Optical Feedback Cooling in Optomechanical Systems

- A brief introduction to input-output formalism

  C. W. Gardiner and M. J. Collett, *Input and output in damped quantum systems: Quantum Stochastic differential equations and the master equation*, PRA **31**, 3761-3774

- Quantum damped harmonic oscillator

- Optomechanical systems, classical description of optical cooling.

- A quantum treatment of optical cooling.

  I. Wilson-Rae et al. *Theory of Ground State Cooling of a Mechanical Oscillator Using Dynamical Backaction* PRL **99**, 093901 (2007)
  –and–
  F. Marquardt et al., *Quantum Theory of Cavity-Assisted Sideband Cooling of Mechanical Motion* PRL **99**, 093901 (2007).

$$H = H_{sys} + H_B + H_int$$
$$H_b = \hbar \int_{-\infty}^{\infty} d\omega\ \omega b^\dagger(\omega) b(\omega)$$
$$H_{int} = i\hbar \int_{-\infty}^{\infty} d\omega\ \kappa(\omega) \left[b^\dagger(\omega) c - c^\dagger b(\omega) \right]$$

This represents a continuum of *independant* bath harmonic oscillators, and hence: $\left[b(\omega), b^\dagger(\omega') \right] = \delta(\omega - \omega')$

Working in the Heisenberg picture:

$$\dot{\hat{A}} = -\frac{i}{\hbar} \left[\hat{A}, H \right] + \underbrace{\frac{\partial \hat{A}}{\partial t}}_{\text{(0 for ``normal'' operators)}}$$

$$\Rightarrow \dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega) c,$$
$$\dot{\hat{A}} = -\frac{i}{\hbar} \left[\hat{A}, H_{sys} \right] + \int d\omega\ \left\{ b^\dagger(\omega) \left[\hat{A}, c \right] - \left[\hat{A}, c^\dagger \right] b(\omega) \right\}$$

The formal solution for $b(\omega)$ is:

$$b(\omega) = e^{-i\omega(t-t_0)} b_0(\omega) + \kappa(\omega) \int_{t_0}^{t} c(t') dt',$$

where $b_0(\omega) \equiv b(\omega)(t = 0)$. Furthermore:

$$\dot{\hat{A}} = -\frac{i}{\hbar} \left[\hat{A}, H_{sys} \right] + \int d\omega\ \left\{ e^{+i\omega(t-t_0)} b_0^\dagger(\omega) \left[\hat{A}, c \right] - \left[\hat{A}, c^\dagger \right] e^{-i\omega(t-t_0)} b_0(\omega) \right\}$$
$$+ \int d\omega \left[\kappa(\omega) \right]^2 \int_{t_0}^{t} dt' \left\{ e^{+i\omega(t-t')} c^\dagger \left[\hat{A}, c \right] - \left[\hat{A}, c^\dagger \right] e^{-i\omega(t-t')} c \right\}$$

Input and output in damped quantum systems: Quantum Stochastic differential equations and the master equation, PRA **31**, 3761-3763

We now make the "first Markov approximation":

$$\kappa(\omega) = \sqrt{\gamma/2\pi}$$

Note the following relations:

$$\frac{1}{2\pi}\int d\omega\ e^{-i\omega(t-t')} = \delta(t-t')$$

$$\int_{t_0}^{t} dt'\ c(t')\delta(t-t') = \frac{1}{2}c(t)$$

We can then rewrite the equations of motion for an arbitrary operator, the "Quantum Langevin Equation ":

$$\dot{\hat{A}} = -\frac{i}{\hbar}\left[\hat{A}, H_{sys}\right] - \left[\hat{A}, c^\dagger\right]\left(\frac{\gamma}{2}c + \sqrt{\gamma}b_{in}(t)\right) + \left(\frac{\gamma}{2}c^\dagger + \sqrt{\gamma}b_{in}^\dagger(t)\right)\left[\hat{A}, c\right],$$

where we have defined $b_{in}(t)$:

$$b_{in}(t) = \frac{1}{\sqrt{2\pi}}\int d\omega\ e^{-i\omega(t-t_0)}b_0(\omega)$$

Note also that: $\left[b_{in}(t), b_{in}^\dagger(t')\right] = \delta(t-t')$.

