1. Euler's equation is

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla \rho + \nabla \Phi$$

Using the equation of continuity

$$\frac{\partial \rho}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = 0$$

we get

$$\rho \frac{\partial \vec{v}}{\partial t} + \nabla \rho \cdot \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla \rho + \nabla \Phi$$

$$\frac{\partial}{\partial t} (\rho \vec{v}) + \nabla (\rho \vec{v} \cdot \vec{v}) = -\nabla \rho + \nabla \Phi$$

$$\frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla \rho + \nabla \Phi$$

Let \( \hat{1} \) be the identity tensor.

In 2D the equation reduces to

$$\frac{\partial}{\partial t} (\rho \vec{v}) + \frac{\partial}{\partial x} (\rho v^2 + p + \Phi) = 0$$

Note that \( \Phi \) does not vary across the shock.

After integrating we get,

$$P v_n^2 + P_1 = P_2 + v_n^2 + P_2$$
Ib. the continuity and energy equa
gues

\[ P_1 v_1 = P_2 v_2 \quad \text{(1)} \]

\[ \frac{1}{2} v_1^2 + P_1 = P_2 v_2^2 + P_2 \quad \text{(2)} \]

\[ \frac{1}{2} v_1^2 + \frac{P_1}{p_1} + E_1 = \frac{1}{2} v_2^2 + \frac{P_2}{p_2} + E_2 \quad \text{(3)} \]

but since \[ E_1 = \frac{P_1}{p_1} \quad \text{(3)} \]

(3) can be rewritten as

\[ \frac{1}{2} v_1^2 + \frac{\gamma P_1}{(\gamma - 1) p_1} = \frac{1}{2} v_2^2 + \frac{\gamma P_2}{(\gamma - 1) p_2} \quad \text{(4)} \]

(1) can be rewritten as

\[ P_2 = p_1 \frac{v_1}{v_2} \quad \text{(5)} \]

substitute that into (3)

\[ P_1 v_1^2 + P_1 = P_1 v_2 v_2 + P_2 \Rightarrow P_2 = P_1 v_1 (2v_1 - v_2) + P_1 \quad \text{(6)} \]

substitute (4) into (3)

\[ \frac{1}{2} v_1^2 + \frac{\gamma P_1}{(\gamma - 1) p_1} = \frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \left[ \frac{P_1}{p_2} v_1 (v_1 - v_2) + \frac{P_1}{p_2} \right] = \frac{1}{2} v_2^2 \]

\[ \frac{1}{2} + \frac{\gamma P_1}{(\gamma - 1) p_1 v_1^2} = \frac{1}{2} \left( \frac{v_2}{v_1} \right)^2 + \frac{\gamma}{\gamma - 1} \left[ \frac{v_2}{v_1} - \left( \frac{v_2}{v_1} \right)^2 + \frac{P_1}{p_1 v_1^2} \left( \frac{v_2}{v_1} \right) \right] \]

\[ c_s^2 = \frac{\gamma P_1}{p_1} \quad \text{is the sound speed and} \quad \frac{\gamma}{\gamma - 1} \quad \text{is the upstream Mach number we have} \quad \frac{M^2}{\gamma} = \frac{p_1 v_1^2}{\gamma P_1} \]
$$\Rightarrow \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{A}_1^2} = \frac{1}{2} \left( \frac{\mathcal{V}_1}{\mathcal{V}_1} \right)^2 + \frac{\mathcal{V}_1}{\mathcal{V}_1} \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right) - \frac{\mathcal{V}_1}{\mathcal{V}_1} \left( \frac{\mathcal{V}_3}{\mathcal{V}_1} \right)^2$$

$$+ \frac{1}{(\gamma-1)\mathcal{A}_2^2} \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right)$$

$$= \frac{\mathcal{V}_1}{2(\gamma-1)} \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right)^2 + \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} \right] \left[ \frac{\mathcal{V}_2}{\mathcal{V}_1} \right] \left( \frac{\mathcal{V}_3}{\mathcal{V}_1} \right)$$

$$\Rightarrow \frac{\gamma+1}{2} \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right)^2 - \left[ \mathcal{V}_1 + \frac{\mathcal{V}_3}{\mathcal{V}_1} \right] \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right) + \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} + \frac{\mathcal{V}_3}{\mathcal{V}_1} \right] = 0$$

