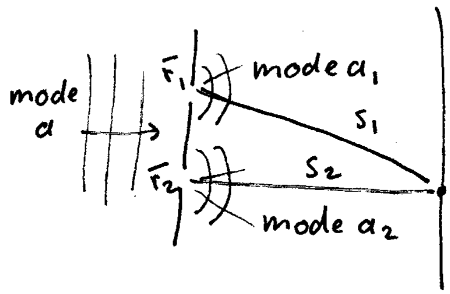
Quantum correlation functions

Young's interference experiment



$I(\bar r, t)$ position on screen

geometric factor $\frac{1}{r}$ $r \sim s_1 \sim s_2$

$$\hat E^+(r,t) = f(r)\left[\hat a_1 e^{iks_1} + \hat a_2 e^{iks_2}\right] e^{-i\omega t}$$

we consider only annih.op' because we discuss detection of photons.

$\downarrow$

$\hat a = \frac{1}{\sqrt 2}(\hat a_1 + \hat a_2)$, the two openings in screen act as beamsplitter

(note there must be a second input mode to make the "beamsplitter" action unitary : $\hat b = \frac{1}{\sqrt 2}(\hat a_1 - \hat a_2)$
we can ignore this mode in the following)

$$I(\bar r, t) = Tr\left\{\rho_{Field}\, \hat E^-(\bar r,t)\, \hat E^+(\bar r,t)\right\}$$

$$= |f(r)|^2 \left\{ Tr(\rho\, \hat a_1^+ \hat a_1) + Tr(\rho\, \hat a_2^+ \hat a_2) + 2\left|Tr(\rho\, \hat a_1^+ \hat a_2)\right| \cos\phi \right.$$

we used $Tr(\rho \hat a_1^+ \hat a_2) = |Tr(\rho\, \hat a_1^+ \hat a_2)|\, e^{i\psi}$
and $\phi = k(s_1 - s_2) + \psi$

Assume n-photon in mode a: $|n\rangle_a\ (|0\rangle_b) = \frac{1}{\sqrt{n!}}\, \hat a^{+n} |0\rangle = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt 2}\right)^n (\hat a_1^+ + \hat a_2^+)^n |0\rangle_1 |0\rangle_2$

eg $n=1$ $\quad |1\rangle_a \rightarrow \frac{1}{\sqrt 2}\left(|1\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2\right) \equiv \frac{1}{\sqrt 2}\, |1,0\rangle + |0,1\rangle$

$$I(\bar r, t) = |f(r)|^2 \left\{ \tfrac12 \langle 1,0|\, \hat a_1^+ \hat a_1 |1,0\rangle + \tfrac12 \langle 0,1|\, \hat a_2^+ \hat a_2 |0,1\rangle + \langle 1,0|\, \hat a_1^+ \hat a_2 |0,1\rangle \cos\phi \right.$$
$$= |f(r)|^2 \left[1 + \cos\phi\right]$$

$$\text{for } \underset{\substack{\downarrow\\ n}}{|2\rangle_a} \longrightarrow I(\hat{r},t) = \underset{\substack{\downarrow\\ n}}{2}\,|f(\bar{r})|^2(\underline{1+\cos\phi})$$

$$\text{for } |\alpha\rangle_a \longrightarrow \;\;" \;\; = |\alpha|^2\,|f(\bar{r})|^2(\overline{1+\cos\phi})$$

interference pattern determined by mode overlap, not by specific occupation of mode.

2^{nd} order Quantum correlation function:

$$g^{(2)}(x_1\, x_2 : x_2\, x_1) = \frac{Tr\{\rho E^-(x_1)E^-(x_2)E^+(x_2)E^+(x_1)\}}{Tr\{\rho E^-(x_1)E^+(x_1)\}Tr\{\rho E^-(x_2)E^+(x_2)\}}$$

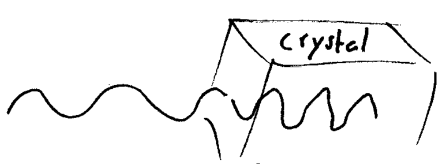
for fixed position

$g^{(2)}$ depends only on $\tau = t_2 - t_1$

$$g^{(2)}(\tau)$$

Many special quantum states of light, e.g "squeezed" light and polarization entangled photons can be generated using nonlinear optical crystals.