In principle, to calculate the behavior of a real system, we need to know the state of the bath as represented by ρ_{in}. However, in practice we almost always work with weakly damped systems, and treat the input noise as *white*, i.e.:

$$\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{in}}\mathrm{b}_{\mathrm{in}}^\dagger(\mathrm{t})\mathrm{b}_{\mathrm{in}}(\mathrm{t}')\right] &\equiv \left\langle b_{in}^\dagger(t)b_{in}(t')\right\rangle \\
&= \bar{N}\delta(t-t') \\
\mathrm{Tr}\left[\rho_{\mathrm{in}}\mathrm{b}_{\mathrm{in}}(\mathrm{t})\mathrm{b}_{\mathrm{in}}^\dagger(\mathrm{t}')\right] &\equiv \left\langle b_{in}(t)b_{in}^\dagger(t')\right\rangle \\
&= (\bar{N}+1)\delta(t-t')
\end{aligned}$$

Harmonic oscillator:

$$\dot{a} = -i\omega_a a - \frac{\gamma_a}{2} a - \sqrt{\gamma_a} b_{in}(t)$$
$$\dot{a}^\dagger = +i\omega_a a^\dagger - \frac{\gamma_a}{2} a^\dagger - \sqrt{\gamma_a} b_{in}^\dagger(t)$$

If the system is linear, we can fourier transform it:

$$-i\Omega\tilde{a} = -i\omega_a \tilde{a} - \frac{\gamma_a}{2}\tilde{a} - \sqrt{\gamma_a}\tilde{b}_{in}(t)$$
$$+i\Omega\tilde{a}^\dagger = +i\omega_a \tilde{a}^\dagger - \frac{\gamma_a}{2}\tilde{a}^\dagger - \sqrt{\gamma_a}\tilde{b}_{in}^\dagger(t),$$

noting that:

$$\tilde{a}(\Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+i\Omega t} a(t) dt$$
$$\tilde{a}^\dagger(\Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega t} a^\dagger(t) dt$$
$$\dot{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\Omega e^{-i\Omega t} \tilde{a}(\Omega) d\Omega$$
$$\left\langle \tilde{b}_{in}^\dagger(\omega)\tilde{b}_{in}(\omega')\right\rangle = \bar{N}\delta(\omega-\omega')$$
$$\left\langle \tilde{b}_{in}(\omega)\tilde{b}_{in}^\dagger(\omega')\right\rangle = \left(\bar{N}+1\right)\delta(\omega-\omega')$$

So...

$$\tilde{a} = \frac{\sqrt{\gamma_a}\tilde{b}_{in}}{i(\Omega-\omega_a)-\frac{\gamma_a}{2}} = \sqrt{\gamma_a}\tilde{b}_{in}\chi_a(\Omega)$$
$$\tilde{a}^\dagger = \frac{\sqrt{\gamma_a}\tilde{b}_{in}^\dagger}{-i(\Omega-\omega_a)-\frac{\gamma_a}{2}} = \sqrt{\gamma_a}\tilde{b}_{in}^\dagger\chi_a^*(\Omega),$$

where $\chi_a^{-1} = i(\Omega-\omega_a) - \frac{\gamma_a}{2}$.

