SEMICLASSICAL AND QUANTUM-ELECTRODYNAMICAL APPROACHES IN NONRELATIVISTIC RADIATION THEORY

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Abstract:

Theoretical aspects of the interaction of atoms with the radiation field are reviewed with emphasis on those features of the interaction requiring field quantization. The approach is nonrelativistic, with special attention given to the theory of spontaneous emission.

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1. Introduction

The theory of the interaction of light with atomic matter looms large in the historical development of quantum mechanics. This circumstance is not accidental, but simply a consequence of the fact that the light emitted or absorbed by atomic matter provides clues into its nature. The synthesis of the quantum theory that resulted from spectroscopic and other optical investigations is a fascinating story which, in many respects, is still developing. The purpose of this article is to review a recent and perhaps yet to be completed chapter of this story, namely, the current debate over the range of validity of semiclassical radiation theory. In this theory, which may assume several forms, electromagnetic radiation is treated as a classically prescribed field while the atomic matter with which it interacts is described according to the dictates of quantum mechanics.

Quantum theory has, of course, been developed and applied with remarkable success in many areas of physics, but it is important to remember that it was originally a theory of the interaction of light with atoms and molecules. In this historical sense, semiclassical radiation theory strikes at the very roots of quantum mechanics. Semiclassical approaches have been successful in explaining several phenomena which had become paradigms for illustrating the failure of classical theory. It is interesting to reflect on what consequences these results might have had for the development of quantum theory had they been known, say, in 1927, the year in which Dirac [1] published the first work on field quantization.

One of the outstanding successes of Dirac's radiation theory was the derivation of the Einstein A-coefficient for spontaneous emission. The concept of spontaneous emission was introduced by Einstein [2] in his classic 1917 derivation of the blackbody spectrum. Einstein's rate-equation approach led to a relation between the rates of spontaneous and stimulated emission for a system in thermal equilibrium, but neither of these rates could at the time be evaluated from first principles. With the new radiation theory Dirac showed, using perturbation theory, that the spontaneous emission rate from atomic level m to n is given by

$$A_{mn} = 4|\boldsymbol{\mu}_{mn}|^2 \omega_{mn}^3 / 3\hbar c^3 , \tag{1.1}$$

where

$$\hbar\omega_{mn} = E_m - E_n , \tag{1.2}$$

and

$$\boldsymbol{\mu}_{mn} = \langle m|e\boldsymbol{r}|n\rangle = e\int \mathrm{d}^3 r\, \phi_m^*(\boldsymbol{r})\, \boldsymbol{r}\, \phi_n(\boldsymbol{r}), \tag{1.3}$$

$\phi_m(\boldsymbol{r})$ being an atomic stationary state with energy eigenvalue E_m. The rate of energy loss by the atom is thus $\hbar\omega_{mn} A_{mn} = 4|\boldsymbol{\mu}_{mn}|^2 \omega_{mn}^4 3c^3$, the same expression one obtains for the energy loss by a classical oscillating dipole with dipole moment $2\mathrm{Re}[\boldsymbol{\mu}_{mn}\exp(\mathrm{i}\omega_{mn}t)]$; this result was also obtained by Kramers and Heisenberg [3] from the Correspondence Principle.

Perhaps it is this close formal similarity to the classical theory which has inspired various attempts to construct a theory of spontaneous emission which does not quantize the radiation field. In any case, half a century after Dirac's paper, there is active interest in a semiclassical theory of spontaneous emission, and the subject has been the topic of one or more sessions in at least three quantum optics and electronics conferences held within the past three years. One of the primary aims of this review is to present the various arguments which have arisen in assessing the most successful semiclassical theory of spontaneous emission, the "neoclassical" theory advanced by Jaynes and his collaborators [4–6]. It has been found that, aside from its basic importance as a ubiquitous natural phenomenon, spontaneous emission forces consideration of such fundamental theoretical questions as the radiation reaction field of a point charge, the vacuum fluctuations of the radiation field, and the "paradox" of Einstein, Podolsky and Rosen [7].

Another aspect of spontaneous emission processes that is currently enjoying much attention was first introduced by Dicke [8], and is usually referred to as "superradiance" [9]. This is the phenomenon whereby a system of atoms or molecules radiates spontaneously in a cooperative manner, so that the emission proceeds at a rate very much higher than would be expected from a model of independently radiating sources. Complementing such superradiant states of the collective atomic system are "subradiant" states in which the spontaneous emission rate is very low. Superradiance has been observed experimentally only recently [10], and a semiclassical treatment [11] seems to account rather well for the experimental results; here again, however, there is some disagreement regarding the validity of a semiclassical theory. A similar effect, the modification of spontaneous emission rates by the presence of mirrors, has recently been observed [12, 13]. This effect can be described as the cooperative emission of the emitter and its mirror images [14]. As in the case of a "free" atom, semiclassical and quantum-electrodynamical treatments are found to give identical expressions for the radiative linewidth.

The concept of stimulated emission also had its debut in Einstein's paper "On the Quantum Theory of Radiation" [2]. In fact, whereas Einstein considered the assumption of spontaneous emission to be a natural one, in complete analogy to the radiation from a Hertz dipole, the assumption of stimulated emission proved to be the key to deriving Planck's law rather than Wien's. Stimulated emission was also the *sine qua non* of the important "negative-oscillator" terms in the Kramers–Heisenberg formula [3]; this result in turn implied the commutation relation $[x, p_x] = i\hbar$. The concept of stimulated emission, so important in the chain of developments leading by 1927 to a considerably complete quantum theory, is also an ingredient of semiclassical theory. The fact that stimulated emission is well described semiclassically is borne out by the theory of the laser. The use of the semiclassical theory in this case is usually justified on the grounds that, for high "photon" densities, the quantum-mechanical description of the field should approach the classical.

One of the apparently important features of the quantum theory of radiation is its prediction of a fluctuating zero-point field of energy density $\frac{1}{2}\hbar\omega$ per mode. A "classical" theory of zero-point effects results from taking such a field as the homogeneous solution of the Maxwell equation, rather than the usual null source-free solution. The field associated with each mode in the plane-wave expansion is given a phase which is a uniformly distributed random variable over $[0,2\pi]$, and it is assumed that there is no phase correlation of different modes. Some implications of this "random electrodynamics" have been investigated by Boyer [15–21]. In particular, such a model of interacting radiation and matter accounts for the blackbody spectrum and the van der Waals forces between neutral polarizable particles. It is interesting to note that these results were obtained from a classical treatment of the "atoms", in the spirit of the Lorentz model. The theory is therefore neither quantum electrodynamical nor semiclassical, in the usual sense. The rather surprising results mentioned above, however, necessitate some discussion of random electrodynamics in this article.

Rather than concluding this Introduction in the usual fashion by outlining the contents of the remainder of the paper, already to be found in the table of contents, only a statement of intent will be made. It is hoped that this article will provide a readable, accurate account of the current status of semiclassical radiation theory as it is used in quantum optics and electronics. The emphasis is on the meaning of semiclassical theory at a basic level, and how it compares with the quantum-electrodynamical theory at this level. Perhaps the reader who has not been particularly interested in the current controversy will learn what all the fuss is about.

2. Quantum-electrodynamical approach to spontaneous emission

2.1. *Introduction*

The approximation of treating spontaneous radiation as a single-atom process is usually a very accurate one. Collisions between atoms in a luminous gas, for example, produce excited atoms which can undergo transitions to lower energy states and in so doing emit light at the Bohr frequencies. The different atoms emit independently, since each atom has the same likelihood of excitation at thermal equilibrium, *regardless* of the states of excitation of its neighbors. Quantum mechanically, the state vector of the atomic system is a direct product of the state vectors describing the individual atoms, with no phase correlations between the transition amplitudes for different atoms. Furthermore, the visible radiation from such a system is usually preponderantly spontaneous; the ratio of the rate of spontaneous to stimulated emission at frequency ν is simply $\exp(h\nu/kT)-1$. For the sodium D lines and the solar temperature $T \sim 6000°\mathrm{K}$, for example, this ratio is about 58.

Aside from its relevance, the problem of single-atom spontaneous emission is of more fundamental importance than is generally imagined. As mentioned earlier, certain aspects of spontaneous emission have a bearing on several fundamental theoretical questions of continuing interest. As the reader has undoubtedly surmised, the problem occupies a central role in most assessments of semiclassical radiation theory.

Treatments of the theory of spontaneous emission usually follow some variant of the original Weisskopf–Wigner [22] approach. While this approach, of course, leads without difficulty to the spontaneous transition rate (1.1), it does not satisfy the student's desire to understand the phenomenon in a physically intuitive manner. Why should an atom, in an otherwise stationary excited state, for no apparent reason make a transition to a state of lower energy? The usual response to this question is to point out that the quantum theory of radiation predicts the existence of a radiation field even in the vacuum, where classically all fields would vanish identically. While this radiation field has expectation value zero, its fluctuations perturb the excited atom in such a manner as to trigger the release of its stored energy in the form of electromagnetic radiation; the physical picture is vaguely suggestive of leaves being jostled from a tree by the wind. Computational support for this picture can be found in Welton's elegant approach to the Lamb shift [23]. Indirect support comes from a semiclassical theory of spontaneous emission, where perforce there are no vacuum field fluctuations, and no spontaneous emission from an atom in a pure excited state is predicted.

One of the most appealing features of Jaynes' neoclassical theory is that spontaneous emission is attributed to a classically familiar concept, namely, the radiation reaction field of an oscillating dipole. Such an interpretation of spontaneous emission was in fact alluded to by Slater [24] in 1924. More recently, Ackerhalt, Knight and Eberly [25] have shown that the radiation reaction interpretation has a quantum-mechanical basis as well. Working in the Heisenberg picture, they find equations formally similar to the neoclassical ones, and the differences between the neoclassical and fully quantum-mechanical theories are clearly displayed [26]. Moreover, radiation reaction is not the only possible conceptual basis for understanding spontaneous emission. It has been shown that an understanding based on the concept of vacuum field fluctuations is equally justified, and in fact that radiation reaction and vacuum field fluctuations provide essentially complementary conceptual bases for the interpretation of spontaneous decay and the accompanying line width and shift [27–29].

Experiments which seem to support the quantum-electrodynamical theory of spontaneous emission *vis-à-vis* the neoclassical theory have in most cases not been entirely definitive. A striking exception, however, which seems to elude any explanation based on a classical electromagnetic theory, is the experiment of Kocher and Commins [30] on the photon polarization correlations in a three-level cascade. The limitations of semiclassical radiation theories that are suggested by the Kocher–Commins experiment have been discussed by Clauser [31]. These limitations reflect the inadequacy of classical theories in general to account for certain quantum-mechanical correlation effects first discussed by Einstein, Podolsky and Rosen [7]. These effects are often considered to represent the most profound differences between classical and quantum physics, and it is remarkable that such subtle effects must be invoked to support the quantum theory of spontaneous emission.

For the purpose of comparing semiclassical and quantum-electrodynamical theories of spontaneous emission, it is convenient to develop the quantum-electrodynamical theory in the Heisenberg picture; this approach has been used for problems in quantum optics by Senitzky [32–39], Rehler and Eberly [9], and Lehmberg [40], among others. The main objective of this section is to underscore the physical mechanism responsible for spontaneous emission, a task for which the Heisenberg-picture approach is well suited [41].

Before beginning any calculations, it is perhaps useful to review the context in which the idea of spontaneous emission arises. One normally approaches the problem in two steps. First the interaction between the electron and the electromagnetic field of the nucleus is considered; at this stage one ignores completely the coupling of the electron to the vacuum radiation field. The result of this analysis is that the atom has certain stationary states of well-defined energy. The second step is to refine this initial result by introducing the interaction between the electron and the free field as a perturbation. This refinement reveals that only the lowest-energy state is a true stationary state with a well-defined energy. All the higher-energy states have a certain (natural) width in energy and decay – spontaneously – by releasing energy as electromagnetic radiation. Moreover, all energy levels are slightly shifted from the values obtained in the initial calculation.

In discussing spontaneous emission, then, the problem of the interaction of the electron with the nucleus may be regarded as solved. Only the refinements produced by the electron's coupling to the radiation field need to be considered in detail. Since the interest here is in a physical interpretation and not in accurate numerical results for the level shifts and widths, the approach will be nonrelativistic. Furthermore, it suffices for this purpose to regard the "atom" as a spinless, one-electron system.

2.2. *The Hamiltonian*

For the purpose of this section, the Hamiltonian for a one-electron atom interacting with the electromagnetic radiation field may be written as

$$H = \frac{1}{2m}\left(p - \frac{e}{c}A(r)\right)^2 + V(r) + \frac{1}{8\pi}\int \mathrm{d}^3r\,(E^{\perp}(r)^2 + B(r)^2), \tag{2.1}$$

where m and e (< 0) are the electron mass and charge, respectively, $p = m\dot{r} + (e/c)A$ is the momentum conjugate to r, A is the Coulomb-gauge vector potential at the electronic position, and $E^{\perp} = -(1/c)\partial A/\partial t$. $A(r, t)$ may be expanded in plane waves normalized in the V as [42]

$$A(r, t) = \sum_{k,\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}[a_{k\lambda}(t)\,\mathrm{e}^{\mathrm{i}k\cdot r}\hat{e}_{k\lambda} + \text{h.c.}], \tag{2.2}$$

where $k \cdot \hat{e}_{k\lambda} = 0$, $\lambda = 1, 2$ being the polarization index, $\hat{e}_{k\lambda} \cdot \hat{e}^*_{k\lambda'} = \partial_{\lambda\lambda'}$, and $\omega_k = kc$; $a_{k\lambda}(t)$ and $a^+_{k\lambda}(t)$ are, respectively, the Heisenberg-picture photon annihilation and creation operators for mode (k, λ):

$$[a_{k\lambda}(t), a^+_{k'\lambda'}(t)] = \delta^3_{k,k'}\delta_{\lambda\lambda'}. \tag{2.3}$$

The quantization volume V is taken at this point to be finite, but eventually will be allowed to become infinite in order to admit all radiation modes. For definiteness, let V be a cube of side L, so that $(k_x, k_y, k_z) = (2\pi/L)(n_x, n_y, n_z)$, where n_x, n_y and n_z assume all integral values.

The Hamiltonian (2.1) may be written

$$H = H_{\mathrm{A}} + H_{\mathrm{F}} + H_{\mathrm{INT}}, \tag{2.4}$$

where

$$H_{\mathrm{A}} = \frac{p^2}{2m} + V(r), \qquad H_{\mathrm{F}} = \sum_{k,\lambda}\hbar\omega_k a^+_{k\lambda}a_{k\lambda},$$

and

$$H_{\mathrm{INT}} = -\frac{e}{mc}A(0, t)\cdot p(t) + \frac{e^2}{2mc^2}A^2(0, t).$$

The dipole approximation of evaluating the field at the position of the nucleus has been adopted; this approximation is expected to be accurate if the orbital "radius" of the electron is much smaller than the relevant transition wavelengths, a condition well satisfied at optical frequencies.

The Schrödinger-picture stationary states of the atom will be labeled $|1\rangle$, $|2\rangle$, $|3\rangle$, ..., where

$$H_{\mathrm{A}}|n(t)\rangle = E_n|n(t)\rangle = E_n\exp\{-\mathrm{i}E_nt/\hbar\}|n(0)\rangle. \tag{2.5}$$

The states $|n(t)\rangle$ span the Hilbert space of the atomic system. In this basis the atomic Hamiltonian H_{A} may be written as

$$H_{\mathrm{A}} = \sum_n E_n |n(t)\rangle\langle n(t)| = \sum_n E_n \sigma_{nn}(t), \tag{2.6}$$

where $\sigma_{nm}(t) = \exp(\mathrm{i}Ht/\hbar)\, \sigma_{nm}(0) \exp(-\mathrm{i}Ht/\hbar)$. In the absence of any perturbation, $\sigma_{nn}(t) = \sigma_{nn}(0)$.

In general, any atomic operator $A(t)$ has the representation

$$A(t) = \sum_m \sum_n \langle n(0)|A(0)|m(0)\rangle\, \sigma_{nm}(t). \tag{2.7}$$

In particular, the linear momentum operator $\boldsymbol{p}(t)$ has the representation

$$\boldsymbol{p}(t) = \sum_m \sum_m \langle n(0)|\boldsymbol{p}(0)|m(0)\rangle\, \sigma_{nm}(t) = -\frac{\mathrm{i}m}{e} \sum_m \sum_n \omega_{mn} \boldsymbol{\mu}_{nm} \sigma_{nm}(t), \tag{2.8}$$

where $\hbar\omega_{mn} = E_m - E_n$ and $\boldsymbol{\mu}_{nm} = \langle n(0)|e\boldsymbol{r}|m(0)\rangle$ is the electric-dipole-moment transition matrix element between states n and m; $\boldsymbol{\mu}_{nm} \neq 0$ if there is an allowed (electric-dipole) transition between states n and m.

