1) (15 points) A symplectic manifold is an ordinary manifold \( M \) with an antisymmetric tensor \( \Omega_{\alpha\beta} = \Omega_{[\alpha\beta]} \) which has an inverse \( \Omega^{\alpha\beta} \Omega_{\beta\gamma} = \delta^\alpha_\gamma \) and satisfies \( \nabla_{[\alpha} \Omega_{\beta\gamma]} = 0 \). (This is the geometrical setting for classical mechanics. \( M \) represents the phase space and the Poisson bracket of two functions is \( \{ f, g \} = \Omega^{\alpha\beta} \nabla_\alpha f \nabla_\beta g \).)

(a) An infinitesimal canonical transformation is a vector field \( \xi \) satisfying \( \mathcal{L}_\xi \Omega = 0 \) where \( \mathcal{L}_\xi \) denotes the Lie derivative. Write this condition in terms of the covariant derivative \( \nabla_\alpha \) associated with a metric \( g_{\mu\nu} \) on \( M \).

(b) Show that every function \( F \) gives rise to an infinitesimal canonical transformation via \( \xi_\alpha = \Omega^{\alpha\beta} \nabla_\beta F \). (Note: Do not confuse this with Goldstein’s discussion of generators of finite canonical transformations which are different.)

2) (20 points) Consider the 2-dimensional metric:

\[
d s^2 = f(u,v)dudv
\]

where \( f \) is a positive function.

(a) What is the signature of this metric?

(b) Show that the curves \( u = \) constant are geodesics (for any choice of the function \( f \)). Are these timelike, spacelike, or null geodesics?

(c) Find the condition on \( f \) so that \( v \) is an affine parameter along the \( u = \) constant geodesics.

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3) (20 points) Consider the 3-dimensional spacetime

\[ ds^2 = - \left( 1 - \frac{r_0^2}{r^2} \right) dt^2 + dr^2 + (r^2 - r_0^2) d\varphi^2 \]

where \( r_0 \) is a constant. We are only interested in the region \( r > r_0 \).

(a) This space has two Killing fields. What are they? What are the conserved quantities along affinely parameterized geodesics?

(b) Compute the gravitational redshift experienced by a radial photon emitted at \( r = r_1 > r_0 \) and later observed at \( r = r_2 > r_1 \).

(c) Write down the equation for timelike geodesics. What is the condition for circular orbits?

4) (20 points) A cosmic string has a stress energy tensor whose only nonzero components are

\[ T_{tt} = -T_{xx} = \mu \delta(y)\delta(z) \]

where \( \mu > 0 \) is the energy per unit length.

(a) Find a static solution to the linearized Einstein equation for this source.

(b) Describe the motion of a test particle that is initially moving parallel to the cosmic string. Work to first order in the perturbation.

(c) Solve Poisson’s equation \( \partial_i \partial^i \phi = 4\pi \rho \) with \( \rho = T_{tt} \) to find the Newtonian gravitational potential for a cosmic string. Compare your answer to (b) with the corresponding prediction of Newtonian gravity.