Solving gives us

$$\frac{\mathcal{V}_2}{\mathcal{V}_1} = \frac{\mathcal{V}_3}{\mathcal{V}_1} = \frac{\mathcal{V}_1}{\mathcal{V}_1} \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} \right] \left( \frac{\mathcal{V}_3}{\mathcal{V}_1} \right) + \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} + \frac{\mathcal{V}_3}{\mathcal{V}_1} \right]$$

$$\Rightarrow \frac{\mathcal{P}_2}{\mathcal{P}_1} - \frac{\mathcal{P}_1}{\mathcal{P}_1} \frac{\mathcal{V}_1}{\mathcal{V}_2} + 1 = \gamma \mathcal{A}_1^2 - \gamma \mathcal{A}_2^2 \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} \right] \left[ \frac{\mathcal{V}_1}{\mathcal{V}_1} \right]$$

$$+ 1 = \frac{2 \gamma \mathcal{A}_2^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$$

For ideal gas, $\mathcal{P} \propto \rho T$

$$\frac{T_2}{T_1} = \frac{\mathcal{P}_2}{\mathcal{P}_1} \cdot \mathcal{P}_1 \mathcal{P}_2 = \left[ \frac{2 \gamma \mathcal{A}_2^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \right] \left[ \frac{2 \gamma \mathcal{A}_2^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \right]$$

$$= \frac{1}{(\gamma+1)^2} \left[ 2 \gamma (\gamma-1) \mathcal{A}_2^2 - (\gamma-1)^2 \mathcal{A}_2^2 \right]$$

$$= \frac{1}{(\gamma+1)^2} \left[ 2 \gamma (\gamma-1) \mathcal{A}_2^2 + 4 \gamma - (\gamma-1)^2 - 2(\gamma-1) \right]$$

$$= \frac{\gamma-1}{\gamma+1} \left[ \frac{2 \gamma (\gamma-1) \mathcal{A}_2^2 - (\gamma-1)^2}{\gamma+1} + \frac{4 \gamma - (\gamma-1)^2}{\gamma+1} \right]$$
Problem 2

For a spherically symmetric system the cooling time as a function of radius is defined by

\[ t_{\text{cool}} = \frac{3n(r)k_bT(r)}{2n_H(r)^2\Lambda(T)} \]

For an isothermal sphere \( T(r) = T_0 = \frac{\mu m_pP_0}{k_b\rho_0} \) and thus \( \Lambda(T) = \Lambda(T_0) = \Lambda_0 \), so

\[ t_{\text{cool}} = \frac{3nk_bT_0}{2n_H^2\Lambda_0} \]

Note that \( n\frac{\rho}{\mu m_p} \), so inserting this identity gives

\[ t_{\text{cool}} = \frac{3n^2k_bT_0\mu m_p}{2n_H^2\rho\Lambda_0} \]

But \( \rho = \rho_0\left(\frac{r}{r_0}\right)^{-2} \) for an isothermal sphere, so

\[ t_{\text{cool}} = \frac{3\mu m_p}{2\rho_0}\left(\frac{n}{n_H}\right)^2\frac{k_bT_0}{\Lambda_0} \]

\[ = t_0\left(\frac{r}{r_0}\right)^2 \]

where

\[ t_0 = \frac{3\mu m_p}{2\rho_0}\left(\frac{n}{n_H}\right)^2\frac{k_bT_0}{\Lambda_0} \]

Therefore the radius at time \( t_{\text{cool}} \) is

\[ r_{\text{cool}} = r_0\left(\frac{t}{t_0}\right)^2 \]

The cooling rate is given by

\[ M_{\text{cool}} = 4\pi r_{\text{cool}}^2 \frac{dr_{\text{cool}}}{dt} \rho(r_{\text{cool}}) \]

\[ = 4\pi r_0^2 \frac{t}{2t_0} r_0\left(\frac{t}{t_0}\right)^{-\frac{3}{2}} \frac{1}{t_0} \rho_0\left(\frac{r_{\text{cool}}}{r_0}\right)^{-2} \]
Integrating the cooling rate from time \( t = 0 \) to now gives the total mass cooled

\[ M_{\text{cool}}(t) = \int_0^t dt' 2\pi r_0^3 \rho_0 t_0^{-\frac{1}{2}} t'^{-\frac{1}{2}} \]

\[ = 2\pi r_0^3 \rho_0 t_0^{-\frac{1}{2}} (2t^\frac{1}{2}) \]

\[ = 4\pi r_0^3 \rho_0 \left( \frac{t}{t_0} \right)^{\frac{1}{2}} \]