The operators $\sigma_{nm}(t)$ will be referred to as the *atomic operators*. They provide the multilevel generalization of Dicke's [8] spin operators for two-level atoms. It is easily seen that the atomic operators obey the commutation rule

$$[\sigma_{ij}(t), \sigma_{kl}(t)] = \delta_{jk}\sigma_{il}(t) - \delta_{il}\sigma_{kj}(t). \tag{2.9}$$

With the representations (2.6) and (2.8) for H_{A} and $\boldsymbol{p}$, respectively, the Hamiltonian (2.4) may be written,

$$H = \sum_n E_n \sigma_{nn}(t) + \sum_{k,\lambda} \hbar\omega_k a^{+}_{k\lambda}(t)\, a_{k\lambda}(t) + \mathrm{i}\hbar \sum_{k,\lambda} \sum_m \sum_n D_{k\lambda nm}\, \sigma_{nm}(t)[a_{k\lambda}(t) + a^{+}_{k\lambda}(t)] + \frac{e^2}{2mc^2} A^2(0, t), \tag{2.10}$$

where

$$D_{k\lambda nm} = \frac{1}{\hbar}\left(\frac{2\pi\hbar}{\omega_k V}\right)^{1/2} \omega_{mn} \boldsymbol{\mu}_{nm} \cdot \hat{\boldsymbol{e}}_{k\lambda}. \tag{2.11}$$

We have taken the polarization vectors $\hat{\boldsymbol{e}}_{k\lambda}$ to be real.

2.3. *Digression on the form of the interaction Hamiltonian*

In writing the Hamiltonian, the "atom" has been taken to be an entity characterized by the two sets of quantities $\{\omega_{nm}\}$ and $\{\boldsymbol{\mu}_{nm}\}$, that is, by an infinite number of point dipoles, each oscillating sinusoidally. The magnitudes of the quantities ω_{mn} and $\boldsymbol{\mu}_{mn}$ characterize the particular (one-electron) atomic species under consideration. This simple model suffices for the treatment of a great many phenomena involving optical electric dipole transitions, and is easily extended to treat other cases, such as the vibrational–rotational transitions of polyatomic molecules.

The question naturally arises whether one may write the interaction Hamiltonian in the form appropriate to the interaction of an electromagnetic field with a collection of oscillating electric dipoles:

$$H_{\mathrm{INT}} = -\sum_m \sum_n \boldsymbol{\mu}_{nm} \cdot \boldsymbol{E}^{\perp}(0, t)\, \sigma_{nm}(t) = -e\boldsymbol{r}(t) \cdot \boldsymbol{E}^{\perp}(0, t). \tag{2.12}$$

The latter form follows immediately from the representation of the electron position operator $\boldsymbol{r}(t)$ prescribed by eq. (2.7), and indicates that, as in the Lorentz model, the electron and nucleus together comprise a single atomic dipole. This contraction of the complete atomic dipole system to a single dipole will be discussed further in a later section, in connection with the quantum-mechanical approach to the Lorentz model.

The form (2.12) of the interaction Hamiltonian was used by Göppert-Mayer [43] for the study of two-photon transitions. It had been used earlier by Kramers and Heisenberg [3], but Göppert-Mayer seems to have been the first to derive the form (2.12) from the minimal coupling form. Her approach, however, was semiclassical, as was the later treatment by Richards [44]. Power and Zienau [45] found in the calculation of the retarded van der Waals force between two neutral molecules [46] that the $-e\boldsymbol{r} \cdot \boldsymbol{E}^{\perp}$ form was computationally advantageous, and they investigated the form of the interaction Hamiltonian in greater detail in a later paper [47]. A noteworthy feature of the work of Power and Zienau on this problem is that the radiation field is quantized.

Consider the Hamiltonian for the case where the field is not quantized:

$$H = \frac{p^2}{2m} + V(r) - \frac{e}{mc}\boldsymbol{A}(0, t) \cdot \boldsymbol{p} + \frac{e^2}{2mc^2} \boldsymbol{A}(0, t)^2. \tag{2.13}$$

In this (semiclassical) treatment, the Hilbert space of the system is spanned by a complete set of atomic state vectors. The unitary operator

$$U(\boldsymbol{r}, t) = \exp[ie\boldsymbol{r} \cdot \boldsymbol{A}(0, t)/\hbar c] \tag{2.14}$$

transforms the Hamiltonian (2.13) to the form

$$\begin{aligned} H' &= U^{+}(\boldsymbol{r}, t)\, H\, U(\boldsymbol{r}, t) \\ &= \frac{1}{2m}\, U^{+}(\boldsymbol{r}, t)\, p^2\, U(\boldsymbol{r}, t) - \frac{e}{mc}\, \boldsymbol{A}(0, t) \cdot U^{+}(\boldsymbol{r}, t)\, \boldsymbol{p}\, U(\boldsymbol{r}, t) + V(r) + \frac{e^2}{2mc^2} \boldsymbol{A}(0, t)^2, \end{aligned} \tag{2.15}$$

since $U(\boldsymbol{r}, t)$ commutes with $\boldsymbol{r}$ and $U^{+}U = UU^{+} = 1$. From the operator identity

$$\exp(A)\, B \exp(-A) = B + [A, B] + \frac{1}{2!}\, [A, [A, B]] + \ldots, \tag{2.16}$$

it follows that $U^{+}(\boldsymbol{r}, t)\, \boldsymbol{p}\, U(\boldsymbol{r}, t) = \boldsymbol{p} + (e/c)\boldsymbol{A}(0, t)$, and therefore that

$$H' = p^2/2m + V(r).$$

The Schrödinger equation $H|\psi\rangle = i\hbar\, \partial|\psi\rangle/\partial t$ is transformed to

$$U^{+}HU|\phi\rangle = i\hbar\, U^{+} \frac{\partial}{\partial t}(U|\phi\rangle) = i\hbar \frac{\partial}{\partial t}\, |\phi\rangle + i\hbar\, U^{+} \frac{\partial U}{\partial t}\, |\phi\rangle, \tag{2.17}$$

where $|\psi\rangle = U|\phi\rangle$. Now the operator

$$U^{+}\frac{\partial U}{\partial t} = \exp[-\mathrm{i}e\boldsymbol{r}\cdot\boldsymbol{A}(0,t)/\hbar c]\,\frac{\mathrm{i}e}{\hbar c}\boldsymbol{r}\cdot\dot{\boldsymbol{A}}(0,t)\exp[\mathrm{i}e\boldsymbol{r}\cdot\boldsymbol{A}(0,t)/\hbar c]$$

$$= \frac{\mathrm{i}e}{\hbar c}\boldsymbol{r}\cdot\dot{\boldsymbol{A}}(0,t) = -\frac{\mathrm{i}e}{\hbar}\boldsymbol{r}\cdot\boldsymbol{E}^{\perp}(0,t), \tag{2.18}$$

so that eq. (2.17) may be written in the form

$$\left[\frac{p^{2}}{2m} + V(r) - e\boldsymbol{r}\cdot\boldsymbol{E}^{\perp}(0,t)\right]|\phi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\phi\rangle, \tag{2.19}$$

which establishes the equivalence of the two forms of the interaction Hamiltonian. Note that eq. (2.18) does not hold when the field is quantized, since in that case $\boldsymbol{A}$ and $\dot{\boldsymbol{A}}$ are noncommuting operators.

In the quantized-field case, the time evolution of the (Schrödinger-picture) state vector is governed by the Hamiltonian (2.13) plus the field Hamiltonian H_{F}. Eq. (2.15) is replaced by

$$H' = \frac{1}{2m}U^{+}p^{2}U - \frac{e}{mc}\boldsymbol{A}\cdot U^{+}\boldsymbol{p}U + V(r) + \frac{e^{2}}{2mc^{2}}A^{2} + \frac{1}{8\pi}\int \mathrm{d}^{3}r'\,[U^{+}(\boldsymbol{r})\boldsymbol{E}^{\perp}(\boldsymbol{r}')^{2}U(\boldsymbol{r}) + (\nabla'\times\boldsymbol{A}(\boldsymbol{r}'))^{2}]$$

$$= \frac{p^{2}}{2m} + V(r) + \frac{1}{8\pi}\int \mathrm{d}^{3}r'\,[\nabla'\times\boldsymbol{A}(\boldsymbol{r}')]^{2} + \frac{1}{8\pi}\int \mathrm{d}^{3}r'\,[U^{+}(\boldsymbol{r})\boldsymbol{E}^{\perp}(\boldsymbol{r}')U(\boldsymbol{r})]^{2}. \tag{2.20}$$

From the (equal-time) canonical commutation relation [48]

$$[A_{i}(\boldsymbol{r}), E_{j}^{\perp}(\boldsymbol{r}')] = -4\pi\mathrm{i}\hbar c\,\delta_{ij}^{\perp}(\boldsymbol{r}-\boldsymbol{r}'), \tag{2.21}$$

and the general identity (2.16), it follows that

$$U^{+}(\boldsymbol{r})E_{j}^{\perp}(\boldsymbol{r}')U(\boldsymbol{r}) = E_{j}^{\perp}(\boldsymbol{r}') - 4\pi e r_{i}\delta_{ij}^{\perp}(\boldsymbol{r}'), \tag{2.22}$$

using the usual (Cartesian-component) summation convention. Then

$$\int \mathrm{d}^{3}r'\,[U^{+}(\boldsymbol{r})\boldsymbol{E}^{\perp}(\boldsymbol{r}')U(\boldsymbol{r})]^{2} = \int \mathrm{d}^{3}r'\,[\boldsymbol{E}^{\perp}(\boldsymbol{r}') - 4\pi\boldsymbol{P}^{\perp}(\boldsymbol{r}')]^{2}, \tag{2.23}$$

where (in the dipole approximation)

$$\boldsymbol{P}(\boldsymbol{r}') = e\boldsymbol{r}\,\delta^{3}(\boldsymbol{r}') \tag{2.24}$$

is the polarization density. Thus,

$$H' = \frac{p^{2}}{2m} + V(r) + H_{\mathrm{F}} - e\boldsymbol{r}\cdot\boldsymbol{E}^{\perp}(0) + 2\pi\int \mathrm{d}^{3}r'\,\boldsymbol{P}^{\perp}(\boldsymbol{r}')^{2}. \tag{2.25}$$

It remains to investigate the ket vector $U^{+}\partial(U|\phi\rangle)/\partial t$ in the quantized-field case. Since the analysis here is in the Schrödinger picture, it follows simply that

$$U^+ \frac{\partial}{\partial t}(U|\phi\rangle) = U^+U \frac{\partial}{\partial t}|\phi\rangle = \frac{\partial}{\partial t}|\phi\rangle, \tag{2.26}$$

whereas, in the semiclassical case, the vector potential has a prescribed t-dependence and $\partial U/\partial t$ is nonvanishing.

The quantized-field counterpart of eq. (2.19) is then

$$\left[\frac{p^2}{2m} + V(r) + H_\text{F} - e\boldsymbol{r}\cdot\boldsymbol{E}^\perp(0) + 2\pi\int d^3r'\, \boldsymbol{P}^\perp(\boldsymbol{r}')^2\right]|\phi\rangle = i\hbar\frac{\partial}{\partial t}|\phi\rangle. \tag{2.27}$$

This establishes the equivalence of the two forms of the interaction Hamiltonian for the quantized-field case. Note, however, that, aside from the obvious difference that $\boldsymbol{E}^\perp(0)$ in eq. (2.27) is a q-number, eq. (2.27) differs from eq. (2.19) by the presence of the term

$$2\pi\int d^3r'\, \boldsymbol{P}^\perp(\boldsymbol{r}')^2$$

in the Hamiltonian. This term owes its existence to the commutation rule (2.21).

It is important to remember that the Hamiltonians

$$H = H_\text{A} + H_\text{F} - \frac{e}{mc}\boldsymbol{A}(0)\cdot\boldsymbol{p} + \frac{e^2}{2mc^2}\boldsymbol{A}(0)^2 \tag{2.28}$$

and

$$H' = H_\text{A} + H_\text{F} - e\boldsymbol{r}\cdot\boldsymbol{E}^\perp(0) + 2\pi\int d^3r'\, \boldsymbol{P}^\perp(\boldsymbol{r}')^2 \tag{2.29}$$

refer to *different* basis states, $|\psi\rangle$ and $|\phi\rangle = U^+|\psi\rangle$ respectively. Furthermore, the operator $\boldsymbol{E}^\perp(0)$ in eq. (2.29) is the transverse electric field; in the treatment of Power and Zienau [47], the $E^\perp(0)$ that appears in the transformed Hamiltonian H' is instead the transverse part of the displacement vector

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}. \tag{2.30}$$

Davidovich [49] has recently presented a detailed discussion of the two forms of the interaction Hamiltonian, and has emphasized the distinction between active and passive points of view in the transformation of H. In particular, he notes that the treatment of Power and Zienau corresponds to the passive viewpoint. The treatment here, in common with that of Woolley [50], corresponds to the active viewpoint, and in the transformed Hamiltonian $\boldsymbol{E}^\perp$ is indeed the transverse electric field operator. The reader is referred to the work of Davidovich [49] for a discussion of previous work and the various misconceptions that have persisted regarding the two forms H and H' of the Hamiltonian. In this article various parenthetical remarks concerning the two forms will be made at appropriate points, although it will be convenient to use the form H' throughout. Thus the Hamiltonian for the interaction of the atom with the radiation field is given by (2.29).

It should be emphasized that a fully classical, canonical transformation of the Hamiltonian (2.1) also leads to the form (2.25). The term $2\pi\int d^3r'\boldsymbol{P}^\perp(\boldsymbol{r}')^2$ differentiates the *semiclassical* form (2.19) of Schrödinger's equation from the fully quantum-mechanical form (2.27).

It remains therefore to obtain, for the final term on the right-hand side of eq. (2.29), the representation prescribed by eq. (2.7). Since

$$P^{\perp}(r')^2 = P(r')\cdot P^{\perp}(r')$$

$$= P_i(r')\left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3r'' \int \mathrm{d}^3k(\delta_{ij} - \hat{k}_i\hat{k}_j)\exp\{\mathrm{i}k\cdot(r' - r'')\}P_j(r'')$$

$$= e^2 r_i r_j \left(\frac{1}{2\pi}\right)^3 \delta^3(r')\int \mathrm{d}^3k(\delta_{ij} - \hat{k}_i\hat{k}_j)\exp(\mathrm{i}k\cdot r'), \tag{2.31}$$

it follows that

$$2\pi\int \mathrm{d}^3r'\, P^{\perp}(r')^2 = e^2 r_i r_j\left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^3k(\delta_{ij} - \hat{k}_i\hat{k}_j) = \frac{e^2}{4\pi^2}\int \mathrm{d}^3k[r^2 - (\hat{k}\cdot r)^2]$$

$$= \frac{2e^2}{3\pi}\, r^2\int \mathrm{d}k\, k^2 = \frac{2e^2}{3\pi}\int \mathrm{d}k\, k^2 \sum_m \sum_n \langle n(0)|r^2|m(0)\rangle\, \sigma_{nm}(t). \tag{2.32}$$

Thus the transformed Hamiltonian takes the form

$$H = \sum_n E_n\, \sigma_{nn}(t) + \sum_{k,\lambda} \hbar\omega_k\, a^{+}_{k\lambda}(t)\, a_{k\lambda}(t) - \mathrm{i}\hbar \sum_{k,\lambda}\sum_l\sum_m C_{k\lambda lm}\, \sigma_{lm}(t)\,(a_{k\lambda}(t) - a^{+}_{k\lambda}(t))$$

$$+ \frac{2e^2}{3\pi}\int \mathrm{d}k\, k^2 \sum_l\sum_m (r^2)_{lm}\, \sigma_{lm}(t), \tag{2.33}$$

where

$$C_{k\lambda lm} = \frac{1}{\hbar}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} \mu_{lm}\cdot \hat{e}_{k\lambda}, \tag{2.34}$$

and the identification

$$E^{\perp}(0,t) = \mathrm{i}\sum_{k,\lambda}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} (a_{k\lambda}(t) - a^{+}_{k\lambda}(t))\,\hat{e}_{k\lambda} \tag{2.35}$$

has been made. The polarization vectors $\hat{e}_{k\lambda}$ are taken to be real, i.e., a linear polarization basis has been used in the plane-wave expansion of the field.

2.4. *Heisenberg equations of motion*

Mathematically the problem of the interaction of the atom with the field is defined by the Heisenberg equations of motion based on the Hamiltonian (2.33), the commutation rule (2.3) for the field operators, and the commutation rule (2.9) for the atomic operators. One further commutation rule is required to define the problem unambiguously, namely, that involving the atomic and field operators. It will be assumed that the interaction is "switched on" at some time in the distant past, when the atom and field are completely independent systems. Before the interaction, therefore, the atomic operators commute with the field operators, and since the time evolution of these operators is unitary, it follows that equal-time commutators of atomic operators with field operators identically vanish. This fact renders the ordering of atomic and field operators in the interaction Hamiltonian irrelevant; regardless of the ordering chosen, the final "answers" must be the same.

Choosing a normal ordering, in which photon annihilation (creation) operators appear to the extreme right (left) in any operator product, we may write the Heisenberg equations of motion for the system in the form

$$\dot{a}_{k\lambda}(t) = -\mathrm{i}\omega_k a_{k\lambda}(t) + \sum_l \sum_m C_{k\lambda lm}\,\sigma_{lm}(t), \tag{2.36}$$

$$\begin{aligned}\dot{\sigma}_{ij}(t) = -\mathrm{i}\omega_{ji}\,\sigma_{ij}(t) - \sum_{k,\lambda} \sum_l \sum_m C_{k\lambda lm}(\delta_{lj}\,\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t))\,a_{k\lambda}(t) + \\ + \sum_{k,\lambda} \sum_l \sum_m C_{k\lambda lm}\,a^{+}_{k\lambda}(t)(\delta_{jl}\,\sigma_{im}(t\,) - \delta_{im}\,\sigma_{lj}(t)) - \frac{2\mathrm{i}e^2}{3\pi\hbar}\int \mathrm{d}k\,k^2 \sum_l \sum_m (r^2)_{lm}(\delta_{jl}\,\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t)),\end{aligned} \tag{2.37}$$

where $\hbar\omega_{ji} = E_j - E_i$.