An oscillating electric field inside the crystal will induce microscopic dipoles inside the crystal (the E field pulls the electrons and positive charges apart)

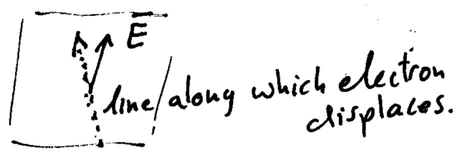


The oscillating dipoles in tern will generate E-M waves

dielectric Polarization $P_i(\bar{E}) = \varepsilon_0\{\chi^{(1)}_{ij}E_j + \chi^{(2)}_{ijk}E_{jk} + \chi^{(3)}_{ijkl}E_jE_kE_l$ etc $\}$

first order susceptibility (Linear) — second order (non linear) susceptibility

Note that the susceptibility is expressed in tensorial form, indicating, for example, that an E field component in the x direction can induce a Polarization component along say the y direction (electrons can in general move displace more freely along certain crystal axes than along other directions).



Since the electrons in the crystal are bound by potentials that in general all only approximate to first order by harmonic potentials the induced dipoles will have a nonlinear response to a large amplitude driving E field. This anharmonic response is to lowest order expressed by the $\chi^{(2)}$ term $(\chi^{(2)}_{ijk} E_j E_k)$

let's consider the simple case $P_i = \chi^{(2)}_{iii} E_i E_i$

$$\downarrow$$

$$P \sim \chi^{(2)} \left(e^{i\omega t} + e^{-i\omega t} \right)^2$$

look at a given position in the crystal. At that position $E \sim (e^{i\omega t} + e^{-i\omega t})$

(in a classical description)

$$P \sim \chi^{(2)} \left(e^{i2\omega t} + 2 + e^{-i2\omega t} \right)$$

P has a contribution that oscillates at 2ω, twice the frequency of the incoming PUMP field.

This "classical" process of frequency doubling is often used in laser systems to generate light at wavelengths were no efficient gain medium exists.

Crystals are in general dispersive; different frequencies have different refractive indices, and crystals have polarization dependent refractive indices (birefringence). This makes it rather complicated to achieve efficient non-linear interactions.

for an efficient process both: $2\omega_{pump} = \omega_{out}$

or $\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3$

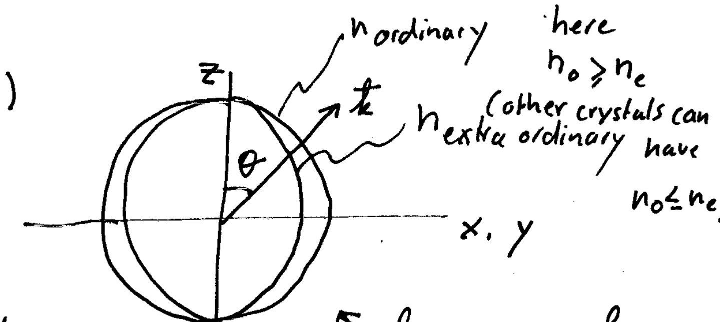
$\hbar\omega_1$ and $\hbar\omega_2$ are both pump photons $\hbar\omega_{pump}$.

and $k_1 + k_2 = k_3$ phase matching.

k-vectors of the two pump photons.

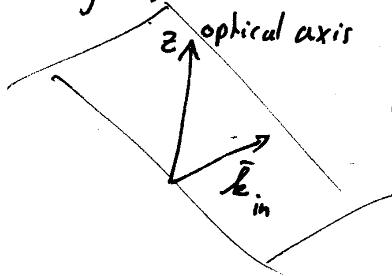
without phase matching the frequency doubled light at different places in the crystal will not constructively interfere to produce a strong frequency doubled signal.

