alpha \beta} \left[ \frac{1}{2} \nabla_\alpha \phi \nabla_\beta \phi - g_{\alpha \beta} \left( \frac{1}{2} g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right) \right]
\]

where the first term comes from varying the explicit metric in the action and the rest comes from varying the volume element. The stress energy tensor for a scalar field is thus

\[
T_{\alpha \beta} = \nabla_\alpha \phi \nabla_\beta \phi - g_{\alpha \beta} \left( \frac{1}{2} \nabla_\mu \phi \nabla_\mu \phi + V(\phi) \right).
\]
Problem 4 (Physics in curved spacetime)

(a) Letting $\nabla_a$ act on the Maxwell equation $J^a = \nabla_b F^{ab}$ gives:

\[
\nabla_a J^a = \nabla_a \nabla_b F^{ab} = \nabla \left[ \nabla_b F^{ab} \right] = \frac{1}{2} [\nabla_a, \nabla_b] F^{ab} = \frac{1}{2} \left( R^b_{c a} F^{ac} + R^a_{c b} F^{cb} \right) = \frac{1}{2} \left( - R^c_{a b} F^{ac} + R^b_{c a} F^{ca} \right) = 0.
\]

To obtain the third line, we used Carroll (3.114). In the last step, we used $R_{a b} F^{a b} = 0$ since $R_{a b}$ is symmetric and $F^{a b}$ is antisymmetric.

(b) The stress energy tensor for a scalar field is

\[
T^{a b} = \nabla^a \phi \nabla^b \phi - g^{a b} \left( \frac{1}{2} \nabla^c \phi \nabla_c \phi + V(\phi) \right).
\]

We now evaluate $\nabla_a T^{a b}$ and use $\nabla a g_{b c} = 0$. The two underlined terms below cancel since for any scalar $\phi$ we have $\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi$:

\[
\nabla_a T^{a b} = (\nabla_a \nabla^a \phi) \nabla^b \phi + \nabla^a \phi (\nabla_a \nabla^b \phi) - g^{a b} (\nabla_a \nabla^c \phi) \nabla_c \phi - g^{a b} \nabla_a V
\]

\[
= (\nabla_a \nabla^a \phi) \nabla^b \phi - \nabla^b V
\]

\[
= \left( \nabla a \nabla^a \phi - \frac{d V}{d \phi} \right) \nabla^b \phi. \quad \text{(outside parentheses)}
\]

The quantity in parentheses vanishes by the field equation for $\phi$, hence $\nabla_a T^{a b} = 0$ and the stress tensor is conserved. The Maxwell stress energy tensor is

\[
T^{a b} = F^{a c} F^b_c - \frac{1}{4} g^{a b} F^{c d} F_{c d}.
\]

We now evaluate $\nabla_a T^{a b}$ and use $\nabla a g_{b c} = 0$:

\[
\nabla_a T^{a b} = (\nabla_a F^{a c}) F^b_c + F_{a c} \nabla^a F^{b c} - \frac{1}{2} F_{c d} \nabla_b F^{c d}
\]

\[
= (\nabla_a F^{a c}) F^b_c + \frac{1}{2} F_{a c} \left[ \nabla^a F^{b c} - \nabla^c F^{b a} \right] - \frac{1}{2} F_{c d} \nabla_b F^{c d}
\]

\[
= (\nabla_a F^{a c}) F^b_c + \frac{1}{2} F_{a c} \left[ \nabla^a F^{b c} + \nabla^c F^{a b} + \nabla^b F^{c a} \right]
\]

\[
= (\nabla_a F^{a c}) F^b_c + \frac{3}{2} F_{a c} \nabla [a F^{b c}].
\]

In the last line, if we now use the Maxwell equation $\nabla_a F^{a c} = -J^c$ on the first term and the Maxwell equation $\nabla [a F^{b c}] = 0$ on the second term, we have $\nabla_a T^{a b} = -J^c F^b_c$. If $J^c = 0$, then $\nabla_a T^{a b} = 0$ and the stress tensor is conserved. If $J^c \neq 0$ then the total stress tensor is the sum of the Maxwell stress tensor and a stress tensor due to the charged matter, and although the Maxwell stress tensor alone is not conserved, the total stress tensor (of the Maxwell field and charged matter) is conserved.