Formally integrating the field equation (2.36), one finds

$$a_{k\lambda}(t) = a_{k\lambda}(0)\exp(-\mathrm{i}\omega_k t) + \sum_l \sum_m C_{k\lambda lm} \int_0^t \mathrm{d}t_1\,\sigma_{lm}(t_1)\exp\{\mathrm{i}\omega_k(t_1 - t)\}. \tag{2.38}$$

Assuming the atom–field interaction to be weak, the atomic operators should evolve very nearly according to their free evolution $\dot{\sigma}_{ij} = -\mathrm{i}\omega_{ji}\,\sigma_{ij}$. Hence one may use in eq. (2.38) the approximation [25, 40]

$$\sigma_{lm}(t_1) \cong \sigma_{lm}(t)\exp\{-\mathrm{i}\omega_{ml}(t_1 - t)\}. \tag{2.39}$$

This will be referred to as the *adiabatic approximation*, and may be shown to be equivalent to what is usually called the Weisskopf–Wigner approximation [29].

Using this approximation in the field equation (2.38), and using the result in eq. (2.37), it is found that

$$\begin{aligned}\dot{\sigma}_{ij}(t) = -\mathrm{i}\omega_{ji}\,\sigma_{ij}(t) &- \sum_{l,m} \sum_{k,\lambda} C_{k\lambda lm}(\delta_{jl}\,\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t))\,a_{k\lambda}(0)\exp(-\mathrm{i}\omega_k t) \\ &+ \sum_{l,m} \sum_{k,\lambda} C_{k\lambda lm}\,a^{+}_{k\lambda}(0)\exp(\mathrm{i}\omega_k t)(\delta_{jl}\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t)) \\ &- \sum_{l,m} \sum_{n,p} \Gamma_{lmnp}(\omega_{pn}, t)(\delta_{jl}\,\sigma_{im}(t)\,\sigma_{np}(t) - \delta_{im}\,\sigma_{lj}(t)\,\sigma_{np}(t)) \\ &+ \sum_{l,m} \sum_{n,p} \Gamma^{*}_{mlnp}(\omega_{pn}, t)(\delta_{jl}\,\sigma_{pn}(t)\,\sigma_{im}(t) - \delta_{im}\,\sigma_{pn}(t)\,\sigma_{lj}(t)) \\ &- \frac{2\mathrm{i}e^2}{3\pi\hbar}\int \mathrm{d}k\,k^2 \sum_{l,m} (r^2)_{lm}(\delta_{jl}\,\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t)),\end{aligned} \tag{2.40}$$

where

$$\Gamma_{lmnp}(\omega_{pn}, t) = \frac{1}{\hbar^2} \sum_{k,\lambda} \left(\frac{2\pi\hbar\omega_k}{V}\right) (\boldsymbol{\mu}_{lm} \cdot \hat{e}_{k\lambda})(\boldsymbol{\mu}_{np} \cdot \hat{e}_{k\lambda}) \int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_{pn})(t_1 - t)\}. \tag{2.41}$$

Note that $\Gamma_{lmnp}(\omega_{pn}, t) = 0$ for $l = m$ or $n = p$.

With the multiplication rule $\sigma_{im}(t)\,\sigma_{np}(t) = \delta_{mn}\,\sigma_{ip}(t)$, eq. (2.40) reads

$$\begin{aligned} \dot{\sigma}_{ij}(t) = -\mathrm{i}\omega_{ji}\,\sigma_{ij}(t) - X_{ij}(t) - \sum_{l,m} \sum_{n,p} \Gamma_{lmnp}(\omega_{pn}, t)[\delta_{jl}\delta_{mn}\sigma_{ip}(t) - \delta_{im}\,\delta_{jn}\,\sigma_{lp}(t)] \qquad (2.42) \\ + \sum_{l,m} \sum_{n,p} \Gamma^*_{mlnp}(\omega_{pn}, t)[\delta_{jl}\,\delta_{in}\,\sigma_{pm}(t) - \delta_{im}\,\delta_{nl}\sigma_{pj}(t)] - \frac{2\mathrm{i}e^2}{3\pi\hbar} \int \mathrm{d}k\, k^2 \sum_{l,m} (r^2)_{lm}(\delta_{jl}\,\sigma_{im}(t) - \delta_{im}\sigma_{lj}(t)), \end{aligned}$$

where

$$\begin{aligned} X_{ij}(t) = \sum_{l,m} \sum_{k,\lambda} C_{k\lambda lm}(\delta_{jl}\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t))\,a_{k\lambda}(0) \exp(-\mathrm{i}\omega_k t) \\ - \sum_{l,m} \sum_{k,\lambda} C_{k\lambda lm}\, a^+_{k\lambda}(0) \exp(\mathrm{i}\omega_k t)(\delta_{jl}\,\sigma_{im}(t) - \delta_{im}\,\sigma_{lj}(t)). \end{aligned} \tag{2.43}$$

Note that the adiabatic approximation (2.39) simplifies the atomic operator equation to the extent that the vacuum or source-free part of the field operator is explicitly contained entirely in the term $X_{ij}(t)$. This is very convenient, especially when expectation values are taken in a state for which the field state is the vacuum, for then $\langle X_{ij}(t)\rangle = 0$.

It is convenient to write eq. (2.42) in the form

$$\begin{aligned} \dot{\sigma}_{ij}(t) = -\mathrm{i}\omega_{ji}\,\sigma_{ij}(t) - X_{ij}(t) \\ - \left[\sum_m (\Gamma_{jmmj}(\omega_{jm}, t) + \Gamma^*_{immi}(\omega_{im}, t)) + \left(\frac{2\mathrm{i}e^2}{3\pi\hbar} \int \mathrm{d}k\, k^2\right)(r^2_{jj} - r^2_{ii})\right] \sigma_{ij}(t) \\ - \sum_m \sum_{p \neq j} \Gamma_{jmmp}(\omega_{pm}, t)\,\sigma_{ip}(t) + \sum_m \sum_p \Gamma_{mijp}(\omega_{pj}, t)\,\sigma_{mp}(t) + \sum_m \sum_p \Gamma^*_{mjip}(\omega_{pi}, t)\,\sigma_{pm}(t) \\ - \sum_m \sum_{p \neq i} \Gamma^*_{immp}(\omega_{pm}, t)\,\sigma_{pj}(t) - \frac{2\mathrm{i}e^2}{3\pi\hbar} \int \mathrm{d}k\, k^2 \left[\sum_{m \neq j} r^2_{jm}\,\sigma_{im}(t) - \sum_{m \neq i} r^2_{mi}\sigma_{mj}(t)\right]. \end{aligned} \tag{2.44}$$

Letting $V \to \infty$, it is easily shown that

$$\Gamma_{lmnp}(\omega_{pn}, t) = \frac{2\boldsymbol{\mu}_{lm} \cdot \boldsymbol{\mu}_{np}}{3\pi\hbar c^3} \int \mathrm{d}\omega\, \omega^3 \int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega - \omega_{pn})(t_1 - t)\}. \tag{2.45}$$

For times t long compared with any of the $|\omega_{pn}|^{-1}$, the well-known approximation [48]

$$\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega-\omega_{pn})(t_1-t)\} \cong \pi\delta(\omega-\omega_{pn}) - \mathrm{i}P\left(\frac{1}{\omega-\omega_{pn}}\right) \tag{2.46}$$

leads to

$$\begin{aligned}\Gamma_{lmnp}(\omega_{pn},t) &\cong \frac{2\omega_{pn}^3}{3\hbar c^3}\,\boldsymbol{\mu}_{lm}\cdot\boldsymbol{\mu}_{np}\,U(\omega_{pn}) - \frac{2\mathrm{i}}{3\pi\hbar c^3}\,\boldsymbol{\mu}_{lm}\cdot\boldsymbol{\mu}_{np}\,P\!\int\frac{\omega^3\mathrm{d}\omega}{\omega-\omega_{pn}}\\ &= \beta_{lmnp} - \mathrm{i}\gamma_{lmnp},\end{aligned} \tag{2.47}$$

where U is the unit step function. Then eq. (2.44) becomes

$$\begin{aligned}\dot{\sigma}_{ij}(t) = &-\mathrm{i}\left[\omega_{ji} - \sum_m(\gamma_{jmmj}-\gamma_{immi}) + \frac{2e^2}{3\pi\hbar c^3}\int\mathrm{d}\omega\,\omega^2(r_{jj}^2-r_{ii}^2)\right]\sigma_{ij}(t)\\ &-\sum_m(\beta_{jmmj}+\beta_{immi})\,\sigma_{ij}(t) - X_{ij}(t) - \frac{2\mathrm{i}e^2}{3\pi\hbar c^3}\int\mathrm{d}\omega\,\omega^2\left[\sum_{m\neq j} r_{jm}^2\sigma_{im}(t) - \sum_{m\neq i} r_{mi}^2\sigma_{mj}(t)\right]\\ &-\sum_m\sum_{p\neq j}(\beta_{jmmp}-\mathrm{i}\gamma_{jmmp})\,\sigma_{ip}(t) - \sum_m\sum_{p\neq i}(\beta_{immp}^*+\mathrm{i}\gamma_{immp}^*)\,\sigma_{pj}(t)\\ &+\sum_m\sum_p[(\beta_{mijp}-\mathrm{i}\gamma_{mijp})\,\sigma_{mp}(t) + (\beta_{mjip}^*+\mathrm{i}\gamma_{mjip}^*)\,\sigma_{pm}(t)].\end{aligned} \tag{2.48}$$

2.5. *The level shift and width*

For $i \neq j$ it is seen that the term

$$\Delta_{ji} = -\sum_m(\gamma_{jmmj}-\gamma_{immi}) + \frac{2e^2}{3\pi\hbar c^3}\int\mathrm{d}\omega\,\omega^2(r_{jj}^2-r_{ii}^2) \tag{2.49}$$

represents a frequency shift for the $j \to i$ transition. Writing

$$\hbar\Delta_{ji} = \Delta E_j - \Delta E_i, \tag{2.50}$$

one may identify

$$\Delta E_j = -\hbar\sum_m\gamma_{jmmj} + \frac{2e^2}{3\pi c^3}\,r_{jj}^2\int\mathrm{d}\omega\,\omega^2 \tag{2.51}$$

as the level shift of state $|j\rangle$. The second term on the right-hand side of eq. (2.51) may be written as

$$\frac{2e^2}{3\pi c^3}\int\mathrm{d}\omega\,\omega^2\sum_m\langle j|\mathbf{r}|m\rangle\cdot\langle m|\mathbf{r}|j\rangle = \frac{2}{3\pi c^3}\int\mathrm{d}\omega\,\omega^2\sum_m|\boldsymbol{\mu}_{jm}|^2\,, \tag{2.51'}$$

from which it follows that

$$\Delta E_j = -\frac{2}{3\pi c^3}\sum_m |\boldsymbol{\mu}_{jm}|^2\left[\mathrm{P}\int\frac{\omega^3\,\mathrm{d}\omega}{\omega-\omega_{jm}} - \int\mathrm{d}\omega\,\omega^2\right]$$
$$= -\frac{2}{3\pi c^3}\sum_m \omega_{jm}|\boldsymbol{\mu}_{jm}|^2\,\mathrm{P}\int\frac{\omega^2\,\mathrm{d}\omega}{\omega-\omega_{jm}}. \tag{2.52}$$

The divergence in ΔE_j may be reduced by subtracting off the expectation value in state $|j\rangle$ of the (unobservable) transverse self-energy of the free electron arising from the A^2 term [47],

$$E_j^{(0)} = \frac{e^2}{2mc^2}\langle j;\{0\}|A^2(0,t)|j;\{0\}\rangle, \tag{2.53}$$

where $\{0\}$ is the vacuum field state. It is easily shown that

$$E_j^{(0)} = \frac{e^2\hbar}{\pi mc^3}\int\mathrm{d}\omega\,\omega. \tag{2.54}$$

Using the Thomas–Reiche–Kuhn sum rule in the form

$$\frac{e^2\hbar}{m} = -\tfrac{2}{3}\sum_m |\boldsymbol{\mu}_{jm}|^2\,\omega_{jm}, \tag{2.55}$$

one obtains

$$E_j^{(0)} = -\frac{2}{3\pi c^3}\sum_m |\boldsymbol{\mu}_{jm}|^2\,\omega_{jm}\int\mathrm{d}\omega\,\omega, \tag{2.56}$$

and therefore the "observable" part of the level shift,

$$\Delta E_j = -\frac{2}{3\pi c^3}\sum_m \omega_{jm}|\boldsymbol{\mu}_{jm}|^2\left[\mathrm{P}\int\frac{\omega^2\,\mathrm{d}\omega}{\omega-\omega_{jm}} - \int\mathrm{d}\omega\,\omega\right]$$
$$= -\frac{2}{3\pi c^3}\sum_m \omega_{jm}^2|\boldsymbol{\mu}_{jm}|^2\,\mathrm{P}\int\frac{\omega\,\mathrm{d}\omega}{\omega-\omega_{jm}}. \tag{2.57}$$

This final expression is just the one to which Bethe [51] applied his famous renormalization in estimating the Lamb shift; the more familiar form is obtained by noting that

$$\omega_{jm}^2|\boldsymbol{\mu}_{jm}|^2 = \frac{e^2}{m^2}|\boldsymbol{p}_{jm}|^2, \tag{2.58}$$

so that

$$\Delta E_j = -\frac{2e^2}{3\pi m^2c^3}\sum_m |\boldsymbol{p}_{jm}|^2\,\mathrm{P}\int\frac{\omega\,\mathrm{d}\omega}{\omega-\omega_{jm}} = \frac{2}{3\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2\sum_m |\boldsymbol{p}_{jm}|^2\,\mathrm{P}\int\frac{E\,\mathrm{d}E}{E_j-E_m-E}, \tag{2.59}$$

where the integration is over "photon" energy.

As noted in the introduction to this section, the problem of spontaneous emission may be approached by first solving the problem for the electron in the Coulomb field of the nucleus, and then including the refinements introduced by the electron's interaction with the vacuum radiation field. It may happen that the solution to the first problem shows that two particular levels are degenerate, while the solution to the second problem reveals that the two levels are non-degenerate. The classic experimental confirmation of this circumstance is, of course, the Lamb shift. It is well known that the solution of the Dirac equation for the hydrogen atom problem shows that states with the same quantum numbers n and j are degenerate; thus, for example, the $2s_{1/2}$ and $2p_{1/2}$ states in hydrogen are predicted to be degenerate according to the Dirac theory for the interaction of the electron with the Coulomb field of the proton. While the possible nondegeneracy of these states had been a rather long-standing conjecture [52], experimental evidence was lacking prior to the work of Lamb and Retherford [53]. Using microwave technology developed during the second world war, they showed that the $2s_{1/2}$ state is higher in energy than the $2p_{1/2}$ state, the frequency separation being about 1060 Mc.

Although the Lamb shift must result from the electron's interaction with the radiation field – the "second part" of the spontaneous emission problem – the solution (eq. (2.59)) is divergent. To extract a meaningful number from eq. (2.59), Bethe [51] invoked Kramer's concept of mass renormalization. While this approach is well known in the context of Bethe's Lamb shift calculation, it will be useful for the purposes of later discussions to review it briefly here.

Note first that the expectation value over state $|j\rangle$ of the *free* electron energy, $\Delta E_j^{(F)}$, is obtained by taking $E_j - E_m = 0$ for all m in the denominator of (2.59):

$$\Delta E_j^{(F)} = -\left(\frac{2}{3\pi}\right)\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \sum_m |\boldsymbol{p}_{jm}|^2 \int \mathrm{d}E. \tag{2.60}$$

Thus the observable part of the level shift of state $|j\rangle$ is

$$\Delta E_j^{(\mathrm{obs})} = \Delta E_j - \Delta E_j^{(F)} = \left(\frac{2}{3\pi}\right)\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \sum_m (E_j - E_m)|\boldsymbol{p}_{jm}|^2 \int_0^{E_{\max}} \frac{\mathrm{d}E}{E_j - E_m - E}, \tag{2.61}$$

where the "cut-off energy" $E_{\max}$ is introduced on the grounds that the nonrelativistic theory is only applicable for photon energies E small compared with the electron rest energy mc^2; this means that the Lamb shift is assumed *a priori* to be essentially a nonrelativistic effect. Then

$$\Delta E_j^{(\mathrm{obs})} = \frac{2}{3\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \sum_m (E_m - E_j)|\boldsymbol{p}_{jm}|^2 \log\left|\frac{E_{\max}}{E_m - E_j}\right|, \tag{2.62}$$

since $E_{\max} \gg |E_m - E_j|$. Replacing the logarithm in the summation by its average value, and using

$$\sum_m |\boldsymbol{p}_{jm}|^2 (E_j - E_m) = -\tfrac{1}{2}\hbar^2 \int \mathrm{d}^3 r\, |\psi_j(r)|^2 \nabla^2 V(r),$$

one obtains finally, for $V(r) = -e^2/r$, the result

$$\Delta E_j^{(\mathrm{obs})} = \frac{2}{3\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \log\left|\frac{E_{\max}}{\langle E_m - E_j\rangle_{\mathrm{av}}}\right| \tfrac{1}{2}\hbar^2 e^2 |\psi_j(0)|^2, \tag{2.63}$$

where for hydrogen $|\psi_n(0)|^2 = 1/\pi n^3 a_0^3$ for s states and zero otherwise, a_0 being the Bohr radius.