We define the spectral density of the number operator as:

$$\begin{aligned}
S_{aa}(\Omega) &= \int dt\ e^{+i\Omega t} \left\langle a^\dagger(t)a(0)\right\rangle \\
&= \frac{1}{2\pi}\int\int\int dt\ d\omega'\ d\omega''\ e^{i(\Omega-\omega')t}\left\langle \tilde{a}^\dagger(\omega')\tilde{a}\omega'')\right\rangle \\
&= \int d\omega'' \left\langle \tilde{a}^\dagger(\Omega)\tilde{a}(\omega'')\right\rangle \\
&= \left\langle \tilde{a}^\dagger(\Omega)\tilde{a}(\Omega)\right\rangle,
\end{aligned}$$

where we have assumed: $\left\langle \tilde{a}^\dagger(\omega)\tilde{a}(\omega')\right\rangle \propto \delta(\omega-\omega')$.

And hence...

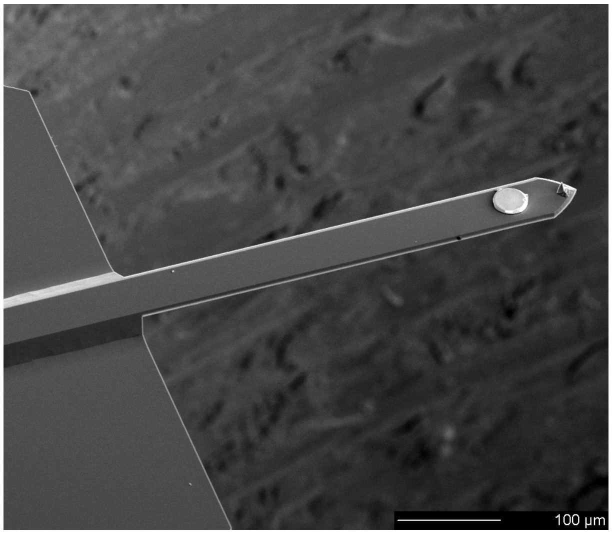
$$\begin{aligned}
S_{aa}(\Omega) &= \gamma_a \left|\chi(\Omega)\right|^2 \left\langle \tilde{b}_{in}^\dagger(\Omega)\tilde{b}_{in}(\Omega)\right\rangle \\
&= \frac{\bar{N}\gamma_a}{(\Omega-\omega_a)^2+\left(\frac{\gamma_a}{2}\right)^2}
\end{aligned}$$

Where we have assumed a white noise input. This is a Lorentzian centered at w_a with FWHM γ_a – this is the expected result for a damped harmonic oscillator.

Note also:

$$\frac{1}{2\pi}\int S_{aa} d\Omega = \bar{N},$$

as you might expect.



Classical motion:

$$m\ddot{x} = -kx - \frac{m\gamma_c}{2}\dot{x} + F_{th}(t) + 2\frac{P(x,t)}{c}$$

Cavity field:

$$P_0(x) \propto \frac{1}{\left(\frac{\gamma_a c}{2}\right)^2 + (x - x_0)^2}$$

$$\dot{P}(t) = -\gamma_a \left[P(t) - P_0(x)\right]$$

$$\approx -\gamma_a \left[P(t) - P_0(x_0) - x \underbrace{\left.\frac{dP_0}{dx}\right|_{x=0}}_{ck_r/2}\right]$$

So...

$$-i\Omega\tilde{P} = -\gamma_a\tilde{P} + \gamma_c\tilde{x}\frac{ck_r}{2}$$

$$\tilde{P} = \left(\frac{c\gamma_a k_r}{2}\right)\frac{\tilde{x}}{\gamma_a - i\Omega}$$

$$= \left(\frac{ck_r}{2}\right)\frac{(1 + i\frac{\Omega}{\gamma_a})\tilde{x}}{1 + \left(\frac{\Omega}{\gamma_a}\right)^2}$$

Optomechanical Hamiltonian:

$$H = \underbrace{\hbar\,\omega_a \left(aa^\dagger + \frac{1}{2}\right)}_{\text{Optical Cavity}} + \underbrace{\hbar\omega_c \left(cc^\dagger + \frac{1}{2}\right)}_{\text{Mechanical Resonator}} + \underbrace{H_{int}}_{\text{Radiation Pressure Interaction}}$$

