1. **The Stellar Slab** Consider an infinite slab with density $\rho(z)$ and potential $\Phi(z)$. All properties of the slab are independent of the horizontal coordinates $x$ and $y$. According to the Jeans theorem, the distribution function of the slab can depend only on the energy,

$$ F(z, v_z) = \int f(x, v) dv_x dv_y = F(E_z), \quad (1) $$

where $E_z = v_z^2/2 + \Phi(z)$. Define the constant in the potential so that $\Phi(0) = 0$. Let us assume that this distribution function is symmetric about $z = 0$ and isothermal, so that

$$ F(E_z) = n_0(2\pi \sigma_z^2)^{-1/2} e^{-E_z/\sigma_z^2}, \quad (2) $$

where $\sigma_z$ is the vertical velocity dispersion and $n_0$ is a constant. Finally, assume that the disk is self-gravitating.

(a) Show that Poisson’s equation may be written

$$ 2 \frac{d^2 \phi}{d\zeta^2} = e^{-\phi}, \quad (3) $$

where $\phi = \Phi/\sigma_z^2$, $\zeta = z/z_0$, $z_0^2 = \sigma_z^2/(8\pi G \rho_0)$, and $\rho_0$ is the density at $z = 0$.

(b) Show that the density distribution

$$ \rho(z) = \rho_0 \text{sech}^2(z/2z_0), \quad (4) $$

where the hyperbolic secant is $2/(e^z + e^{-z})$, satisfies Poisson’s equation.

(c) Show that the surface density of the disk is

$$ \Sigma = \frac{\sigma_z^2}{2\pi G z_0}. \quad (5) $$

(d) Typical disk galaxies have thicknesses that are nearly independent of radius, and $\Sigma(R) \propto e^{-R/R_d}$. If their distribution function is isothermal in $E_z$, how must the vertical dispersion vary with radius?
2. **Self-Gravity and Heat Capacity:** (a) Let $F(\lambda x_1, \ldots, \lambda x_N) = \lambda^n F(x_1, \ldots, x_N)$; we say that $F$ is *homogeneous* of order $n$. Show that

$$\sum x_i \frac{\partial F}{\partial x_i} = nF. \quad (6)$$

*Hint:* Consider $dF(\lambda x_i)/d\lambda$, and then set $\lambda = 1$.

(b) Suppose that a different universe has a gravitational potential between pairs of particles such that $\Phi_{\alpha \beta} = -C|\mathbf{x}_\alpha - \mathbf{x}_\beta|^{-p}$, where $p$ and $C$ are positive constants. (For convenience we have set all the masses to unity.) Show that the scalar virial theorem is

$$0 = 2T + pW, \quad (7)$$

where $T$ and $W$ are the kinetic and gravitational potential energies.

(c) For what values of $p$ does the system have negative heat capacity?

3. **Neutrino Dark Matter:** The purpose of this problem is to explore whether neutrinos can constitute the bulk of the dark matter in the Universe. (a) Neutrinos obey Fermi-Dirac statistics, so they have an equilibrium “fine-grained” (or actual) distribution function (leftover from their interactions in the early Universe) of the form

$$f_{fg}(p) = \frac{2}{h^3} \frac{1}{\exp(pc/k_B T_0) + 1}, \quad (8)$$

where $p$ is the momentum (you may assume them to be non-relativistic, if they are to make up the dark matter) and $T_0$ is their temperature. What is the *maximal* phase-space density?

(b) If neutrinos make up the dark matter inside galaxies, phase mixing must have relaxed the “coarse-grained” distribution to an approximately isothermal, Maxwell-Boltzmann form (to explain the flat rotation curves):

$$f_{cg}(r, p) = \frac{n(r)}{(2\pi m_\nu^2 \sigma^2)^{3/2}} \exp \left( \frac{-p^2}{2m_\nu^2 \sigma^2} \right), \quad (9)$$

where $m_\nu$ is the neutrino mass. What is the maximum $f_{cg}$ at a given radius?

(c) Now suppose that galaxies are singular isothermal spheres, and find a *lower* bound on $m_\nu$ as a function of $r$ by demanding that $f_{cg} < f_{fg}$ (remember that this is the inevitable result of phase mixing!). This is called the *Tremaine-Gunn bound*. Evaluate the bound for $\sigma = 150$ km s$^{-1}$ and $r = 5$ kpc (express your answer in terms of eV).

We cannot (yet) measure the mass of a neutrino, but neutrino oscillations let us measure the mass differences between the different flavors. These differences are orders of magnitude smaller than your limit should be, which makes it unlikely for neutrinos to comprise the bulk of the dark matter.