Using $E_{\mathrm{max}} = mc^2$, Bethe found the expression (2.63) for $\Delta E_j^{(\mathrm{obs})}$ to be about 1040 Mc for the 2s state, in excellent agreement with the experimental result of Lamb and Retherford. It should be noted, however, that the expression (2.62) for $\Delta E_j^{(\mathrm{obs})}$ depends on the choice of the rather arbitrary cut-off energy E_{max}. The fully relativistic treatment yields a cut-off-independent result which is remarkably close to the experimentally observed Lamb shift.

To see that Bethe's approach does indeed correspond to mass renormalization, one needs only to write eq. (2.60) in the form

$$\Delta E_j^{(\mathrm{F})} = -\frac{2}{3\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \langle j|\boldsymbol{p}^2|j\rangle \int\limits_0^{E_{\mathrm{max}}} \mathrm{d}E = C\langle j|\boldsymbol{p}^2|j\rangle, \tag{2.64}$$

and note that in the atomic Hamiltonian the mass that appears in the kinetic energy term is an "observable" mass, which must therefore be considered as already containing the mass corresponding to kinetic energy $\Delta E^{(\mathrm{F})}$ as well as a "bare" mass. That is,

$$\frac{p^2}{2m} = \frac{p^2}{2m_{\mathrm{bare}}} + \Delta E^{(\mathrm{F})} = \left(\frac{1}{2m_{\mathrm{bare}}} + C\right)p^2 \cong \frac{p^2}{2m_{\mathrm{bare}}(1 - 2m_{\mathrm{bare}}C)}, \tag{2.65}$$

or,

$$m \cong (1 - 2m_{\mathrm{bare}}C)m_{\mathrm{bare}} \cong m_{\mathrm{bare}} + \frac{4e^2}{3\pi\hbar c^3} E_{\mathrm{max}}. \tag{2.66}$$

When the renormalized mass m is used in the atomic Hamiltonian, therefore, the energy $\Delta E^{(\mathrm{F})} = Cp^2$ is already accounted for, and since it arises again in the "second part" of the spontaneous emission problem, this spurious addition must be subtracted away; if this subtraction is not performed, the "electromagnetic mass" $m - m_{\mathrm{bare}}$ will have been added to m_{bare} twice. This is equivalent to saying that the interaction of the electron with the radiation field can never be "switched off", but in computing the radiative corrections we switch it on "again".

Returning again to eq. (2.48) it is seen that the linewidth, or the imaginary part of the level shift for the $j \to i$ transition, is

$$\beta_{ji} = \sum_m (\beta_{jmmj} + \beta_{immi}) = \frac{2}{3\hbar c^3}\left(\sum_{E_j > E_m} |\boldsymbol{\mu}_{jm}|^2 \omega_{jm}^3 + \sum_{E_i > E_m} |\boldsymbol{\mu}_{im}|^2 \omega_{im}^3\right). \tag{2.67}$$

It is a simple matter to show explicitly that Δ_{ji} and β_{ji} do indeed correspond to the line shift and width respectively of the $j \to i$ transition. This will not be done here, since lineshape questions are discussed in a later section. The results obtained above imply that the adiabatic approximation is the equivalent of second-order perturbation theory.

The last four terms on the right-hand side of eq. (2.48) oscillate at frequencies different from that of $\sigma_{ij}(t)$, and their effect is analogous to that of the counter-rotating term in the Bloch–Siegert problem. Denegeracy has been ignored, although it will be admitted for a special problem in a later subsection. The emphasis in this section, and throughout this article, is on physical interpretation rather than on detailed derivations.

2.6. *Radiation reaction as the mechanism responsible for spontaneous emission*

The use of the term "mass renormalization" above suggests that the concept of radiation reaction may be applied to the theory of spontaneous emission. This idea was in fact alluded to by Slater [24] in 1924:

> The part of the field originating from the given atom itself is supposed to induce a probability that that atom lose energy spontaneously, while radiation from external sources is regarded as inducing additional probabilities that it gain or lose energy, much as Einstein has suggested ....

Slater's discussion was in the context of a "virtual radiation" theory, but the idea that radiation reaction is the mechanism responsible for spontaneous emission is nevertheless unmistakable. While this interpretation can hardly be inferred from the Weisskopf–Wigner-type approach found in the literature, its viability is easily demonstrated.

Ackerhalt, Knight and Eberly [25] have made the following observation: If expectation values are taken on both sides of eq. [2.48] over a state for which the field state is the vacuum, $\langle X_{ij}(t)\rangle = 0$, and the vacuum fluctuations apparently do not play any explicit role in the damping of the atomic dipole and energy expectation values. Only the second term on the right-hand side of the field equation (2.38) therefore plays an explicit role, and this term represents the solution of the inhomogeneous Maxwell equation with the atomic polarization as the source, evaluated at the atom; that is, this term is the radiation reaction term. Using it in eq. (2.35), it follows that

$$\begin{aligned} E^{\perp}_{\mathrm{RR}}(t) &= \mathrm{i}\sum_{k,\lambda}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2}\left\{\sum_{l,m} C_{k\lambda lm}\int_0^t \mathrm{d}t_1\,\sigma_{lm}(t_1)\exp\{\mathrm{i}\omega_k(t_1-t)\} - \mathrm{h.c.}\right\}\hat{e}_{k\lambda} \\ &= \sum_{l,m}\left(\frac{2}{3c^3}\dddot{\sigma}_{lm}(t) + \sigma_{lm}(t)\,\frac{4}{3\pi}\int_0^K \mathrm{d}k\,k^2 - \ddot{\sigma}_{lm}(t)\,\frac{4}{3c^2\pi}\int_0^K \mathrm{d}k\right)\boldsymbol{\mu}_{lm} \\ &= \sum_{l,m}\left(\frac{2}{3c^3}\dddot{\sigma}_{lm}(t) + \frac{4K^3}{9\pi}\,\sigma_{lm}(t) - \frac{4K}{3\pi c^2}\,\ddot{\sigma}_{lm}(t)\right)\boldsymbol{\mu}_{lm}, \end{aligned} \tag{2.68}$$

where $K = E_{\max}/\hbar c = mc/\hbar$ for Bethe's choice of cut-off energy. The representation (2.7) for the electron coordinate $\boldsymbol{r}$ shows that eq. (2.68) is equivalent to

$$E^{\perp}_{\mathrm{RR}}(t) = \frac{2e}{3c^3}\,\dddot{\boldsymbol{r}}(t) - \frac{4eK}{3\pi c^2}\,\ddot{\boldsymbol{r}}(t) + \frac{4K^3 e}{9\pi}\boldsymbol{r}(t). \tag{2.69}$$

Eq. (2.69) differs from what one would expect on the basis of the classically familiar form of the radiation reaction field, where there is no counterpart of the third term on the right-hand side:

$$\boldsymbol{\varepsilon}^{\perp}_{\mathrm{RR}}(t) = \frac{2e}{3c^3}\dddot{\boldsymbol{r}}(t) - \frac{4eK}{3\pi c^2}\,\ddot{\boldsymbol{r}}(t) \tag{2.70}$$

But as emphasized by Davidovich [49], the transformation of the Hamiltonian (2.1) to the form (2.29) must be accompanied by a change in basis states. The electric field (2.69), although it is indeed the transverse electric field operator in the active view of the transformation effected by $U(\boldsymbol{r}, t)$, refers to a set of basis states transformed from that associated with the original

Hamiltonian (2.1); if eq. (2.70) refers to the set $\{|\psi\rangle\}$, then eq. (2.69) must be referred to the set $\{U^+|\psi\rangle\}$. Suppose, for example, that we are interested in the expectation value of the radiation reaction electric field operator in the state

$$|\phi\rangle = |\{0\}\rangle \otimes |\phi_A\rangle, \tag{2.71}$$

where $|\{0\}\rangle$ is the vacuum state of the radiation field and $|\phi_A\rangle$ is some atomic state. Then what must equal $\langle\phi|E^{\perp}_{RR}(t)|\phi\rangle$ is not $\langle\phi|\boldsymbol{\varepsilon}^{\perp}_{RR}(t)|\phi\rangle$ but rather

$$\langle\psi|UE^{\perp}_{RR}(t)U^+|\psi\rangle, \qquad \text{where } |\psi\rangle = U|\phi\rangle.$$

From eq. (2.22) it is seen that this means that

$$\langle E^{\perp}_{RR}(t)\rangle = \langle\boldsymbol{\varepsilon}^{\perp}_{RR}(t)\rangle + 4\pi\langle P^{\perp}(0)\rangle, \tag{2.72}$$

so that when the transformation in basis states is accounted for, it is clear that we are in fact dealing with just the usual radiation reaction field.

The Heisenberg equation of motion for the electron coordinate $r(t)$ is simply

$$m_{\mathrm{obs}}\ddot{r}(t) = -\nabla V(r) + eE^{\perp}_0(0, t) + (2e^2/3c^3)\dddot{r}(t), \tag{2.73}$$

where E_0 is the homogeneous solution of the (operator) Maxwell equation. Despite the formal resemblance which eq. (2.73) bears to the corresponding classical equation, it is not of much immediate use for calculational purposes, since it tells us nothing about individual atomic levels. For this purpose, the representation (2.7) for $r(t)$ must be used, and then the equations (2.48), which give information about the shifts and widths of individual levels, follow. When this is done, however, it is seen that the field no longer enters by itself, but only as one factor in a product with an atomic operator. This in turn forces us to pay close attention to the order in which the creation and annihilation parts of the field appear. If vacuum expectation values are to be taken, the normal ordering is of obvious convenience, since the term $X_{ij}(t)$ in eq. (2.48) has vanishing vacuum expectation value. The only terms pertaining to the field which survive the quantum-mechanical ensemble averaging are therefore the positive- and negative-frequency parts of the radiation reaction field. The damping of the atomic energy and dipole moment expectation values, and the accompanying radiative frequency shifts, therefore, are due entirely to the radiation reaction of the electron upon itself. This extension of the classical concept of radiation reaction to the quantum-electrodynamical regime is the essential contribution of Ackerhalt, Knight and Eberly [25].

While the interpretation of spontaneous decay and the Lamb shift as radiation reaction effects satisfies one's desire for a classical-like explanation, it must be remembered that the subject of radiation reaction has been a difficult one. The classical equation for a point charge subjected only to the field it creates is

$$m_{\mathrm{obs}}\ddot{r}(t) = (2e^2/3c^3)\dddot{r}(t), \tag{2.74}$$

an equation which admits a "runaway" solution. A discussion of attempts to eliminate divergent mass terms and to explain away the runaway solution is quite beyond the scope of this work; these attempts are legion, but in some respects a satisfactory understanding of radiation reaction is still lacking. An excellent introduction to the subject may be found in the textbook by

Panofsky and Phillips [54]. The reader is also referred to the recent quantum-mechanical approach of Sharp and Moniz [55], who give the point charge a spread of the order of the Compton radius and argue that, quantum mechanically, there is no divergent mass term or runaway solution. The approach used here follows the usual mass renormalization prescription; furthermore, the adiabatic approximation is equivalent to replacing the $\dddot{r}$ term on the right-hand side of eq. (2.70) by an $\dot{r}$ term, which is the usual approach for an oscillating charge, and avoids the runaway difficulty.

2.7. *The role of vacuum fluctuations in spontaneous emission*

The suggestion [25] that spontaneous emission be attributed to radiation reaction is in marked contrast to the usual heuristic picture in which the atom is forced to radiate by the vacuum field fluctuations. Regarding the Lamb shift, the usual picture is strikingly supported by Welton's simple derivation [23]. It is of interest then to show how the two points of view may be reconciled [27–29]. For this purpose, it is convenient to consider the simple two-state model of the atom. The main points of this section are equally demonstrable in the multistate formalism developed above.

Let $|1\rangle$ and $|2\rangle$ be respectively the lower and upper states of the two-state atom, with (real) transition dipole moment $\boldsymbol{\mu} = \langle 1|e\boldsymbol{r}|2\rangle$ connecting them. Furthermore, let $\omega_0 = (E_2 - E_1)/\hbar$, and suppose that $|1\rangle$ is a (nonradiative) ground state. The Heisenberg equations of motion for the two-state atom are, from eqs. (2.36) and (2.37),

$$\dot{a}_{k\lambda}(t) = -\mathrm{i}\omega_k a_{k\lambda}(t) + C_{k\lambda}(\sigma(t) + \sigma^+(t)), \tag{2.75}$$

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\sigma(t) + \sum_{k,\lambda} C_{k\lambda}\sigma_z(t)(a_{k\lambda}(t) - a^+_{k\lambda}(t)), \tag{2.76}$$

$$\dot{\sigma}_z(t) = 2\sum_{k,\lambda} C_{k\lambda}(\sigma(t) - \sigma^+(t))(a_{k\lambda}(t) - a^+_{k\lambda}(t)), \tag{2.77}$$

where $\sigma(t) = \sigma_{12}(t)$, $\sigma_z(t) = \sigma_{22}(t) - \sigma_{11}(t)$, and $C_{k\lambda} = C_{k\lambda 12}$. In writing eq. (2.76), we have omitted a term

$$-\frac{2\mathrm{i}e^2}{3\pi\hbar c^3}\int \mathrm{d}\omega\, \omega^2(r^2_{22} - r^2_{11})\,\sigma(t)$$

on the right-hand side. This term is strictly zero in the two-state model, since $r^2_{22} = r_{21}r_{12} = r^2_{11}$. Furthermore, the energy scale has been changed so that the atomic Hamiltonian is simply

$$H_{\mathrm{A}} = \tfrac{1}{2}\hbar\omega_0\sigma_z(t), \tag{2.78}$$

that is, the states $|1\rangle$ and $|2\rangle$ are of energies $-\frac{1}{2}\hbar\omega_0$ and $\frac{1}{2}\hbar\omega_0$, respectively. The operators σ_z, $\sigma_x = \sigma + \sigma^+$, and $\sigma_y = \mathrm{i}(\sigma - \sigma^+)$ obey the Pauli spin $-\frac{1}{2}$ algebra, in the usual notation. With the expression (2.35) for $E^\perp$, eqs. (2.76) and (2.77) take the form

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\,\sigma(t) - \frac{\mathrm{i}}{\hbar}\,\boldsymbol{\mu}\cdot \boldsymbol{E}^\perp(0, t)\,\sigma_z(t), \tag{2.79}$$

$$\dot{\sigma}_z(t) = -\frac{2i}{\hbar}\,\boldsymbol{\mu}\cdot\boldsymbol{E}^{\perp}(0,t)\,(\sigma(t) - \sigma^{+}(t)), \tag{2.80}$$

or

$$\dot{\sigma}_x(t) = -\omega_0\,\sigma_y(t), \tag{2.81}$$

$$\dot{\sigma}_y(t) = \omega_0\,\sigma_x(t) + \frac{2}{\hbar}\,\boldsymbol{\mu}\cdot\boldsymbol{E}^{\perp}(0,t)\,\sigma_z(t), \tag{2.82}$$

$$\dot{\sigma}_z(t) = -\frac{2}{\hbar}\,\boldsymbol{\mu}\cdot\boldsymbol{E}^{\perp}(0,t)\,\sigma_y(t), \tag{2.83}$$

where

$$\boldsymbol{E}^{\perp}(0,t) = \boldsymbol{E}_0^{\perp}(0,t) + \boldsymbol{E}_{\mathrm{RR}}^{\perp}(t). \tag{2.84}$$

$\boldsymbol{E}_0^{\perp}(\boldsymbol{r},t)$ is the solution of the homogeneous operator Maxwell equation

$$\nabla^2\boldsymbol{E}^{\perp}(\boldsymbol{r},t) - \frac{1}{c^2}\,\ddot{\boldsymbol{E}}^{\perp}(\boldsymbol{r},t) = 0, \tag{2.85}$$

while from eq. (2.68) we have

$$\boldsymbol{E}_{\mathrm{RR}}^{\perp}(t) = \left[\frac{2}{3c^3}\,\dddot{\sigma}_x(t) - \frac{4K}{3\pi c^2}\,\ddot{\sigma}_x(t) + \frac{4K^3}{9\pi}\,\sigma_x(t)\right]\boldsymbol{\mu}. \tag{2.86}$$

In a semiclassical approach to the two-state atom, the electric field is a c-number, a prescribed function of time. The atomic variables σ_x, σ_y, and σ_z may then be regarded as expectation values, and eqs. (2.81) to (2.83) have the well-known geometrical interpretation [56] corresponding to their equivalent vectorial form,

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{R}(t) \times \boldsymbol{\Omega}(t), \tag{2.87}$$

where $\boldsymbol{R}(t) = (\sigma_x(t), \sigma_y(t), \sigma_z(t))$ and $\boldsymbol{\Omega}(t) = ((2/\hbar)\boldsymbol{\mu}\cdot\boldsymbol{E}(t), 0, -\omega_0)$. This point is discussed in detail in section 3.