We now assume we are in the adiabatic limit: $\frac{L_0}{c} \ll \frac{1}{w_c}$. In this case we can approximate the interaction term with:

$$\omega_a \to \omega_a \frac{L_0}{L_0 + x} \approx w_a \left(1 - \frac{x}{L_0}\right)$$

$$\approx w_a \left[1 - \underbrace{\frac{\sqrt{\frac{\hbar}{2m\omega}}}{L_0}}_{g} \left(c^\dagger + c\right)\right]$$

$$\Rightarrow H \approx \hbar\omega_a \left[1 - g\left(c^\dagger + c\right)\right] aa^\dagger + \hbar\omega_c cc^\dagger$$

Now we include the bath couplings and a driving term:

$$H = \hbar w_a a^\dagger a \left[1 - \frac{\sigma_0}{L}\left(c + c^\dagger\right)\right] + \hbar w_c c^\dagger c + \underbrace{\hbar A e^{-i\omega t}\left(a + a^\dagger\right)}_{\text{Optical driving term}} + \underbrace{H_{B,a} + H_{int,a}}_{\text{Optical ``bath''}} + \underbrace{H_{B,c} + H_{int,c}}_{\text{Mechanical bath}}$$

We can calculate the equations of motion:

$$\dot{a} = -i\omega_a a\left[1 - g\left(c + c^\dagger\right)\right] - iAe^{-i\omega t} - \frac{\gamma_a}{2}a - \sqrt{\gamma_a}a_{in}$$

$$\dot{c} = -i\omega_c c + i\omega_a g a^\dagger a - \frac{\gamma_c}{2}c - \sqrt{\gamma_c}c_{in},$$

where $a_{in} \equiv b_{a,in}$ and $c_{in} \equiv b_{c,in}$.

We make the substitution $a \rightarrow e^{-iwt}(\bar{a} + d)$, where $\bar{a}$ is a constant and d represents the quantum fluctuations of the system. The equation of motion for d is:

$$\dot{d} = -i\Delta(\bar{a}+d) + i\omega_a g \underbrace{(\bar{a}+d)}_{\approx\bar{a}}\left(c^\dagger + c\right) - iA - \frac{\gamma}{2}(\bar{a}+d) - \underbrace{e^{+iwt}\sqrt{\gamma_a}a_{in}}_{\rightarrow\sqrt{\gamma_a}d_{in}},$$

where $\Delta = \omega - \omega_a$ is the optical detuning.

If we assume $\langle d^\dagger \rangle \ll |\bar{a}|^2$, we can linearize the equation. Furthermore we see the natural choice for A:

$$A \rightarrow -\left(i\frac{\gamma_a}{2} + \Delta\right)\bar{a}$$

$$\dot{d} \cong -i\Delta d + i\alpha(c + c^\dagger) - \frac{\gamma_a}{2}d - \sqrt{\gamma_a}d_{in}$$

$$\alpha \equiv \bar{a}\omega_a g$$

Additionally:

$$\dot{c} = -i\omega_c c + i\omega_a g \underbrace{\left(\bar{a}^*\bar{a} + \bar{a}^* d + d^\dagger\bar{a} + d^\dagger d\right)}_{\approx|\bar{a}|^2 + \bar{a}^* d + \bar{a}d^\dagger} - \frac{\gamma_c}{2}c - \sqrt{\gamma_c}c_{in}$$

$$\cong -i\omega_c' c + i\left(\alpha^* d + \alpha d^\dagger\right) - \frac{\gamma_c}{2}c - \sqrt{\gamma_c}c_{in},$$

where I have ignored the $|\bar{a}|^2$ term, which is just a displacement of the mean position of the mechanical resonator.

