Problem 1 (Fermi-Walker Transport)

(a) Let \( v^\beta = u^\beta \). Then
\[
2u^{[\beta}A^\gamma]u_\gamma = 2u^{[\beta}(u^{[\alpha}\nabla_{\alpha}u^{\gamma])u_\gamma = \frac{1}{2}u^\beta u^\alpha \nabla_{\alpha}(u_\gamma u^\gamma) - u_\gamma u^\gamma u^\alpha \nabla_{\alpha}u^\beta = u^\alpha \nabla_{\alpha}u^\beta
\]

since \( u_\gamma u^\gamma = -1 \).

(b) Take \( v^\beta, w^\beta \) Fermi-Walker transported along \( C \). Then
\[
u^\alpha \nabla_{\alpha}(v_\beta w^\beta) = w_\beta u^\alpha \nabla_{\alpha}v^\beta + v_\beta u^\alpha \nabla_{\alpha}w^\beta = 2w_\beta u^{[\beta}A^\gamma]v_\gamma + 2v_\beta u^{[\beta}A^\gamma]w_\gamma = 4v_{(\beta}w_{\gamma)}u^{[\beta}A^\gamma] = 0
\]

so the inner product is preserved along the curve.

Problem 2

We are given the three dimensional metric
\[
ds^2 = -(e^x + 1)dt^2 + 2dxdt + dx^2 + 2(t^2 + 1)dydt
\]

(a) The metric is invariant under changing \( y \), so \( \partial/\partial y \) is a Killing field. It is \textit{null} since \( g_{yy} = 0 \).

(b) Setting \( x = 0, y = \lambda, \) and \( t = \lambda \) we get
\[
ds^2 = -2d\lambda^2 + 2(\lambda^2 + 1)d\lambda^2 = 2\lambda^2 d\lambda^2
\]

So the length is
\[
L = \sqrt{2} \int_0^1 \lambda d\lambda = \sqrt{2}/2
\]

(c) The vectors \textit{tangent} to the \( x = \) constant surfaces are \( \partial/\partial t \) and \( \partial/\partial y \). So we start with a general vector
\[
n = A \partial/\partial t + B \partial/\partial x + C \partial/\partial y
\]
This is orthogonal to $\partial/\partial y$ if $0 = n \cdot \partial/\partial y = A (t^2 + 1)$ which implies $A = 0$. This is orthogonal to $\partial/\partial t$ if $0 = n \cdot \partial/\partial t = B + C (t^2 + 1)$ which implies $B = -C (t^2 + 1)$ (where we have used the fact that $A = 0$). So

$$n = C[-(t^2 + 1)\partial/\partial x + \partial/\partial y]$$

Normalizing this vector $n \cdot n = 1$ implies that $C = -1/(t^2 + 1)$ so

$$n = \frac{\partial}{\partial x} - \frac{1}{t^2 + 1} \frac{\partial}{\partial y}$$

Note that $-n$ also works, but this points in a direction that decreases $x$.

**Problem 3**

We are given the spacetime

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]$$

(a) Using $t$ as a parameter along the curve, the tangent vector has components $\xi^\mu = (1, 1/a, 0, 0)$ in the natural coordinate basis associated with $(t, x, y, z)$. The norm of this vector is $g_{\mu\nu}\xi^\mu\xi^\nu = -1 + a^2(1/a^2) = 0$, so the curve is *null*.

(b) This question can be answered without computing Christoffel symbols. The problem has rotational symmetry in the $y, z$ plane around the curve. Consider the acceleration $\xi^\mu \nabla_\mu \xi^\nu$. If this vector had any component in the $y, z$ plane, it would break this rotational symmetry. So the acceleration can only point in the $t, x$ plane. But $\xi_\nu(\xi^\mu \nabla_\mu \xi^\nu) = (1/2)\xi^\mu \nabla_\mu(\xi \cdot \xi) = 0$, so the acceleration is orthogonal to $\xi^\mu$. But $\xi^\mu$ is null, so the acceleration must be proportional to $\xi^\mu$. This is the statement that the curve is a geodesic. To see if $t$ is an affine parameter, we consider $\xi \cdot \partial/\partial x = a(t)$. Since $\partial/\partial x$ is a Killing field, the fact that this inner product is not constant means that the geodesic is not affinely parameterized.

**Problem 4**

Consider

$$ds^2 = \left(1 - \frac{M}{r} \right) \left(-dt^2 + dr^2\right) + r^2\left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$
(a) This spacetime has a timelike Killing vector $K_\mu = (1, 0, 0, 0)$. A static observer has 4-velocity proportional to this Killing vector, so that the 4-velocities of the source and the observer are

$$u_{e,o}^\mu = \frac{K^\mu}{\sqrt{-g_{\sigma\tau}K_\sigma K_\tau}} \bigg|_{r_{e,o}} = \frac{K^\mu}{\sqrt{-g_{tt}}} \bigg|_{r_{e,o}}$$

For a photon with 4-momentum $P^\mu$, the energy measured by the observers at $r_{e,o}$ is

$$E_{e,o} = -u_{e,o}^\mu P_\mu = \frac{-K^\mu P_\mu}{\sqrt{-g_{tt}}} \bigg|_{r_{e,o}}$$

Since $K^\mu$ is a Killing field, $K^\mu P_\mu$ is constant along the geodesic. The ratio of energies measured at $r_e$ and $r_o$ is then

$$\frac{E_o}{E_e} = \frac{\sqrt{-g_{tt}(r_e)}}{\sqrt{-g_{tt}(r_o)}} = \left[ \frac{1 - M/r_e}{1 + M/r_e} \left( \frac{1 + M/r_o}{1 - M/r_o} \right) \right]^{1/2}$$

To first order in $M/r_e$ and $M/r_o$, this is

$$\frac{E_o}{E_e} \approx 1 - \frac{M}{r_e} + \frac{M}{r_o}$$

which matches the result from Schwarzschild, Carroll (5.104).

(b) Let $P^\mu = mu^\mu$ be the four-momentum of the particle. Then the conserved energy is

$$E = -P^\mu K_\mu = mt \left( \frac{1 - M/r}{1 + M/r} \right)$$

and the conserved angular momentum is

$$L = P^\mu (\partial/\partial \varphi)_\mu = mr^2 \dot{\varphi}$$

where the dot denotes derivative with respect to proper time. Note that since the problem asked for the conserved quantities (and not, e.g., the energy per unit mass) it is important to include the rest mass. It suffices to consider motion in the equatorial plane $\theta = \pi/2$. Then $P^\mu = (\dot{t}, \dot{r}, \dot{\varphi})$. Thus

$$-1 = P^\mu P_\mu/m^2 = \left( \frac{1 - M/r}{1 + M/r} \right) (-\dot{t}^2 + \dot{r}^2) + r^2 \dot{\varphi}^2$$
Replacing $\dot{t}$ and $\dot{\phi}$ with $E$ and $L$, we get

\[-1 = - \left( \frac{1 + M/r}{1 - M/r} \right) \frac{E^2}{m^2} + \left( \frac{1 - M/r}{1 + M/r} \right) \dot{r}^2 + \frac{L^2}{m^2 r^2} \]

So

\[\dot{r}^2 + \left( \frac{1 + M/r}{1 - M/r} \right) \left( 1 + \frac{L^2}{m^2 r^2} \right) - \left( \frac{1 + M/r}{1 - M/r} \right) E^2 \frac{m^2}{m^2} = 0\]