Since atomic and field operators commute, eqs. (2.82) and (2.83) may be written in the equivalent forms

$$\dot{\sigma}_y(t) = \omega_0\,\sigma_x(t) + \frac{2}{\hbar}\,\boldsymbol{\mu}\cdot[\boldsymbol{E}^{(-)}(0,t)\,\sigma_z(t) + \sigma_z(t)\,\boldsymbol{E}^{(+)}(0,t)], \tag{2.88}$$

and

$$\dot{\sigma}_z(t) = -\frac{2}{\hbar}\,\boldsymbol{\mu}\cdot[\boldsymbol{E}^{(-)}(0,t)\,\sigma_y(t) + \sigma_y(t)\,\boldsymbol{E}^{(+)}(0,t)], \tag{2.89}$$

where $\boldsymbol{E}^{(+)}$ and $\boldsymbol{E}^{(-)}$ are respectively the positive- and negative-frequency parts of the total quantized (transverse) electric field (2.84). Thus,

$$E^{(+)}(0,t) = E_0^{(+)}(0,t) + E_{\mathrm{RR}}^{(+)}(t), \tag{2.90}$$

where

$$E_0^{(+)}(0,t) = \mathrm{i} \sum_{k,\lambda} \left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} a_{k\lambda}(0) \exp(-\mathrm{i}\omega_k t)\, \hat{e}_{k\lambda}, \tag{2.91}$$

and $E^{(-)}(0,t) = E^{(+)}(0,t)^{+}$. When we solve for $E_{\mathrm{RR}}^{(\pm)}(t)$ in the adiabatic approximation, and take vacuum expectation values, we obtain, of course, the two-state versions of eqs. (2.48):

$$\langle\dot{\sigma}(t)\rangle = -\mathrm{i}(\omega_0 - \Delta - \mathrm{i}\beta)\langle\sigma(t)\rangle - \mathrm{i}(\Delta - \mathrm{i}\beta)\langle\sigma^{+}(t)\rangle, \tag{2.92}$$

$$\langle\dot{\sigma}_z(t)\rangle = -2\beta(1 + \langle\sigma_z(t)\rangle), \tag{2.93}$$

where

$$\beta = 2|\boldsymbol{\mu}_{12}|^2\,\omega_0^3/3\hbar c^3$$

and

$$\Delta = \frac{2|\boldsymbol{\mu}_{12}|^2}{3\pi\hbar c^3}\,\omega_0^2\,\mathrm{P}\!\int\! \mathrm{d}\omega \left(\frac{\omega}{\omega-\omega_0} - \frac{\omega}{\omega+\omega_0}\right). \tag{2.94}$$

Actually, the calculation results in a frequency shift term given by

$$\Delta' = \frac{2|\boldsymbol{\mu}_{12}|^2}{3\pi\hbar c^3}\,\mathrm{P}\!\int\! \mathrm{d}\omega \left(\frac{\omega^3}{\omega-\omega_0} - \frac{\omega^3}{\omega+\omega_0}\right), \tag{2.95}$$

which then leads to eq. (2.94) in the same way that eq. (2.49) leads to (2.57). There is a slight problem in this step, resulting from the fact that the term involving $r_{22}^2 - r_{11}^2$ vanishes for a two-state atom; but this difficulty is simply a defect of the two-state model and need not concern us.

The normal ordering once again leads to equations for the atomic energy and dipole moment expectation values in which the vacuum field seems to play no explicit role, the radiation reaction electric field apparently being responsible for the line shift and width. It must be emphasized, of course, that greater prominence is imparted to the effect of radiation reaction only after vacuum expectation values are taken. The vacuum field, i.e., the solution of the homogeneous operator wave equation, certainly plays a crucial role, even if it is hidden by taking vacuum expectation values. That role is to assure the preservation of equal-time commutators, without which the equations of motion are meaningless. For example, if the vacuum field is deleted from eq. (2.48) by setting the term $X_{ij}(t)$ equal to zero, the *operators* $\sigma_{ij}(t)$ with $i \neq j$ would decay to zero, so that the electron coordinate operator

$$\boldsymbol{r}(t) = \sum_i \sum_j \langle i|\boldsymbol{r}|j\rangle\, \sigma_{ij}(t)$$

would decay to zero – the electron spirals into the nucleus! The importance of retaining operators with vanishing expectation values in operator equations of motion has been emphasized very clearly by Senitzky [33].

But the vacuum field can be assigned a more explicit role in spontaneous emission. Obviously the vacuum field did not contribute to the expectation values above because of the use of normal ordering. This circumstance suggests that in a different ordering one may find an explicit contribution of the vacuum field to vacuum expectation values, and this is indeed the case [27–29]. Naturally, the final result must be the same as that obtained using the more convenient normal ordering, since equal-time atomic and field operators commute. Thus the problem becomes one of interpretation.

Suppose we consider the two-state-atom Heisenberg equations of motion in "anti-normal" order:

$$\dot{a}_{k\lambda}(t) = -\mathrm{i}\omega_k a_{k\lambda}(t) + C_{k\lambda}(\sigma(t) + \sigma^+(t)), \tag{2.96}$$

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0 \sigma(t) - \sum_{k,\lambda} C_{k\lambda}(\sigma_z(t)\, a^+_{k\lambda}(t) - a_{k\lambda}(t)\, \sigma_z(t)), \tag{2.97}$$

$$\dot{\sigma}_z(t) = -2 \sum_{k,\lambda} C_{k\lambda} a_{k\lambda}(t)\, (\sigma^+(t) - \sigma(t)) + 2 \sum_{k,\lambda} C_{k\lambda}(\sigma^+(t) - \sigma(t))\, a^+_{k\lambda}(t). \tag{2.98}$$

Formally integrating eqs. (2.96) and (2.98), and using the adiabatic approximation,

$$a_{k\lambda}(t_1)\, \sigma^+(t_1) \cong a_{k\lambda}(t)\, \sigma^+(t) \exp\{-\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\}, \text{ etc.,} \tag{2.99}$$

we find upon substitution into eq. (2.97) that

$$\begin{aligned}\dot{\sigma}(t) = {} & -\mathrm{i}\omega_0 \sigma(t) - \sum_{k,\lambda} C_{k\lambda}(\sigma_z(0)\, a^+_{k\lambda}(t) - a_{k\lambda}(t)\, \sigma_z(0)) \\ & + 2 \sum_{k,\lambda} \sum_{\mu,\nu} C_{k\lambda}\, C_{\mu\nu} \Big[a_{k\lambda}(t)\, a_{\mu\nu}(t)\, \sigma(t) \int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_\mu + \omega_0)(t_1 - t)\} \\ & - a_{k\lambda}(t)\, a^+_{\mu\nu}(t)\, \sigma(t) \int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_\mu - \omega_0)(t_1 - t)\} - a_{\mu\nu}(t)\, a^+_{k\lambda}(t)\, \sigma(t) \int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_\mu - \omega_0)(t_1 - t)\} \\ & + a^+_{\mu\nu}(t)\, a^+_{k\lambda}(t)\, \sigma(t) \int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_\mu - \omega_0)(t_1 - t)\} \Big],\end{aligned} \tag{2.100}$$

where "counter-rotating" terms, i.e., terms containing $\sigma^+(t)$, have been neglected. This approximation is justified on the grounds that $\sigma(t)$ and $\sigma^+(t)$ oscillate principally as $\exp(-\mathrm{i}\omega_0 t)$ and $\exp(\mathrm{i}\omega_0 t)$, respectively.

To remain in second order in the coupling constant, the field operators in the last term on the right-hand side of eq. (2.100) should be replaced by their free-field parts. When this is done, and vacuum expectation values are taken, we find that, within the adiabatic approximation,

$$\langle \dot\sigma(t)\rangle = -\mathrm{i}\omega_0\langle\sigma(t)\rangle - \sum_{k,\lambda} C_{k\lambda}(\langle\sigma_z(0)\, a^{+}_{k\lambda}(t)\rangle - \langle a_{k\lambda}(t)\,\sigma_z(0)\rangle) \tag{2.101}$$

$$- 2\sum_{k,\lambda} C^2_{k\lambda}\langle\sigma(t)\rangle\langle a_{k\lambda}(0)\, a^{+}_{k\lambda}(0)\rangle\left[\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\} + \int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_k + \omega_0)(t_1 - t)\}\right],$$

where, of course, $\langle a_{k\lambda}(0)\, a^{+}_{k\lambda}(0)\rangle = 1$.

In second order the second term on the right-hand side of eq. (2.101) is

$$- \sum_{k,\lambda} C_{k\lambda}(\langle\sigma_z(0)\, a^{+}_{k\lambda}(t)\rangle - \langle a_{k\lambda}(t)\,\sigma_z(0)\rangle) \tag{2.102}$$

$$\cong - \sum_{k,\lambda} C^2_{k\lambda}\left[\langle\sigma_z(0)\,\sigma(t)\rangle\int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_k + \omega_0)(t_1 - t)\} - \langle\sigma(t)\,\sigma_z(0)\rangle\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\}\right],$$

where, again, "counter-rotating" terms have been omitted. Using (2.102) in (2.101), we have

$$\langle\dot\sigma(t)\rangle = -\mathrm{i}\omega_0\langle\sigma(t)\rangle - \sum_{k,\lambda} C^2_{k\lambda}\left[\langle\sigma_z(0)\,\sigma(t)\rangle\int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_k + \omega_0)(t_1 - t)\}\right.$$

$$- \langle\sigma(t)\,\sigma_z(0)\rangle\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\} + 2\langle\sigma(t)\rangle\langle a_{k\lambda}(0)\, a^{+}_{k\lambda}(0)\rangle\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\}$$

$$\left. + 2\langle\sigma(t)\rangle\langle a_{k\lambda}(0)\, a^{+}_{k\lambda}(0)\rangle\int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k + \omega_0)(t_1 - t)\}\right]$$

$$\cong -\mathrm{i}\omega_0\langle\sigma(t)\rangle - \langle\sigma(t)\rangle\sum_{k,\lambda} C^2_{k\lambda}\left[(2\langle a_{k\lambda}(0)\, a^{+}_{k\lambda}(0)\rangle - 1)\right.$$

$$\left.\times\left(\int_0^t \mathrm{d}t_1 \exp\{-\mathrm{i}(\omega_k + \omega_0)(t_1 - t)\} + \int_0^t \mathrm{d}t_1 \exp\{\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\}\right)\right], \tag{2.103}$$

since $\sigma(t)\,\sigma_z(0) \cong \sigma(t)$ and $\sigma_z(0)\,\sigma(t) \cong -\sigma(t)$, to zeroth order in the coupling constant. Using the approximation (2.46), and letting $V \to \infty$, we obtain from eq. (2.103) the result,

$$\langle\dot\sigma(t)\rangle = -\mathrm{i}(\omega_0 - \Delta - \mathrm{i}\beta)\langle\sigma(t)\rangle, \tag{2.104}$$

which, in the approximation of neglecting counter-rotating terms, is just eq. (2.92). However, looking back to eq. (2.103), we may now interpret Δ (and similarly β) as

$$\Delta = 2\Delta_{\mathrm{VF}} - \Delta_{\mathrm{RR}}, \tag{2.105}$$

where $\Delta_{\mathrm{VF}} = \Delta$ is the contribution from the vacuum field fluctuations and $\Delta_{\mathrm{RR}} = \Delta$ is the contribution from the radiation reaction of the electron back on itself.

Thus it has been found that when normal ordering is used, the entire contribution to the radiative frequency shift comes from the radiation reaction, whereas when antinormal ordering is used, 2Δ comes from the vacuum field fluctuations, with another $-\Delta$ from the radiation reaction [27, 28]. If we had used a symmetric ordering (half normal and half anti-normal) we would have found the entire contribution to the shift and width of the spectral line stemming from vacuum field fluctuations [28]. For other possible choices of ordering, neither the radiation reaction nor vacuum fluctuations term produces the entire shift; rather, each contributes a portion complementary to the other.

We have shown that vacuum fluctuations can be considered a physical basis for the radiative frequency shift, as Welton [23] has suggested; it has also been shown that radiation reaction is an equally valid basis. Thus the two interpretations of the width and shift of the emitted line are equivalent, the interpretation depending as it does on the particular ordering chosen for *commuting* atomic and field operators. The two interpretations "are merely two sides of the same quantum-mechanical coin, with each ... being an oversimplification motivated by the ordering scheme adopted" [27].

With regard to the decay of $\langle\sigma_z(t)\rangle$ (cf. eq. (2.93)), it was stated in ref. [28] that "there is no ordering which would attribute the entire decay entirely to a vacuum fluctuation effect". This contradicts the analysis of ref. [27], where it is argued that both the $\langle\sigma(t)\rangle$ and $\langle\sigma_z(t)\rangle$ equations can be interpreted entirely on the basis of vacuum field fluctuations. In order to see how the difference arises, the analysis of ref. [27] relevant to the $\langle\sigma_z(t)\rangle$ equation will be briefly discussed here, using the notation of this work.

Since it does not affect our results, we take the rotating-wave-approximation version of the equations of motion,

$$\dot{\sigma}_z(t) = -2\sum_{k,\lambda} C_{k\lambda}(a_{k\lambda}(t)\,\sigma^+(t) + \sigma(t)\,a^+_{k\lambda}(t)), \tag{2.106}$$

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\,\sigma(t) + \sum_{k,\lambda} C_{k\lambda}a_{k\lambda}(t)\,\sigma_z(t), \tag{2.107}$$

and

$$\dot{a}_{k\lambda}(t) = -\mathrm{i}\omega_k a_{k\lambda}(t) + C_{k\lambda}\sigma(t), \tag{2.108}$$

in anti-normal order as in ref. [27]. From eq. (2.107),

$$\sigma(t) = \sigma(0)\exp(-\mathrm{i}\omega_0 t) + \sum_{k,\lambda} C_{k\lambda}\int_0^t \mathrm{d}t_1\, a_{k\lambda}(t_1)\,\sigma_z(t_1)\exp\{\mathrm{i}\omega_0(t_1 - t)\}$$

$$\cong \sigma(0)\exp(-\mathrm{i}\omega_0 t) + \sum_{k,\lambda} C_{k\lambda}a_{k\lambda}(t)\,\sigma_z(t)\int_0^t \mathrm{d}t_1\exp\{-\mathrm{i}(\omega_k - \omega_0)(t_1 - t)\} \tag{2.109}$$

in the adiabatic approximation

$$a_{k\lambda}(t_1)\,\sigma_z(t_1) \cong a_{k\lambda}(t)\,\sigma_z(t)\exp\{-\mathrm{i}\omega_k(t_1 - t)\}.$$

Substituting eq. (2.109) into (2.106), one obtains

$$\dot{\sigma}_z(t) \cong -2\sum_{k,\lambda} C_{k\lambda}(a_{k\lambda}(t)\,\sigma^+(0)\exp(\mathrm{i}\omega_0 t) + \sigma(0)\,a^+_{k\lambda}(t)\exp(-\mathrm{i}\omega_0 t))$$

$$-2\sum_{k,\lambda}\sum_{\mu,\nu} C_{k\lambda}C_{\mu\nu}\,\sigma_z(t)\,a_{k\lambda}(t)\,a^+_{\mu\nu}(t)\int_0^t \mathrm{d}t_1\exp\{\mathrm{i}(\omega_k-\omega_0)(t_1-t)\}$$

$$-2\sum_{k,\lambda}\sum_{\mu,\nu} C_{k\lambda}C_{\mu\nu}\,\sigma_z(t)\,a_{\mu\nu}(t)\,a^+_{k\lambda}(t)\int_0^t \mathrm{d}t_1\exp\{-\mathrm{i}(\omega_k-\omega_0)(t_1-t)\}. \tag{2.110}$$

To stay in second order, the field operators in the last two terms should be replaced by their free-field parts. Making this replacement and taking vacuum expectation values, we have

$$\langle\dot{\sigma}_z(t)\rangle = -2\sum_{k,\lambda} C_{k\lambda}(\langle a_{k\lambda}(t)\,\sigma^+(0)\rangle\exp(\mathrm{i}\omega_0 t) + \langle\sigma(0)\,a^+_{k\lambda}(t)\rangle\exp(-\mathrm{i}\omega_0 t)) \tag{2.111}$$

$$-2\sum_{k,\lambda} C^2_{k\lambda}\langle\sigma_z(t)\rangle\langle a_{k\lambda}(0)\,a^+_{k\lambda}(0)\rangle\left(\int_0^t \mathrm{d}t_1\exp\{\mathrm{i}(\omega_k-\omega_0)(t_1-t)\} + \int_0^t \mathrm{d}t_1\exp\{-\mathrm{i}(\omega_k-\omega_0)(t_1-t)\}\right),$$

where again we have written $\langle a_{k\lambda}(0)\,a^+_{k\lambda}(0)\rangle$ rather than unity to show explicitly the effect of the vacuum field. Then the second term in eq. (2.111), due to vacuum field fluctuations, may be written as $-4\beta\langle\sigma_z(t)\rangle$ if we use the approximation (2.46),

$$\langle\dot{\sigma}_z(t)\rangle = -2\sum_{k,\lambda} C_{k\lambda}(\langle a_{k\lambda}(t)\,\sigma^+(0)\rangle\exp(\mathrm{i}\omega_0 t) + \langle\sigma(0)\,a^+_{k\lambda}(t)\rangle\exp(-\mathrm{i}\omega_0 t)) - 4\beta\langle\sigma_z(t)\rangle. \tag{2.112}$$

Senitzky [27] now considers the initial atomic state $|+\rangle$, the upper energy state, and considers the short-time limit

$$\langle\dot{\sigma}_z(t)\rangle \cong -4\beta\langle\sigma_z(0)\rangle = -4\beta, \tag{2.113}$$

consistent with the perturbation expansion. This gives the correct Einstein A-coefficient for the decay rate of the excited atomic state, resulting entirely from vacuum field fluctuations.