What about a fourier transform?

$$-i\Omega\tilde{d}(+\Omega) = -i\Delta\tilde{d}(+\Omega) + i\alpha\big[\tilde{c}(+\Omega) + \underbrace{\tilde{c}^\dagger(-\Omega)}_{!}\big]$$
$$- \frac{\gamma_a}{2}\tilde{d}(+\Omega) - \sqrt{\gamma_a}\tilde{d}_{in}(+\Omega)$$
$$-i\Omega\tilde{c}(+\Omega) = -i\omega_c\tilde{c}(+\Omega) + i\big[\alpha^*\tilde{d}(+\Omega) + \underbrace{\alpha\tilde{d}(-\Omega)}_{!}\big]$$
$$- \frac{\gamma_c}{2}\tilde{c}(+\Omega) - \sqrt{\gamma_c}\tilde{c}_{in}(+\Omega)$$

So...

$$\chi_d^{-1}(\Omega)\tilde{d}^+ + i\alpha(\tilde{c}^+ + \tilde{c}^{\dagger -}) = \sqrt{\gamma_a}\tilde{d}_{in}^+$$
$$\chi_c^{-1}(\Omega)\tilde{c}^+ - i(\alpha^*\tilde{c}^+ + \alpha\tilde{c}^{\dagger -}) = \sqrt{\gamma_a}\tilde{c}_{in}^+$$

Let's define a system operator, $\mathbb{A}$:

$$\mathbb{A} \equiv \begin{pmatrix} d \\ d^\dagger \\ c \\ c^\dagger \end{pmatrix}; \quad \tilde{\mathbb{A}} = \begin{pmatrix} \tilde{d}^+ \\ \tilde{d}^{\dagger -} \\ \tilde{c}^+ \\ \tilde{c}^{\dagger -} \end{pmatrix}$$

$$\begin{pmatrix} \chi_d^{-1}(\Omega) & 0 & i\alpha & i\alpha \\ 0 & \chi_d^{-1*}(-\Omega) & -i\alpha^* & -i\alpha^* \\ i\alpha^* & i\alpha & \chi_c^{-1}(\Omega) & 0 \\ -i\alpha^* & -i\alpha & 0 & \chi_c^{-1*}(-\Omega) \end{pmatrix} \tilde{A}^T = \begin{pmatrix} \sqrt{\gamma_a}\tilde{d}_{in}^+ \\ \sqrt{\gamma_a}\tilde{d}_{in}^{\dagger -} \\ \sqrt{\gamma_c}\tilde{c}_{in}^+ \\ \sqrt{\gamma_c}\tilde{c}_{in}^{\dagger -} \end{pmatrix}$$

where:

$$\chi_a^{-1} = i(\Omega - \Delta) - \frac{\gamma_a}{2}$$
$$\chi_c^{-1} = i(\Omega - \omega_c) - \frac{\gamma_c}{2}$$

The solution is:

$$\tilde{d} = \chi_a \left(\sqrt{\gamma_d} \left[\tilde{d}_{in} + \chi_a^{*-} \xi_c \left(|\alpha|^2 \tilde{d}_{in} - \alpha^{*2} \tilde{d}_{in}^{\dagger -} \right) \right] + \right.$$
$$\left. i\sqrt{\gamma_c} \alpha^* \left[\chi_c^{*-} \tilde{c}_{in}^{\dagger -} - \chi_c \tilde{c}_{in} \right] \right) / \left(1 + |\alpha|^2 \xi_a \xi_c \right)$$
$$\tilde{c} = \chi_c \left(\sqrt{\gamma_c} \left[\tilde{c}_{in} + |\alpha|^2 \chi_c^{*-} \xi_a \left(\tilde{c}_{in} - \tilde{c}_{in}^{\dagger -} \right) \right] + \right.$$
$$\left. i\sqrt{\gamma_d} \chi_c^{*-} \left[\alpha^* \chi_a^{*-} \tilde{d}_{in}^{\dagger -} - \alpha \chi_a \tilde{d}_{in} \right] \right) / \left(1 + |\alpha|^2 \xi_a \xi_c \right)$$

where: $\xi_{a/c}(\Omega) = \chi_{a/c}^*(-\Omega) - \chi_{a/c}(\Omega)$.