**Problem 3**

a

Let the radius of the expanding shell be

\[ r_{sh} \propto L_w^\alpha \rho_0^\beta t^\gamma. \]

The dimensions of \( r_{sh}, L_w, \rho_0, \) and \( t \) are

\[ [r_{sh}] = L , \quad [L_w] = M L^2 T^{-3} , \quad [\rho_0] = M L^{-3} , \quad [t] = T. \]

Plugging the dimensions into the first proportionality,

\[ L = M^\alpha L^{2\alpha} T^{-3\alpha} M^\beta L^{-3\beta} T^\gamma. \]

Because dimensions must be conserved, the above equation can be broken up by dimension. This will generate three equations for \( \alpha, \beta \) and \( \gamma \) which can be written as follows

\[ \alpha + \beta = 0 , \quad 2\alpha - 3\beta = 1 , \quad \gamma - 3\alpha = 0. \]

After solving the above system of equations, it is clear

\[ \alpha = \frac{1}{5} , \quad \beta = -\frac{1}{5} , \quad \text{and} \quad \gamma = \frac{3}{5} . \]

Plugging these exponents into the first proportionality,

\[ r_{sh} \propto L_w^{1/5} \rho_0^{-1/5} t^{3/5} \]

Because \( v_{sh} = \frac{dr_{sh}}{dt} \),

\[ v_{sh} \propto L_w^{1/5} \rho_0^{-1/5} t^{-2/5} \]
b

From a, \( r_{sh} = C_1 L_w^{1/5} \rho_0^{-1/5} t^{3/5} \), where \( C_1 \) is a numeric coefficient and assumed to be 1 for the remainder of this problem as we are only interested in an order of magnitude estimate. Rearranging and solving for \( t \),

\[
   t_{interaction} = \frac{\rho_0^{1/3}}{L_w} L_w^{2/5} r_{sh}^{5/3}
\]

If 3 \( M_\odot \) of stars are formed every year, and one supernova occurs after 100 \( M_\odot \) of stars have formed, that means that one could expect 0.03 SN/yr. \( L_w \) is the total mechanical power from all supernovae and assuming 1 foe (10^{51} ergs) is released per supernovae,

\[
   L_w \approx 0.03 \frac{SN}{yr} \cdot 10^{51} \frac{ergs}{SN} \approx 9.5 \times 10^{41} ergs/s.
\]

Assuming a number density of \( n_0 \approx 100 \text{ atoms/cm}^3 \), and assuming that most of the gas is atomic hydrogen,

\[
   \rho_0 \approx 10^{-22} \text{ g/cm}^3.
\]

Assuming \( r_{sh} \approx 10H \) where \( H \) is the scale height of the disk of the galaxy (150 pc from the text),

\[
   r_{sh} \approx 5 \times 10^{21} \text{ cm}.
\]

Plugging these numbers into the first equation for \( t_{interaction} \),

\[
   t_{interaction} \approx 2 \times 10^7 \text{ yr}.
\]

From the text, the orbital period of the Sun around the galactic center is

\[
   t_{orbit} \approx 2.5 \times 10^8 \text{ yr}.
\]

Thus the \textbf{disk-halo interaction time is shorter} than the orbital period of the Sun.

c

From a, \( v_{sh} \propto L_w^{1/5} \rho_0^{-1/5} t^{-2/5} \). The problem states to assume \( \rho = \rho_0 (r/r_0)^{-n} \). Plugging this into the equation for \( v_{sh} \),

\[
   v_{sh} \propto L_w^{1/5} (\rho_0 (r/r_0)^{-n})^{-1/5} t^{-2/5}.
\]

If the goal is to find the value of \( n \) that allows for constant shock speeds, that means that \( v_{sh} \) is constant, and assuming that \( L_w, \rho_0, \) and \( r_0 \) are constants as well, the proportionality becomes

\[
   r^{n/5} \propto t^{2/5}
\]

and from this it is clear to see that \( n = 2 \).