Lecture #2: An Aside on Variational Principles

Note Title 1/6/2010

You know that the stress tensor for matter fields is defined by the equation

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{ab}}$$

But what does the RHS mean? We'll go over this briefly today, but you will learn best by simply diving into the HW & practicing on your own.

I- Functional Derivatives

From classical mechanics, you are used to the idea that dynamics follows from an action principle of the form

$$S = \int L\,dt$$

This form ensures that EOM's are local in time.

& that requiring S to be stationary $\Rightarrow$ Euler-Lagrange equations.

The idea is that S is a function on the space of possible histories ($\mathcal{H}$) of the system [i.e., on the space of functions $x^a(t)$] & that we want the gradient of this function to vanish.

Note: As you now know, the gradient of a function is a one-form. Since S is a function on $\mathcal{H}$, its gradient is a one-form on $\mathcal{H}$. Note: $\mathcal{H}$ is an infinite-dimensional manifold.

You also know that a one-form is a linear functional on the space of tangent vectors.

For a function f on a finite-dimensional manifold, we would write

Basis for the space of one-forms

$$\underbrace{df}_{\text{a one-form}} = \sum_a (\partial_a f)\, dx^a$$

or maybe $\underbrace{\delta f}_{\substack{\text{the change}\\ \text{in } f \text{ under}\\ \text{displacement}}} = \sum_a (\partial_a f)\, \delta x^a$ (*)

Arbitrary tangent vector δx^a to the manifold

Note: Really the same equation, but interpreted differently.

Also note: $\partial_a f = \delta f \Big|_{\delta x^b = \delta_a^b}$

On infinite-dim manifolds, physicists tend to use the latter form (*), but no real difference.

The point: if we calculate δS to first order under an arbitrary displacement (say, $\delta x^a(t)$) in $\mathcal{H}$, we have calculated the gradient.

In particle mechanics, we find

$$\delta S = \int dt\, [EL_a(t)]\, \delta x^a(t)$$

Basically the same form as (*) under $\sum_a \to \int dt \sum_a$

(i.e., replace finite discrete basis by continuous basis.)

Thus, $EL_a(t)$ can be thought of as the components of the gradient of S in the particular basis $\delta x^a(t) = \delta^a_b\, \delta(t - t_0)$

Like basis of position-eigenstates in Q.M.

Since $$EL_b(t_0) = \left.\delta S\right|_{\delta x^a = \delta^a_b\,\delta(t-t_0)}$$

For this reason, we say that "the functional derivative of S wrt $x^a(t)$" is EL_a:

$$\frac{\delta S}{\delta x^a(t)} = EL_a(t).$$

We see that the quantity $\frac{\delta S}{\delta x^a}(t)$ is in fact defined to be the quantity that satisfies

$$\delta S = \int dt\, \frac{\delta S}{\delta x^a(t)}\, \delta x^a(t)$$

for (sufficiently) arbitrary smooth $\delta x^a(t)$.

→ we often assume some boundary condition on the $\delta x^a(t)$, or perhaps even compact support, to allow us to perform various integrations by parts.

II. Actions for scalar fields in S.R.

The story is much the same for field theories.

Let's consider the example of a scalar field ϕ in special relativity. (Lorentz-covariant)

Need an action $S = \int dt\, L$

But desire both i) Lorentz-covariance

& ii) Locality in spacetime

$\Rightarrow$ take $L = \int d^3x\, \mathcal{L}$, ← Lagrange density

or $S = \int d^4x\, \mathcal{L}$ Note: A Lorentz-scalar!

w/ $\mathcal{L} = \mathcal{L}(\phi, \partial\phi, \text{etc.})$

Actually, we usually take $\mathcal{L}$ to depend only on ϕ & $\partial\phi$ to guarrantee that EOM's have only 2nd derivatives.
(may also assume that no more than two derivatives in each term.)
Let's assume this here.

What can $\mathcal{L}$ possibly be? What Lorentz scalars can we build from ϕ, $\partial\phi$?

1) $(\partial\phi)^2 := \partial_a\phi\,\partial^a\phi$

2) functions of ϕ (e.g. $V(\phi)$, a potential)

$\Rightarrow$ most general choice is

$$\mathcal{L} = \underbrace{-\tfrac{1}{2}}_{\text{conventional}}\, g(\phi)\,(\partial\phi)^2 + V(\phi)$$

But now define $\tilde{\phi} = \int \sqrt{g(\phi)}\, d\phi$

Can think of $g(\phi)$ as a 1-d metric in ϕ-space. $\tilde{\phi}$ is then "proper distance"!

So that $\partial_a\tilde{\phi} = \partial_a\phi\,\dfrac{\partial\tilde{\phi}}{\partial\phi} = \sqrt{g(\phi)}\,\partial_a\phi$

(Note: need $g(\phi) \geq 0$ for energy to be bounded below)

So we can change variables (coords on field-space!) to find

$$\mathcal{L} = \underbrace{-\tfrac{1}{2}(\partial\tilde{\phi})^2}_{\text{"canonically normalized kinetic term."}} + \tilde{V}(\tilde{\phi})$$

note: If more than one field, field space is multi-dimensional

$\Rightarrow$ Curvature is possible & generic metric can no longer be removed via change of coordinates...

Let's compute $\frac{\delta S}{\delta \phi(x)}$. (rename $\tilde{\phi} \to \phi$)

$$\delta S = \delta \int d^4x \left[-\frac{1}{2} (\partial_a \phi)(\partial_b \phi) \eta^{ab} + V(\phi) \right]$$

$\delta \partial_a \phi = \partial_a (\delta \phi)$

$$= \int d^4x \left[-\frac{1}{2} (\partial_a \delta\phi)(\partial_b \phi)\eta^{ab} - \frac{1}{2} (\partial_a \phi)(\partial_b \delta\phi)\eta^{ab} + V'(\phi)\, \delta\phi \right]$$

used symmetry of η_{ab}

$$= \int d^4x \left[-(\partial^a \phi)\, \partial_a \delta\phi + V'(\phi)\, \delta\phi \right]$$

now, int by parts to ~~get~~ desired form (& assume $\delta\phi$ has compact support to kill boundary term)

$$\to \quad \delta S = \int d^4x \left[\partial^2 \phi + V'(\phi) \right] \delta\phi$$

$$\Rightarrow \quad \frac{\delta S}{\delta \phi}(x) = \partial^2 \phi(x) + V'(\phi(x))$$