In ref. [28], however, the step from eq. (2.112) to (2.113) has not been considered entirely justified, simply because for the case in which the initial atomic state is the lower energy state $|-\rangle$, we would have

$$\langle\dot{\sigma}_z(t)\rangle \cong -4\beta\langle\sigma_z(0)\rangle = 4\beta, \tag{2.114}$$

corresponding to spontaneous absorption from the vacuum! To rectify this situation we must consider in more detail the first term on the right-hand side of eq. (2.112). Using

$$a_{k\lambda}(t) \cong a_{k\lambda}(0)\exp(-\mathrm{i}\omega_k t) + C_{k\lambda}\sigma(t)\int_0^t \mathrm{d}t_1\exp\{\mathrm{i}(\omega_k-\omega_0)(t_1-t)\},$$

we have

$$2\sum_{k,\lambda} C_{k\lambda}(\langle a_{k\lambda}(t)\,\sigma^+(0)\rangle \exp(i\omega_0 t) + \langle \sigma(0)\,a^+_{k\lambda}(t)\rangle \exp(-i\omega_0 t)) \cong$$

$$2\sum_{k,\lambda} C^2_{k\lambda}(\langle \sigma(t)\,\sigma^+(0)\rangle \exp(i\omega_0 t)\int_0^t dt_1 \exp\{i(\omega_k-\omega_0)(t_1-t)\}$$

$$+\langle \sigma(0)\,\sigma^+(t)\rangle \exp(-i\omega_0 t)\int_0^t dt_1 \exp\{-i(\omega_k-\omega_0)(t_1-t)\})$$

$$\cong 2\sum_{k,\lambda} C^2_{k\lambda}\langle \sigma(t)\,\sigma^+(t)\rangle\left(\int_0^t dt_1 \exp\{i(\omega_k-\omega_0)(t_1-t)\} + \int_0^t dt_1 \exp\{-i(\omega_k-\omega_0)(t_1-t)\}\right)$$

$$= 4\beta\langle \sigma(t)\,\sigma^+(t)\rangle = -2\beta(-1+\langle \sigma_z(t)\rangle) \tag{2.115}$$

when the approximation (2.46) is again used.

Note that this term is due to radiation reaction. Using the result (2.115) in eq. (2.112), we have finally,

$$\langle \dot\sigma_z(t)\rangle = -2\beta(1+\langle \sigma_z(t)\rangle), \tag{2.116}$$

which is just eq. (2.93). Since the term (2.115) stems from the radiation reaction part of the field, the decay of $\langle \sigma_z(t)\rangle$ in the anti-normally ordered calculation is attributable to both vacuum field fluctuations and radiation reaction.

We have shown, therefore, that for the decay of $\langle \sigma_z(t)\rangle$ we need both the vacuum field and the source field. It may be shown that this statement remains valid regardless of the ordering scheme used. The fact that atomic fluctuations are necessary for spontaneous emission has been emphasized by Fain [57]. He points out that a ground-state atom does not undergo spontaneous absorption due to the vacuum field fluctuations because in this state they are exactly cancelled by the atomic fluctuations. This fact shows the importance of having both types of fluctuations contributing to the overall physical process.

2.8. *Discussion*

The unorthodox approach to spontaneous emission used in this section is well suited for a comparison of the quantum-electrodynamical and semiclassical theories. The Heisenberg-picture method provides insight into the dynamics of spontaneous emission, and it is unlikely that the problem of the interpretation of the radiative corrections can be resolved as neatly by any other approach. For the purpose of comparing quantum-electrodynamical and semiclassical theories, the problem of interpretation is of paramount importance. The Heisenberg equations of motion are superficially so similar to the semiclassical equations of the next section that the quest for the range of validity of semiclassical radiation theories must proceed quite beyond the calculation of various numerical magnitudes. In this context, Einstein's remark that "every theory is true, provided you suitably associate its symbols with observed quantities" [58], does not seem entirely facetious.

Whereas radiation reaction and vacuum fluctuations in spontaneous emission are, according to Senitzky [27], two sides of the same quantum-mechanical coin, it is evident that the corresponding semiclassical coin can only be one-sided: There can be no vacuum fluctuations in a purely semiclassical treatment. Quantum mechanically, the vacuum field appears, naturally, simply as the homogeneous solution of the Maxwell equation. In a classical treatment of the field, however, it is customary to take as zero the homogeneous solution of the corresponding equation: If there are no sources, there are no fields. The homogeneous solution of the quantum-mechanical operator Maxwell equation cannot be so dispensed, as it is required for the preservation of commutation relations and therefore the Heisenberg uncertainty relations. We have seen, for example, that without the fluctuating influence of the vacuum field, an atomic electron would apparently collapse into the nucleus. The replacement of the usual (vanishing) classical homogeneous solution with a fluctuating zero-point field of energy density $\frac{1}{2}\hbar\omega$ associated with each plane-wave mode leads to an interesting stochastic theory of the radiation field. This theory, which is suggested by the quantum-electrodynamical theory but seems to have an independent *raison d'être*, is discussed in section 4.

For further discussion of recent work on the theory of spontaneous emission the reader is referred especially to the monograph by Agarwal [59]. The role of the radiation reaction field in spontaneous emission has also been discussed by Bullough [60] and Agarwal [61], who has also noted the intimate connection between radiation reaction and vacuum fluctuations.

The recent interest in theories of spontaneous emission may be traced to the success of the semiclassical approach in Lamb's laser theory [62]. This triumph of the semiclassical approach raises questions as to its precise limits of applicability. It is natural to study the problem of spontaneous emission as well as other problems which played an important role in the historical development of the quantized-field theory. Bohr seems to have been the first person to recognize in the phenomenon of spontaneous emission a renunciation of classical notions. Whereas Einstein [2] referred to transitions "without excitation by an external cause", Bohr considered them to be purely *spontaneous*, i.e., acausal [63]. Indeed, it is now recognized that the study of the phenomenon responsible for most of the light around us requires consideration of the most subtle differences between the classical and quantum theories.

3. Semiclassical approach to spontaneous emission

3.1. *Schrödinger's radiation theory*

In classical electrodynamics the electric and magnetic fields are prescribed functions of time and space. The fields are uniquely determined at any space-time point by the specification of their sources. Realistically there is, of course, some uncertainty in the fields, since the sources cannot be specified precisely in every detail. In the usual sense, however, this uncertainty is not regarded as "fundamental", for should it happen that the sources are precisely specified at an initial space-time point, their equations of motion in principle uniquely determine them, and therefore the fields they generate, at all other space-time points. Quantum mechanically, there is a fundamental uncertainty in the specification of the sources, i.e., the Heisenberg uncertainty principle is operative; thus there must be a fundamental uncertainty in the fields generated by the sources. The fields therefore must also be treated according to the dictates of quantum mechanics. In other words, the field variables must be regarded as operators in a Hilbert space: The field is "quantized".

It is well known that the Hamiltonian for the free electromagnetic field corresponds to that for an assembly of uncoupled simple harmonic oscillators, each mode representing a single oscillator. In analogy to the atomic operators of the preceding section, we may define the field operators

$$\sigma_{mn}(t) = U^{+}(t)\,\sigma_{mn}(0)\,U(t) = |m\rangle\langle n|\exp\{-\mathrm{i}(E_n - E_m)t/\hbar\} \tag{3.1}$$

for each mode, where

$$H_{\mathrm{F}}|n\rangle = n\hbar\omega|n\rangle, \tag{3.2}$$

$U(t) = \exp(-(\mathrm{i}/\hbar)H_{\mathrm{F}}t)$, and ω is the mode (circular) frequency. Since the integer n specifying the state of the mode in this (Fock-state) representation may assume any positive integral value, it proves convenient to define the operator sum

$$a(t) = \sum_{n=1}^{\infty} \sqrt{n}\,\sigma_{n-1,n}(t). \tag{3.3}$$

From the commutation rule

$$[\sigma_{nm}, \sigma_{ij}] = \delta_{im}\sigma_{nj} - \delta_{jn}\sigma_{im}, \tag{3.4}$$

which has the same form as the corresponding rule (2.9) for the atomic operators, it follows that

$$[a(t), a^{+}(t)] = 1, \tag{3.5}$$

that is, a and a^{+} are just the usual "photon" annihilation and creation operators. The important point is that, while the field and atomic σ-operators obey the same commutation rules, the field σ-operators are more conveniently collected in the manner of eq. (3.3).

The equations for a and a^{+} have the same formal structure as the equations for c-numbers α and α^{*} in classical electrodynamics. Thus we may regard classical electrodynamics as a theory in which the field σ-operators are c-numbers. But classical electrodynamics is a very successful theory, so the c-number treatment of the field σ-operators must reflect some "element of truth". The reason for this circumstance is well known from an intimately related point of view: An arbitrarily large number n of "photons" may occupy the same state, and when this situation obtains, it is accurate to regard the photon wave function as defining a classical field distribution.

The treatment of the atomic σ-operators as c-numbers corresponds, of course, to the original Schrödinger interpretation of the wave function. When Schrödinger calculated the wave function $\psi(\boldsymbol{r}, t)$ for an electron in the Coulomb field of the nucleus, he did no suppose it to have any probabilistic connotations; $|\psi|^2$ represented the spatial distribution of a "smeared-out" electron. The electric dipole moment for the electron-nucleus system takes the form

$$\boldsymbol{\mu}(t) = e\int \mathrm{d}^3r\,\boldsymbol{r}\,|\psi(\boldsymbol{r}, t)|^2 = e\langle\psi(\boldsymbol{r}, t)|\boldsymbol{r}|\psi(\boldsymbol{r}, t)\rangle \tag{3.6}$$

in the Dirac notation, with analogous results for higher multipole terms. In the notation of the previous section, we may write eq. (3.6) in the form

$$\boldsymbol{\mu}(t) = \sum_m \sum_n e\, \mathbf{r}_{mn} \langle \sigma_{mn}(t) \rangle . \tag{3.7}$$

Schrödinger had created the first semiclassical theory of the interaction of light with atoms. Given the wave function, charge distributions could be calculated, and these charge distributions then fell into the domain of classical electrodynamics.

Aside from its conceptual appeal, the Schrödinger semiclassical theory had the advantage of providing a simple explanation for the absence of spontaneous emission from the ground state. Thus for the ground state $\psi_0(\mathbf{r}) \exp(-iE_0 t/\hbar)$ the charge density $|\psi|^2$ is a constant in time and the system is nonradiative; a serious problem that had confronted the old Bohr model had been given, according to Schrödinger [64], "A surprisingly simple solution". Unfortunately, the solution may be applied to *any* stationary atomic state, so that Schrödinger's semiclassical theory did not predict spontaneous radiation from an atom excited to a pure excited state. Obviously Schrödinger must have recognized this, but did not regard it in any way as a defect of his theory. In fact, the notion of "quantum jumps" was, throughout his career, completely foreign to Schrödinger. The very notion of discrete energy levels was considered by Schrödinger to be superfluous; term *differences* were what mattered. "Partial oscillations" of the *wavelike* electron, which correspond, loosely, to the quantities $\langle \sigma_{mn}(t) \rangle$ in eq. (3.7), held more physical significance than the idea of quantum jumps, and as Schrödinger wrote in a letter to Planck in 1926, "The energetic property of the individual partial oscillation is its frequency" [65].

Despite its conceptual advantages, the Schrödinger interpretation of the wave function – and the radiation theory it implies – is not entirely satisfactory. Of the unsatisfactory aspects of his theory, Schrödinger was fully aware. On the other hand, he never fully accepted any probabilistic interpretation of the wave function. The difficulties besetting the Schrödinger interpretation are reviewed in the next section. This section is devoted to a theory of spontaneous emission which proceeds from the Schrödinger interpretation of the wave function. This theory, advanced by Jaynes and his collaborators [4–6], has been aptly called the "neoclassical" theory of spontaneous emission [6]. The following remark by an historian of the quantum theory suggests one important reason for the need to carefully delineate the range of validity of the neoclassical theory: "Should Jaynes' neoclassical approach, which so far seems to be only in an initial stage of its development, prove to be viable on future evidence, Schrödinger's semiclassical interpretation of quantum mechanics may well be destined to command much higher respect than it does today" [66].

The proponents of the neoclassical theory suggest that the semiclassical approach might not be merely an accurate approximation scheme in certain instances where the Schrödinger and probabilistic interpretations of the wave function yield practically equivalent results. They suggest that the semiclassical approach might be valid even in the limit of vanishing external fields, or in effect that radiation fields might in every instance be amenable to c-number treatments. In discussing the neoclassical theory *vis-à-vis* the quantum-electrodynamical, care must be taken to avoid the logical inconsistency of attacking the former with concepts appropriate only to the latter. Thus it is quite irrelevant, for example, to argue that the neoclassical theory violates the Heisenberg uncertainty principle [67], since neoclassical theory is essentially based precisely on the abrogation of such quantum-mechanical notions. In this section it is convenient to employ a certain terminology which may in some instances be strictly appropriate only to the quantum theory. The reader is urged to bear in mind that certain terms may have radically different connotations in the two theories. It is hoped that the essence of neoclassical theory will be sufficiently clear that the problem of semantics will not be a serious one.

3.2. *The neoclassical equations for spontaneous emission*

In the neoclassical theory the expectation values of atomic transition dipole moments are taken to be actual, classically radiating dipole moments. The radiation reaction of such a dipole on itself leads to damping of its oscillation or "spontaneous emission" of radiation. The Hilbert space for the system of radiation and matter is defined entirely by the Hilbert space of the free atom, and in this Hilbert space the field is a scalar. Thus the term "expectation value" can only refer to the atomic state $|\phi_A\rangle$, the field having no quantum-mechanical state-description. The dipole source is given by eq. (3.7).

It is important to note that in the neoclassical theory *all* electromagnetic fields are c-numbers, including the field generated by the *quantum-mechanical* atom. Now in the previous section, expectation values of certain operator products arose; these were products of atomic operators with field operators. The field operators were each a sum of two parts, one corresponding to the free field and the other to the radiation reaction field of the electron back on itself. It is clear then that a simple decorrelation of these operator-product expectation values must result in the corresponding equations of the neoclassical theory. Starting with the atomic operator equations, one replaces all operators with their expectation values; expectation values of operators taken with respect to the atomic Hilbert space are the only quantities that can be sensibly compared with corresponding quantities of the neoclassical theory. The terms containing the free or vacuum field contribution in particular must be replaced by c-numbers characterizing the prescribed (classical) applied field. For the problem of pure spontaneous emission in the vacuum, there are no applied fields and these terms are taken to be zero. There are no "vacuum" fields in the neoclassical theory as it has been enunciated to date; the vanishing solution of the homogeneous Maxwell equation is chosen.

The neoclassical and quantum-electrodynamical theories are most conveniently compared by considering the two-state model of the atom. From the discussion above, it follows that the decorrelated versions of eqs. (2.81)–(2.83) must yield the equations of the neoclassical model of a two-state atom interacting with radiation. The resulting equations have the form

$$\dot{x}(t) = -\omega_0 y(t), \tag{3.8}$$

$$\dot{y}(t) = \omega_0 x(t) + (2/\hbar)\,\boldsymbol{\mu}\cdot \boldsymbol{E}^{\perp}(0,t)\, z(t), \tag{3.9}$$

$$\dot{z}(t) = -(2/\hbar)\boldsymbol{\mu}\cdot \boldsymbol{E}^{\perp}(0,t)\, y(t), \tag{3.10}$$

where $\boldsymbol{E}^{\perp}(0,t)$ is now taken to be a classical (c-number) field and $x(t)$, $y(t)$, $z(t)$ have been written in place of $\langle\sigma_x(t)\rangle$, $\langle\sigma_y(t)\rangle$, and $\langle\sigma_z(t)\rangle$. These equations are effectively the starting equations of Jaynes and Stroud [5].