Consider a uniaxial crystal (z-axis)

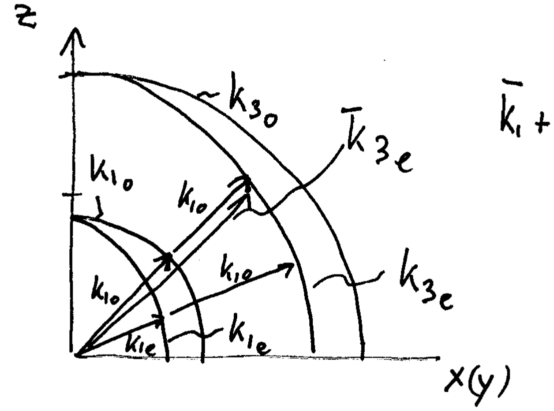


principle plane in a crystal is defined by optical axis of crystal (here taken to be z) and the incoming light k vector

for a given frequency this plot shows the birefringent properties



a beam with $\bar{E}$ normal to principle plane is called ordinary (o-beam), and $\bar{E}$ in plane is called extra-ordinary (e-beam)



$\bar{k}_1 + \bar{k}_2 = \bar{k}_3$ for phase matching

$|k| = \frac{n\omega}{c}$

for normal dispersion

$n_{\omega_3} > n_{\omega_1}$

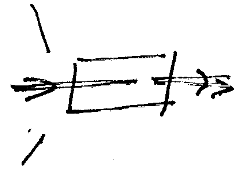
with $\omega_3 > \omega_1$

$\rightarrow k_{3\,\text{ordinary}} > 2k_{1\,\text{ordinary}}$

$\rightarrow k_{3\,\text{extra ordinary}} > 2k_{1\,\text{ext.or.}}$

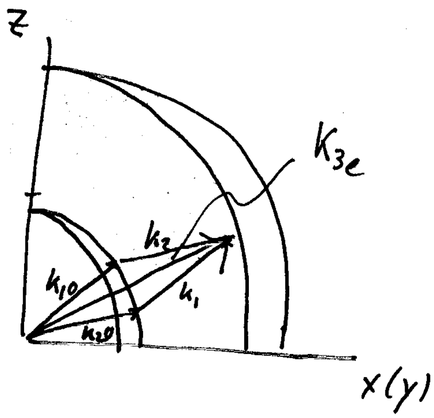
type I phase matching colinear

oo → e

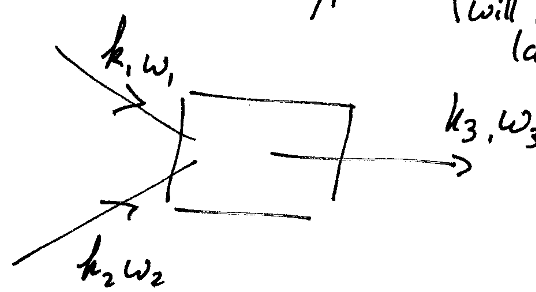


type II phase matching colinear

oe → e



oo → e non colinear type I (type II non-colinear will be discussed later)



if $\omega_1 \neq \omega_2$
but still $\omega_1 + \omega_2 = \omega_3$

$$P \sim \chi^2 \underbrace{\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)\left(e^{i\omega_2 t} + e^{-i\omega_2 t}\right)}$$

$$\underbrace{e^{i(\omega_1+\omega_2)t} + e^{-i(\omega_1+\omega_2)t}}_{\text{sum frequency generation}} + \underbrace{e^{i(\omega_1-\omega_2)t} + e^{-i(\omega_1-\omega_2)t}}_{\text{difference freq. gen.}}$$

· Let's now consider the reversed process of freq. doubling

<u>classically</u> a field at ω_1 can only generate higher harmonics $2\omega_1, 3\omega_1, \dots$

not $\frac{1}{2}\omega_1$!

quantum mechanically $\underset{\substack{\uparrow \\ \text{at given position in crystal}}}{E(t)} \sim \left(e^{i\omega t} + e^{-i\omega t}\right)$ is replaced by

$$\hat{E} \sim \left(\hat{a}\, e^{-i\omega t} - \hat{a}^{+} e^{+i\omega t}\right)$$

describes the "mode" irrespectively of it's occupation.