What is the input?

$\left\langle \tilde{d}_{in}^{\dagger} \tilde{d}_{in} \right\rangle = 0$ Ground state (a is coherent state!)

$\left\langle \tilde{c}_{in}^{\dagger} \tilde{c}_{in} \right\rangle = n_{th}$ Thermal state

$$S_{cc}(\Omega) = \underbrace{\frac{|\chi_c|^2}{\left|1 + |\alpha|^2 \xi_a \xi_c\right|^2}}_{\textit{Modified mechanical response}} \times \left[\underbrace{\gamma_a \left|\alpha \chi_a^-\right|^2}_{\textit{Zero point fluc.}} + \right.$$
$$\left. \underbrace{\bar{n}_{th} \gamma_c \left|1 - |\alpha|^2 \chi_c^- \xi_a\right|^2}_{\textit{Main thermal excitation}} + \underbrace{\left(\bar{n}_{th} + 1\right) \gamma_c \left||\alpha|^2 \chi_c^- \xi_a\right|^2}_{\textit{Back action}} \right]$$

From Gardiner and Collett:

$$a_{out} = a_{in} + \sqrt{\gamma_a} a$$

So the optical field leaving the cavity is:

$$S_{aa,out}(\Omega) = \int_{-\infty}^{\infty} dt e^{i\Omega t} \left\langle a_{out}^{\dagger}(t) a_{out}(0) \right\rangle$$

$$S_{aa,out}(\Omega + \omega_a) = \int_{-\infty}^{\infty} dt e^{i\Omega t} \Big\langle \left[d_{in}^{\dagger}(t) + \sqrt{\gamma_a} \left(\bar{a}^* + d^{\dagger}(t) \right) \right] \times$$

$$\left[d_{in}(0) + \sqrt{\gamma_a} \left(\bar{a} + d(0) \right) \right] \Big\rangle$$

$$= \left\langle \tilde{d}_{in}^{\dagger}(\Omega) \tilde{d}_{in}(\Omega) \right\rangle + \int_{-\infty}^{\infty} dt\, e^{i\omega' t} \times$$

$$\left[\sqrt{\gamma_a} \left(\left\langle \tilde{d}_{in}^{\dagger}(\Omega) \tilde{d}(\omega') \right\rangle + \left\langle \tilde{d}^{\dagger}(\Omega) \tilde{d}_{in}(\omega') \right\rangle \right) + \right.$$

$$\left. \gamma_a \left(|\bar{a}|^2 + \left\langle \tilde{d}^{\dagger}(\Omega) \tilde{a}(\omega') \right\rangle \right) \right]$$

$$\frac{S_{aa,out}(\Omega + \omega_a)}{\gamma_a} = 2\pi \left|\bar{a}\right|^2 \delta(\Omega) + \frac{1}{\left|1 + |\alpha|^2 \xi_a \xi_c\right|^2} \times$$

$$\left[\gamma_a \left| \alpha^2 \chi_a \chi_a^- \xi_c \right|^2 + \underbrace{\bar{n}_{th} \gamma_c \left| \alpha \chi_a \chi_c \right|^2}_{\text{Anti-Stokes (Cooling)}} + \right.$$

$$\left. \underbrace{\left(\bar{n}_{th} + 1 \right) \gamma_c \left| \alpha \chi_a \chi_c^- \right|^2}_{\text{Stokes (Heating)}} \right]$$

$$\Delta = -\omega_c$$
$$n_{th} = 1000$$
$$\bar{n} = \frac{1}{2\pi}\int d\Omega \ S_{cc}(\Omega)$$