As in the quantum-electrodynamical approach, the field $\boldsymbol{E}^{\perp}(0,t)$ is the sum of the radiation reaction field and an external field. Since there are no "vacuum" fields in neoclassical theory, the only field that can enter into the above equations for the problem of spontaneous emission into free space is the radiation reaction field $\boldsymbol{E}^{\perp}_{\mathrm{RR}}(t)$ of the electron:

$$\boldsymbol{E}^{\perp}(t) = \boldsymbol{E}^{\perp}_{\mathrm{RR}}(t) = \left[\frac{2}{3c^3}\,\dddot{x}(t) - \frac{4K}{3\pi c^2}\,\ddot{x}(t)\right]\boldsymbol{\mu}\,. \tag{3.11}$$

Jaynes and Stroud [5] now follow the usual approximation of replacing $\dddot{x}(t)$ by $-\omega_0^2\dot{x}(t)$:

$$E_{\mathrm{RR}}^{\perp}(t) \cong \left[-\frac{2\omega_0^2}{3c^3}\,\dot{x}(t) + \frac{4\omega_0^2 K}{3\pi c^2}\,x(t)\right]\boldsymbol{\mu}\,. \tag{3.12}$$

This replacement eliminates the possibility of run-away solutions and is closely akin to the adiabatic approximation of the preceding section. It may be regarded as the semiclassical equivalent of the Weisskopf–Wigner approximation. Substitution of the result (3.12) into eqs. (3.9) and (3.10) yields the equations

$$\dot{y}(t) = \omega_0 x(t) + \frac{4\omega_0^3\mu^2}{3\hbar c^3}\,y(t)\,z(t) + \frac{8K\,\omega_0^2\mu^2}{3\pi\hbar c^2}\,x(t)\,z(t), \tag{3.13}$$

$$\dot{z}(t) = -\frac{4\mu^2\,\omega_0^3}{3\hbar c^3}\,y^2(t) - \frac{8\omega_0^2 K}{3\pi\hbar c^2}\,x(t)\,z(t). \tag{3.14}$$

For more immediate comparison with the quantum-electrodynamical results of the preceding section it is convenient to define the quantity $S(t) = \frac{1}{2}(x(t) - \mathrm{i}y(t))$, the equation of motion for which is

$$\dot{S}(t) = -\mathrm{i}\omega_0 S(t) + \beta(S(t) - S^*(t))\,z(t) - \frac{4\mathrm{i}K\mu^2\,\omega_0^2}{3\pi\hbar c^2}\,(S(t) + S^*(t))\,z(t), \tag{3.15}$$

where again $\beta = 2\mu^2\,\omega_0^3/3\hbar c^3$. $S(t)$ corresponds to the expectation value $\langle\sigma(t)\rangle$ used in the quantum-electrodynamical approach. To compare with the equations of Jaynes and Stroud [5], we define $S(t) = u(t)\exp(-\mathrm{i}\omega_0 t)$, obtain the equation of motion for $u(t)$, and then make the "counter-rotating" approximation. The equations of motion for the real and imaginary parts of $u(t)$, denoted again by $x(t)$ and $-y(t)$, respectively, are then

$$\dot{x}(t) = \beta\,x(t)\,z(t) - \gamma\,y(t)\,z(t), \tag{3.16}$$

and

$$\dot{y}(t) = \beta\,y(t)\,z(t) + \gamma\,x(t)\,z(t), \tag{3.17}$$

where $\gamma = 4K\mu^2\,\omega_0^2/3\pi\hbar c^2$. In the same approximation we obtain the equation for $z(t)$:

$$\dot{z}(t) = -\beta(1 - z^2(t)), \tag{3.18}$$

where use has been made of the fact that $x^2 + y^2 + z^2 = 1$. Equations (3.16)–(3.18) are the neoclassical equations of Jaynes and Stroud [5].

The decorrelation of atomic and field operators in the fully quantum-mechanical theory yields the neoclassical *equations* but does not define the neoclassical theory. The neoclassical theory is based on the Schrödinger interpretation of the wave function. This interpretation leads to equations of motion which can be obtained by decorrelating expectation values of products of atomic and field operators in the quantum-electrodynamical equations. But the fact that the neoclassical equations can be obtained by invalid manipulations of the quantum-electrodynamical equations does not imply that neoclassical electrodynamics is invalid – unless, of course, the validity of quantum electrodynamics is presupposed.

3.3. *Implications of the neoclassical equations*

In contrast to the equations derived by quantizing the radiation field, the neoclassical equations for the atomic dipole moment and energy are nonlinear. One consequence of this nonlinearity is that the theory predicts a spontaneous generation of harmonics of the fundamental frequency ω_0. This is seen from an approximate integration of the neoclassical equations (3.8), (3.13) and (3.14). In addition to the component of the dipole moment which oscillates with frequency ω_0, one finds components also at $3\omega_0$, $5\omega_0$, etc.; that only odd harmonics are generated follows from the dipole selection rule. These additional components are very weak – of second and higher order in β/ω_0 – but their existence supports a statement made by Heisenberg in his objection to Schrödinger's semiclassical theory. Heisenberg argued that, according to the Schrödinger theory, "the radiation emitted by an atom could be expanded into a Fourier series in which the frequencies of the overtones are integral multiples of a fundamental frequency. The frequencies of the atomic spectral lines, however, according to quantum mechanics, are never such integral multiples of a fundamental frequency ..." [68].

Another prediction of the neoclassical theory is that an atom excited to a pure stationary state cannot radiate. For an atom which has been pumped into the upper energy state, $z(0) = 1$ and $x(0) = y(0) = 0$, so that $\dot{z}(0) = 0$. Exactly as in the Schrödinger theory, the neoclassical theory accounts for the absence of radiation from the ground state, but predicts in addition that every atomic stationary state is nonradiative. These results are to be expected, since the neoclassical theory, in its physical content, is identical to the Schrödinger semiclassical theory. The formalism has changed, but the theory remains the same.

The quantum and neoclassical radiation theories also differ in their predictions of the emission lineshape. The quantum theory predicts that the emission lineshape is approximately Lorentzian, with half-width β. The approximation lies in the derivation of the exponential decay law. The problem of the interaction of an atom with an infinite number of field modes cannot be handled exactly; the adiabatic approximation, which is essentially a Markovian or "Weisskopf–Wigner" approximation, is invoked for tractability. The "ladder" aspect of the Weisskopf–Wigner approximation has been elucidated in diagrammatic language by Knight and Allen [69]. Fermi [70] provided a simple coupled-oscillator model which captures the physical essence of the Weisskopf–Wigner approximation. Two linearly coupled oscillators sinusoidally exchange the energy that initially is assumed to be contained entirely in one of the oscillators. As more oscillators are allowed to be coupled into the system, the period with which the initially excited oscillator repossesses its initial energy monotonically increases until finally, as the number of oscillators tends to infinity, the energy is lost forever to the initially excited oscillator; its "emission" process becomes irreversible. Analogously, if we allow a two-state, excited atom to be coupled to but a single field mode, the excitation energy is exchanged sinusoidally in time between the atom and the field [71]; when all the modes of the radiation field are admitted, the emission becomes irreversible. This irreversibility does not require the decay to be exponential. In fact, one can improve upon what is usually referred to as the Weisskopf–Wigner approximation and obtain small corrections to the exponential decay law [72, 73].

It may be useful to digress still further on the (approximate) lineshape predicted by quantum electrodynamics. One can easily derive, using the adiabatic approximation to the atomic operator equations, the spectrum

$$P(\omega) = \frac{\beta\omega_0^n\,\omega^m/\pi\omega_0^3}{(\omega-\omega_0)^2+\beta^2}\,, \tag{3.19}$$

where $n = 2$, $m = 1$ or $n = 0$, $m = 3$ according to whether the $\boldsymbol{A}\cdot\boldsymbol{p}$ or $\boldsymbol{r}\cdot\boldsymbol{E}$ form of the interaction Hamiltonian is used. As was the case with the derivation of the radiative frequency shift in section 2, the two different results agree only on the energy shell $\omega = \omega_0$. From his experimental work on the radiative frequency shift in hydrogen, Lamb [53] concluded that the spectrum (3.19) with $n = 0$, $m = 3$ is the correct one. Davidovich [49], emphasizing that the transformation from the one form of the Hamiltonian to the other must be accompanied by a transformation of basis states, shows that this spectrum can also be obtained from the $\boldsymbol{A}\cdot\boldsymbol{p}$ Hamiltonian if this basis transformation is accounted for. Specifically, if in deriving the spectrum from the $\boldsymbol{r}\cdot\boldsymbol{E}$ Hamiltonian, the expectation values refer to the initial state

$$|\phi\rangle = |+\rangle \otimes |\{0\}\rangle, \tag{3.20}$$

where $|\{0\}\rangle$ is the field state in which there are no transverse photons, then the same spectrum can be obtained from the $\boldsymbol{A}\cdot\boldsymbol{p}$ Hamiltonian if one uses instead the initial state

$$|\psi\rangle = U(\boldsymbol{r}, t)|\phi\rangle, \tag{3.21}$$

where $U(\boldsymbol{r}, t)$ is defined by eq. (2.14). Physically, the state $|\psi\rangle$ defines the situation in which the total field energy density outside the atom vanishes, not just the energy density of the transverse field.

The transformation of the minimal coupling form of the Hamiltonian to the $\boldsymbol{r}\cdot\boldsymbol{E}$ form is easily accomplished in the semiclassical theory; this transformation was reviewed in section 2. The form of the interaction Hamiltonian used in this section is the $\boldsymbol{r}\cdot\boldsymbol{E}$ form, and so the lineshape predicted by the neoclassical theory must be compared with the quantum-electrodynamical lineshape

$$P(\omega) = \frac{\beta\omega^3/\pi\omega_0^3}{(\omega-\omega_0)^2+\beta^2}\,, \tag{3.22}$$

with the understanding, of course, that both lineshapes are approximate.

The quantity γ that appears in the neoclassical equations (3.16) and (3.17) will be shown below to represent a radiative frequency shift. For the purpose of obtaining the neoclassical emission lineshape it is convenient to temporarily take $\gamma \equiv 0$. In this case the equations (3.16)–(3.18) have the solutions [5]

$$x(t) = \cos\phi\ \mathrm{sech}\beta(t-t_0), \tag{3.23}$$

$$y(t) = \sin\phi\ \mathrm{sech}\beta(t-t_0), \tag{3.24}$$

and

$$z(t) = -\tanh\beta(t-t_0), \tag{3.25}$$

where ϕ and t_0 define the assumed initial values. The dipole terms have a hyperbolic secant dependence on t, and therefore their Fourier transforms are again hyperbolic secants. Thus the neoclassical theory predicts a hyperbolic secant squared lineshape, as opposed to the Lorentzian lineshape of the quantum theory. We shall not discuss the detailed aspects of the two spectra here; the interested reader should consult the paper by Jaynes and Stroud [5] or the review by Jaynes [6].

It is worthwhile, however, to study in detail the implications of the neoclassical theory for the radiative frequency shift, especially since the explanation of the Lamb shift is usually regarded as a crowning achievement of quantum electrodynamics. We shall not actually discuss the implications of the neoclassical theory for the Lamb shift: Enough of a contrast with the quantum-electrodynamical theory arises in the simple two-state model. Thus we shall discuss in detail the predictions of the two theories for the radiative frequency shift of a two-state atom.

From the "counter-rotating" approximation version of eq. (3.15),

$$\dot{S}(t) = -\mathrm{i}\omega_0 S(t) - \mathrm{i}\gamma S(t)\, z(t) + \beta S(t)\, z(t), \tag{3.26}$$

we see that the transition frequency ω_0 undergoes a time-dependent shift. The radiative frequency shift is

$$\Delta(t) = \gamma z(t). \tag{3.27}$$

Near the upper state, the transition frequency is approximately $\omega_0 + \gamma$, whereas near the lower state it is approximately $\omega_0 - \gamma$. This is in marked contrast to the unchirped radiative frequency shift predicted by quantum electrodynamics. In that case, the left-hand side and the first two terms on the right-hand side of eq. (3.26) are replaced by

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\sigma(t) - \mathrm{i}\Delta^{(-)}\sigma_z(t)\,\sigma(t) - \mathrm{i}\Delta^{(+)}\sigma(t)\,\sigma_z(t) + \dots \tag{3.28}$$

where

$$\Delta^{(\pm)} = \frac{2\mu^2\omega_0^2}{3\pi\hbar c^3}\,\mathrm{P}\int \frac{\omega\,\mathrm{d}\omega}{\omega \pm \omega_0}. \tag{3.29}$$

Since $\sigma_z(t)\,\sigma(t) = -\sigma(t)$ and $\sigma(t)\,\sigma_z(t) = \sigma(t)$, it follows that

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\,\sigma(t) + \mathrm{i}(\Delta^{(-)} - \Delta^{(+)})\,\sigma(t) + \dots = -\mathrm{i}(\omega_0 - \Delta)\,\sigma(t) + \dots, \tag{3.30}$$

where $\Delta = \Delta^{(-)} - \Delta^{(+)}$ is given by eq. (2.94). If, on the other hand, $\sigma(t)$ and $\sigma_z(t)$ are manipulated as c-numbers, we find instead that

$$\dot{S}(t) = -\mathrm{i}\omega_0 S(t) - \mathrm{i}(\Delta^{(-)} + \Delta^{(+)})\,S(t)\,z(t) + \dots, \tag{3.31}$$

where we have written S and z in place of σ and σ_z to emphasize the relation of the decorrelated quantum-mechanical equations to the neoclassical equations. In fact,

$$\Delta^{(+)} + \Delta^{(-)} = \frac{2\mu^2\omega_0^2}{3\pi\hbar c^3}\,\mathrm{P}\int_0^{Kc}\left(\frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0}\right)\mathrm{d}\omega \cong \frac{4\mu^2\omega_0^2 K}{3\pi\hbar c^2} = \gamma, \tag{3.32}$$

since $Kc \gg \omega_0$. Once again, improper manipulation of the quantum-mechanical equations leads to the neoclassical equations.

In the quantum-electrodynamical approach, $-\hbar\Delta^{(-)}$ and $-\hbar\Delta^{(+)}$ represent the *level* shifts of the upper and lower states, respectively. According to this view, the neoclassical theory takes as the frequency shift of the transition from the upper to the lower state the sum rather than the difference of the upper and lower state level shifts (divided by $\hbar$). Once again a feature of the Schrödinger semiclassical theory appears: Evidently transition frequencies, and not (superfluous) energy levels, are the essential "energetic properties" of the neoclassical atom.

But there is something more subtle contained in the identification of γ with the magnitude of the radiative frequency shift. Consider the Bethe renormalization applied to the two-state atom; the procedure is not directly applicable to the two-state model, since the Thomas–Reiche–Kuhn sum rule does not apply, but as long as this is borne in mind, it does not destroy the argument made here. The level shift of the upper state is $-\hbar\Delta^{(-)}$. Recognizing that the "electromagnetic mass" is already contained in the expression for the energy of this level, and that switching on the radiation reaction field adds its contribution again, we must subtract its contribution to $-\hbar\Delta^{(-)}$. Thus the observable level shift is, from eq. (2.64),

$$-\hbar\Delta^{(-)}_{\text{obs}} = -\hbar\Delta^{(-)} - C\langle \boldsymbol{p}^2\rangle, \tag{3.33}$$

the expectation value referring to the upper state. In the two-state model, $\langle \boldsymbol{p}^2\rangle$ is

$$\langle +|\boldsymbol{p}^2|+\rangle = \langle +|\boldsymbol{p}|-\rangle\langle -|\boldsymbol{p}|+\rangle = \frac{m^2}{e^2}\,\omega_0^2\mu^2, \tag{3.34}$$

where m is the observable mass. From eq. (2.64) we have

$$C = -\frac{2}{3\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{mc}\right)^2 \hbar K c = -\frac{2e^2K}{3\pi m^2c^2}, \tag{3.35}$$

so that

$$\begin{aligned}-\hbar\Delta^{(-)}_{\text{obs}} &= -\hbar\Delta^{(-)} + \frac{2e^2\omega_0^2 m^2\mu^2 K}{3\pi m^2 e^2 c^2}\\ &= -\hbar\Delta^{(-)} + \frac{2\mu^2\omega_0^2 K}{3\pi c^2} = -\hbar\Delta^{(-)} + \tfrac{1}{2}\hbar\gamma = -\frac{2\mu^2\omega_0^3}{3\pi c^3}\log\left|\frac{\hbar K c}{E_+ - E_-}\right|,\end{aligned} \tag{3.36}$$

which is just the two-state analogue of Bethe's expression (2.62).

Two important points emerge from this simple analysis. First, it appears that what is subtracted off in the quantum-electrodynamical approach is taken to be physically meaningful in the neoclassical theory. Namely, the quantity that is associated with an electromagnetic mass in Bethe's approach is associated instead with a radiative frequency shift in the neoclassical theory – there is no mass renormalization in neoclassical theory. To elaborate somewhat on this point, consider the equations (3.8) and (3.9) for the case where the atom is near the ground state, $z \cong -1$. Then for the atomic dipole $\boldsymbol{\mu}x(t)$, we have the equation

$$\ddot{x}(t) + \omega_0^2 x(t) = \frac{2\omega_0}{\hbar}\,\boldsymbol{\mu}\cdot \boldsymbol{E}(0,t), \tag{3.37}$$

or

$$\ddot{x}(t) + \omega_0^2 x(t) \cong \frac{4\omega_0\mu^2}{3\hbar c^3}\,\dddot{x}(t) - \frac{8\omega_0\mu^2 K}{3\pi\hbar c^2}\,\ddot{x}(t), \tag{3.38}$$

when use is made of eq. (3.11). Making the approximation (3.12), we have

$$\ddot{x}(t) + \omega_0^2 \left[1 - \frac{8\omega_0 \mu^2 K}{3\pi\hbar c^2}\right] x(t) = -\frac{4\omega_0^3 \mu^2}{3\hbar c^3} \dot{x}(t). \tag{3.39}$$

Neoclassical theory would associate the term

$$\omega_0 \left[1 - \frac{8\omega_0 \mu^2 K}{3\pi\hbar c^2}\right]^{1/2} - \omega_0 \cong -\frac{4\mu^2 \omega_0^2 K}{3\pi\hbar c^2} = -\gamma \tag{3.40}$$

with a radiative frequency shift; in fact, as mentioned earlier, $-\gamma$ is indeed the radiative frequency shift for an atom near its ground state, according to neoclassical theory. According to the approach of Bethe, however, the second term on the right-hand side of eq. (3.38) is to be associated with an electromagnetic mass. Since the energy levels (given, for example, by the Bohr formula) were derived using an already-renormalized mass, the energy

$$p^2/2m_{\mathrm{obs}} - p^2/2m_{\mathrm{bare}} = \Delta E^{(\mathrm{F})} = -\tfrac{1}{2}\gamma\hbar \tag{3.41}$$

must be subtracted away. The "extra" factor of two in the eq. (3.40) merely represents the fact that the radiative frequency shift in the neoclassical theory is the sum of the level shifts of the lower and upper states.