If the incoming mode with ω interacts with an outcoming mode with 2ω we have an interaction hamiltonian of the form

$$\hat{H}_{int} \sim \chi^{(2)} \underbrace{\left(\hat{a}(2\omega)\, e^{-i2\omega t} - \hat{a}^{+}(2\omega)\, e^{+i2\omega t}\right)\left(\hat{a}(\omega)\, e^{-i\omega t} - \hat{a}^{+}(\omega)\, e^{+i\omega t}\right)^2}_{\text{collecting the energy conserving terms}}$$

$$\underbrace{\hat{a}(2\omega)\,\hat{a}^{+}(\omega)\,\hat{a}^{+}(\omega)}_{\substack{\text{frequency} \\ \text{down conversion}}} - \underbrace{\hat{a}^{+}(2\omega)\,\hat{a}(\omega)\,\hat{a}(\omega)}_{\substack{\text{frequency} \\ \text{doubling}}}$$

frequency down conversion $\longleftrightarrow$ frequency doubling

also called (degenerate) <u>spontaneous</u> parametric down conversion $\longleftarrow$ frequency down conversion

(in analogy to the purely QM process of spont. emission)

the modes that are relevant for the interaction are selected by the phase matching conditions. (here we considered a colinear phase matching)

• total hamiltonian for degenerate (spontaneous) parametric down conversion (colinear)

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \hbar\omega_p \hat{b}^\dagger \hat{b} + i\hbar\chi^{(2)}\left(\hat{a}^2 \hat{b}^\dagger - \hat{a}^{\dagger 2}\hat{b}\right)$$

mode b is the pump mode : $\omega_p = 2\omega$

Let's consider ω_p mode in coherent state $|\beta \, e^{-i\omega_p t}\rangle$

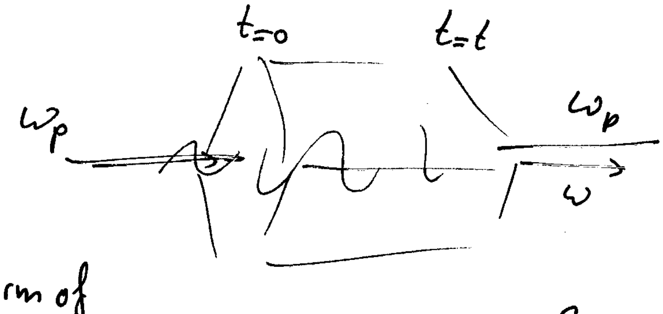
semiclassical approach $\left.\begin{array}{l}\hat{b} \to \beta\, e^{-i\omega_p t} \\ \hat{b}^\dagger \to \beta^* e^{i\omega_p t}\end{array}\right\}$ pump field is considered a classical field that remains unchanged ($\chi^{(2)}$ very weak)

$$\hat{H}_{int}(t) = i\hbar\left[\eta^* \hat{a}^2 \overset{1}{\cancel{e^{i(\omega_p - 2\omega)t}}} - \eta \hat{a}^{\dagger 2} \overset{1}{\cancel{e^{-i(\omega_p - 2\omega)t}}}\right]$$

$\eta = \chi^{(2)}\beta$ for $\omega_p = 2\omega$

$$\hat{U}(t) = e^{-iH_{int}t/\hbar} = e^{(\eta^* t \hat{a}^2 - \eta t \hat{a}^{\dagger 2})} \quad *$$

describes unitary evolution of fields



* has the form of a single mode " squeeze " operator $\hat{S}(\xi) \equiv e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$

$\xi = re^{i\theta}$ r: squeeze parameter

2 photon generalization of displacement operator

$$|\psi_s\rangle = \hat{S}(\xi)|\psi\rangle$$

initial state of mode a at $t=0$ (typically the vacuum state)

We want to describe the effect of $\hat{S}$ on the variance of the quadrature components of mode a

variance : $\langle(\Delta\hat{X}_1)^2\rangle = \langle\hat{X}_1^2\rangle - \langle\hat{X}_1\rangle^2$

recall $\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^+)$

$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^+)$

with $E_x = \mathcal{E}_0 (\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t}) \sin(kz)$

$E_x(t) = 2\mathcal{E}_0 \sin(kz) \times [\hat{X}_1 \cos(\omega t) + \hat{X}_2 \sin(\omega t)]$

recall example : mode a occupied by number state $|n\rangle$

$$\langle(\Delta\hat{X}_1)^2\rangle = \langle n|\hat{X}_1^2|n\rangle - \langle n|\hat{X}_1|n\rangle^2 = \frac{1}{4}(2n+1)$$

$\langle n|\hat{X}_1|n\rangle = \underbrace{\langle n|\frac{(\hat{a}+\hat{a}^+)}{2}|n\rangle}_{=0}$

$\langle n|\hat{X}_1^2|n\rangle = \frac{1}{4}\langle n|\hat{a}^2 + \hat{a}^{+2} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+|n\rangle = \frac{1}{4}(2n+1)$

same for X_2

for $|n\rangle = |0\rangle$ variance in X_1 and X_2 are both $\frac{1}{4}$.

now take mode a occupied by squeezed state $\hat{S}(\xi)\underbrace{|\psi\rangle_{initial}}_{}= |\psi_s\rangle$

state before squeezing, for example $|0\rangle$, $|n\rangle$ or $|\alpha\rangle$

variance in X_1 requires $\langle \hat{X}_1^2 \rangle$ and $\langle \hat{X}_1 \rangle^2$

$\langle \hat{X}_1^2 \rangle \rightarrow$ * $\frac{1}{4} \langle \psi_s | \hat{a}^2 + \hat{a}^{+2} + \hat{a}^+ \hat{a} + \hat{a}\hat{a}^+ | \psi_s \rangle$

$\langle \hat{X}_1 \rangle \rightarrow \frac{1}{2} \langle \psi_s | \hat{a} + \hat{a}^+ | \psi_s \rangle$

calculate $S^+(\xi)\, \hat{a}\, S(\xi)$

and $S^+(\xi)\, \hat{a}^+ S(\xi)$

using Baker Hausdorf

$$S(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})} \equiv e^{-X}$$

$$S^+(\xi) = \hat{S}(-\xi) \equiv e^{X}$$

$$e^{\hat{X}} \hat{Y} e^{-\hat{X}} = \hat{Y} + [\hat{X}, \hat{Y}] + \frac{1}{2!}[\hat{X}, [\hat{X}, \hat{Y}]] + \frac{1}{3!}[\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]] + \ldots$$

$$\hat{Y} = \hat{a}$$

$$[\hat{X}, \hat{Y}] = [-\tfrac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2}), a] = \tfrac{1}{2}\xi [\hat{a}^{+2}, a] = \ldots = -\xi \hat{a}^+$$

$$[\hat{X}, [\hat{X}, \hat{Y}]] = \ldots \tfrac{1}{2}|\xi|^2 \hat{a}$$

$$S^+(\xi)\, a\, S(\xi) = \hat{a} \cosh(r) - \hat{a}^+ e^{i\theta} \sinh(r)$$

$$\ldots a^+ \ldots = \hat{a}^+ \cosh(r) - \hat{a} e^{-i\theta} \sinh(r)$$

calculate terms like

$$* \quad \langle \psi_s | \hat{a}^2 | \psi_s \rangle = \langle \psi | S^+ \hat{a} S \underbrace{S^+ \hat{a} S} | \psi \rangle = \ldots$$

$\ldots \cosh^2(r) \ldots + \sinh^2(r) \ldots + \cosh(r) \sinh(r) \cos\theta$ terms for $|\psi\rangle = |0\rangle$

$$\langle \psi | S^{+} a S | \psi \rangle =$$

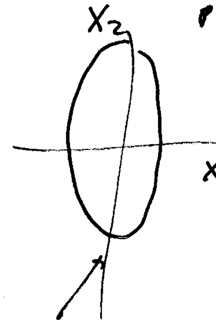
$$\langle 0 | \underbrace{S^{+} a S} | \psi \rangle = 0$$

$$\langle 0 | \hat{a}(\dots) - \hat{a}^{+}(\dots) | 0 \rangle$$

$$\langle (\Delta \hat{X}_1)^2 \rangle = \frac{1}{4} e^{-2r}$$

$\theta = 0$

$$\langle (\Delta \hat{X}_2)^2 \rangle = \frac{1}{4} e^{+2r}$$



$\theta = \pi$ squeezed in X_2

$$\hat{D}(\alpha)\, S(\xi)|0\rangle = |\alpha, \xi\rangle \leftarrow \text{displaced squeezed state}$$

$\hat{D}(\alpha)$: displacement operator

$S(\xi)$: squeeze operator