The second point to be made in this connection is that the term subtracted away in Bethe's approach is an expectation value over the level in question: Quantum mechanically the electron, in a loose sense, is in only one state at a time, and not associated simultaneously with many "partial oscillations". The interpretation of the stationary states of the atom in the neoclassical theory reminds one of the normal modes of vibration of a classical oscillating system. If the state of motion of this system is represented in terms of its normal modes, each normal mode contribution (say, to the transverse oscillation of a plucked string) is contained in the motion of the system at any instant of time, and the relative weights of the different modes in the normal-mode expansion determine the state of oscillation at this instant. The corresponding quantum-mechanical expansion of the wave function in terms of the stationary states is interpreted in a very different way. There the relative weights of the different "normal modes" determine the *probability* of the system to be found in each mode; a measurement can only reveal that the system is in *one* particular stationary state – the measurement indeed prepares that state.

The fact that an expectation value of $\Delta E^{(\mathrm{F})}$ is subtracted away in Bethe's calculation seems to bear on a problem which long concerned Schrödinger [58], namely, whether in the microscopic realm the concept of energy should be considered to be a statistical one.

3.4. *Discussion*

The semiclassical equations (3.8)–(3.10) are useful for a description of many phenomena in quantum optics and electronics [74]. They are applicable when only two atomic or molecular states are sufficient to characterize the system in its interaction with an applied field, i.e., when the applied field is tuned to a single transition. To account for the relaxation of the two-state system one normally introduces phenomenological damping terms into these equations. The resulting equations are sometimes termed "phenomenological Bloch equations", since Bloch [75] used equations of the same form in the theory of (spin $-\frac{1}{2}$) nuclear magnetic resonance. The utility of these equations for problems in maser physics was emphasized by Feynman, Vernon and Hellwarth [56] in 1957, although their use in such problems was not entirely unknown earlier [6]. Indeed, with the wisdom of hindsight, it is surprising that these equations were not well known even much earlier. As early as 1927, Landau [76] had studied spontaneous emission with a density-matrix approach and field quantization; his approach was similar in spirit to that used here in section 2. The inclusion of a classical applied field might have led to the Bloch equations.

The neoclassical theory of spontaneous emission proceeds from the idea of allowing the atom to react back on the applied field. This is accomplished by using for the electric field in eqs. (3.9) and (3.10) the sum of the transition dipole's radiation reaction field and the applied field. It should be noted that the "phenomenological" Bloch equations can be simply derived from the quantum-mechanical operator approach when the only relaxation mechanism is radiative decay and when the initially excited field is described by a Glauber coherent state [77]. The derivation in this case follows easily from eqs. (2.88)–(2.90). The radiation reaction field leads to the radiative damping, while expectation values of products of normally-ordered field operators $E_0^{(\pm)}$ with atomic operators yield c-numbers multiplying atomic-operator expectation values. Thus, for example, if we take expectation values on both sides of eq. (2.88) we obtain

$$\langle\dot{\sigma}_y(t)\rangle = \omega_0\langle\sigma_x(t)\rangle + \frac{2}{\hbar}\,\boldsymbol{\mu}\cdot[\langle E_{\mathrm{RR}}^{(-)}(t)\,\sigma_z(t)\rangle + \langle\sigma_z(t)\,E_{\mathrm{RR}}^{(+)}(t)\rangle] + \frac{2}{\hbar}\,\boldsymbol{\mu}\cdot[\langle E_0^{(-)}(t)\,\sigma_z(t)\rangle + \langle\sigma_z(t)\,E_0^{(+)}(t)\rangle]\,. \tag{3.42}$$

The terms involving the radiation reaction field lead to the radiative damping and frequency shift in the now familiar way. The assumption of an initially coherent state of the field implies that

$$\langle E_0^{(-)}(t)\,\sigma_z(t)\rangle + \langle\sigma_z(t)\,E_0^{(+)}(t)\rangle = (\boldsymbol{\varepsilon}_0^{(-)}(t) + \boldsymbol{\varepsilon}_0^{(+)}(t))\langle\sigma_z(t)\rangle = \boldsymbol{\varepsilon}_0(t)\,\langle\sigma_z(t)\rangle, \tag{3.43}$$

where $\boldsymbol{\varepsilon}_0^{(\pm)}$ are c-numbers defined by

$$E_0^{(+)}(0,t)\,|\text{initial field state}\rangle = \boldsymbol{\varepsilon}_0^{(+)}(0,t)\,|\text{initial field state}\rangle \tag{3.44}$$

and the conjugate relation. Thus the Bloch equations with radiative damping follow from expectation values when the applied field, or more precisely the initial field state, is coherent. Other damping mechanisms can be included in the same way by including their contributions to the interaction Hamiltonian.

Bloch equations are also adequate for the treatment of the interaction of resonant light with a system of many atoms or molecules. In the semiclassical theory of the laser, for example, the electric field used in the Bloch equations is taken to be the solution of the Maxwell equation

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\ddot{\boldsymbol{E}} = \frac{4\pi}{c^2}\ddot{\boldsymbol{P}} \tag{3.45}$$

the polarization density $\boldsymbol{P}$ being just the sum of all the individual atomic polarizations; the atomic polarizations in turn are determined by using the field satisfying eq. (3.45) in the Bloch equations. From a quantum-mechanical viewpoint, the coupled Maxwell–Bloch equations may be regarded as equations for expectation values, the field being assumed to be fully coherent in the Glauber sense. Equivalently these equations are based on the assumption that each atom bears no correlation with the fields from the remaining atoms of the laser medium. Shirley [78] has shown that this approximation is valid to order $1/N$, where N is the number of atoms (confined so that all the atoms are exposed to the same field amplitude); i.e., the terms that account for any correlations are negligible when the number of atoms is large. Neoclassical theory carries the assumption one step further: It is assumed that the atom is not even correlated with its own radiation reaction field. Here the term "correlation" defines the situation in which the expectation value of a product of two operators is not factorable into the product of the expectation values of these operators. The lack of correlation in this sense does *not* preclude in any way the possibility of having the fields from different atoms bearing definite phase relationships, so that the different atoms may interact "coherently" through their fields. This point has been emphasized by Senitzky [32].

The wave function describing the entire system of atoms of the laser medium is thus interpreted in the Schrödinger sense; the approximation is a good one because it is possible for many atoms of the medium to occupy the same state. But the assumption that the Schrödinger interpretation applies to a *single* atom is a radical one, and represents, not an approximation to the quantum theory, but rather the creation of a new theory. For the case of a single atom, there is no "safety in large numbers!". On the other hand, the assumption is certainly not unreasonable, and in many instances would appear to be quite reasonable indeed. Thus, for example, the equation of continuity satisfied by $|\psi|^2$ would strongly suggest a "hydrodynamic" interpretation of the wave function; only when the statistical interpretation is accepted does the equation of continuity present itself as an obvious statement of the "conservation of probability". To cite another example, the forces acting on each nucleus in a molecule can be calculated by classical electrostatics, given the charges and positions of all other nuclei and electrons of the molecule, by using the Schrödinger interpretation of the electronic wave function. This is the Hellmann–Feynman theorem [79].

The most immediately obvious differences between the quantum-electrodynamical and neoclassical theories are found in their predictions of the dynamics of the radiative decay and frequency shift. The rate at which spontaneous emission proceeds in the neoclassical theory depends on the probability amplitudes of both the upper and lower states of the transition. Thus if the atom is initially described by the state

$$|\psi\rangle = c_1\,|-\rangle + c_2\,|+\rangle, \tag{3.46}$$

the square of its transition dipole moment is proportional to $|c_1|^2|c_2|^2$, and the transition rate, in the sense of short-time perturbation theory, is $|c_1|^2|c_2|^2$ times the Einstein A-coefficient. The emission rate according to quantum electrodynamics, on the other hand, is $|c_2|^2 A$. To discuss this difference in more detail, it is useful to consider the simple model of a two-state atom coupled to a single mode of the radiation field. In the rotating-wave (or "counter-rotating") approximation, the Heisenberg equations of motion for the problem are

$$\dot{\sigma}(t) = -\mathrm{i}\omega_0\,\sigma(t) + g\,\sigma_z(t)\,a(t), \tag{3.47}$$

$$\dot{\sigma}_z(t) = -2g(a^+(t)\,\sigma(t) + \sigma^+(t)\,a(t)), \tag{3.48}$$

and

$$\dot{a}(t) = -\mathrm{i}\omega a(t) + g\,\sigma(t), \tag{3.49}$$

where ω is the circular frequency of the field mode and the coupling constant g is real. Consider the time evolution of the field operator $a(t)$ in Heisenberg-picture short-time ($gt \ll 1$) perturbation theory:

$$a(t) \cong a(0)\,\mathrm{e}^{-\mathrm{i}\omega t} + gt\,\sigma(0)\,\mathrm{e}^{-\mathrm{i}\omega t} + \tfrac{1}{2}g^2t^2\,\sigma_z(0)\,a(0)\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.50}$$

From this equation it follows that

$$\langle a^+(t)\,a(t)\rangle \cong \langle a^+(0)\,a(0)\rangle + gt\langle a^+(0)\,\sigma(0)\rangle + gt\langle \sigma^+(0)\,a(0)\rangle + g^2t^2\langle \sigma_z(0)\,a^+(0)\,a(0)\rangle + g^2t^2\langle \sigma^+(0)\,\sigma(0)\rangle. \tag{3.51}$$

The initial atomic state is given by eq. (3.46); for simplicity we assume that the initial field state is the coherent state $|\alpha\rangle$:

$$a(0)|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.52}$$

so that eq. (3.51) may be written as

$$\langle a^+(t)\,a(t)\rangle \cong |\alpha|^2 + gt(\alpha^* c_1^* c_2 + \alpha c_1 c_2^*) + g^2t^2|\alpha|^2(|c_2|^2 - |c_1|^2) + g^2t^2|c_2|^2. \tag{3.53}$$

The first term on the right-hand side of eq. (3.53) corresponds to the initial field energy. The second term represents the interaction of the field with the initial atomic dipole expectation value and vanishes unless the atom is initially in a coherent superposition of its two states. The third term corresponds to stimulated emission or absorption, while the fourth term clearly represents spontaneous emission. Similarly we find

$$\langle a^+(t)\rangle\langle a(t)\rangle \cong |\alpha|^2 + gt(\alpha^* c_1^* c_2 + \alpha c_1 c_2^*) + g^2t^2|\alpha|^2(|c_2|^2 - |c_1|^2) + g^2t^2|c_1|^2|c_2|^2, \tag{3.54}$$

from which we obtain

$$\langle a^+(t)\,a(t)\rangle - \langle a^+(t)\rangle\langle a(t)\rangle \cong g^2t^2|c_2|^4, \tag{3.55}$$

to second order in gt. Senitzky [32] points out that the nonvanishing of the right-hand side of eq. (3.55), i.e., the incoherence which has been imparted to the field, is due to the addition of energy to the field by spontaneous emission. Senitzky [32] defines

$$S_{\mathrm{coh}} = g^2t^2|c_1|^2|c_2|^2 \tag{3.56}$$

to be the "coherent" part of the spontaneous emission. From eq. (3.54) it is seen that S_{coh} is the short-time contribution to the field energy from spontaneous emission that would be predicted by the neoclassical theory. The "incoherent" part of the spontaneous emission is

$$S_{\mathrm{incoh}} = g^2t^2|c_2|^4, \tag{3.57}$$

the "total" spontaneous emission corresponding to

$$S = S_{\mathrm{coh}} + S_{\mathrm{incoh}} = g^2t^2|c_2|^2. \tag{3.58}$$

Senitzky [35] has pointed out that semiclassical theories can predict only "coherent" spontaneous emission. The "incoherent" part of the spontaneous emission requires the quantization of the radiation field; the semiclassical approach cannot account for the nonvanishing of

$\langle a^{+}(t)\,a(t)\rangle - \langle a^{+}(t)\rangle\,\langle a(t)\rangle$. Jaynes and Cummings [71] have found the exact solution of these equations for $\langle a^{+}(t)\,a(t)\rangle$, and the short-time limit of their solution corresponds to the results presented here.

Heisenberg [80] also noted that a semiclassical treatment of spontaneous emission leads to the result that the emission depends on the probability amplitudes of both the initial and final states. He pointed out that this feature may be eliminated by a simple erasure which would appear to correspond closely to the addition of Senitzky's incoherent part to the expression obtained semiclassically.

The neoclassical theory of the radiative frequency shift for a two-state atom has already been discussed in detail. As opposed to the quantum-electrodynamical result, the neoclassical theory predicts a "dynamic" or chirped frequency shift. Moreover, the neoclassical treatment of the shift is completely at odds with the usual notion of mass renormalization. The term that is treated as an electromagnetic mass in the equation for a free point charge is taken to represent a radiative frequency shift for a bound electron, rather than being subtracted away as a spurious "extra" electromagnetic mass. Such a subtraction would in fact eliminate radiative frequency shifts in the neoclassical theory. No very decisive conclusions can be drawn from a simple two-state atomic model, however. All the virtual transitions must be included in a realistic treatment of the radiative frequency shift. A neoclassical theory of the Lamb shift is complicated by the nonlinearity of the equations and is not available at this time.

4. Evidence for the validity of the quantum-electrodynamical theory of spontaneous emission

4.1. *Historical perspective*

The first unified theory of the interaction of light with atomic matter was advanced by Lorentz [81]. Combining Maxwell's equations for the electromagnetic field with Newton's laws of motion for bound electrons, Lorentz created an impressive theoretical edifice which accounted for a vast array of electromagnetic phenomena known in his time. Because it could account for neither the blackbody spectrum nor the existence of discrete atomic energy levels, however, the edifice collapsed in part. Today it stands, with some interpretative modifications, as an accurate approximation to the semiclassical theory of nonresonance phenomena.

The Lorentz model, being essentially a theory of charged harmonic oscillators interacting with light, cannot account for stimulated emission. The existence of the phenomenon of stimulated emission was inferred by Einstein [2] in his derivation of Planck's blackbody law; if the term corresponding to stimulated emission is omitted in Einstein's rate-equation approach Wien's law, not Planck's, results. The concept of stimulated emission later played a central role in what was perhaps a more significant development, the derivation by Kramers [82] and Heisenberg [3, 83] of the dispersion law. Their work took into account Einstein's "negative absorption", and Kuhn's considerations [84] on the classical limit of the dispersion law led to the famous f-sum rule. This sum rule, the first sum rule in physics, is essentially a statement of the commutation relation $[x, p_x] = i\hbar$. In fact, the dispersion law played a decisive role in the work of Born and Jordan [85] in which the concept of noncommutativity made its debut into physics. On the experimental side, Ladenburg and Kopfermann [86] made the first direct observation of stimulated emission. They studied the refractive index near a spectral line of a rare gas contained in a discharge tube. By measuring the refractive index as a function of current through the discharge – or equivalently as a function of the degree of atomic excitation – they were able to infer that there was indeed a contribution to the index from "negative absorption" or stimulated emission. Further work could conceivably have led to the first light amplifier.

The rates of spontaneous and stimulated emission were connected by Einstein's well-known relation [2]. The expression for the A-coefficient itself was known through reasoning based on the Correspondence Principle [3], but an actual derivation founded on a rigorous formalism was lacking until Dirac [1] applied the Born–Jordan commutation rules to the radiation field. Thus we have some historical "evidence" for the necessity of field quantization for a rigorous treatment of spontaneous emission: In spite of intense work on the theory of the interaction of light with atomic matter, no such treatment was given prior to the creation of Dirac's radiation theory.

It is well known that quantum theory at its inception was basically a theory of the interaction between light and atomic matter. The formalism of quantum electrodynamics, as the above remarks suggest, had its origins in attempts to understand on an atomistic foundation the ubiquitous phenomenon of refraction. These origins, humble by today's standards, should be considered as representative of the beautiful inductive reasoning which characterized the work of the founders of quantum theory. But they also suggest a starting point for an examination of the foundations of the quantum theory of radiation: From an historical point of view, at least, the examination should begin with an investigation into the nonrelativistic theory of the interaction of light with atoms. The absence of any adequate treatment of spontaneous emission prior to Dirac's theory suggests, more specifically, that the investigation should begin with the theory of spontaneous emission. This is precisely the starting point of the neoclassical theory. Clearly, it is essential to understand fully the limits of applicability, if any, of the neoclassical theory. The fact that the theory predicts spontaneous emission is itself a significant accomplishment. That this theory does not invoke field quantization does not necessarily suggest that it might point the way to a resolution of the conceptual difficulties besetting some aspects of quantum electrodynamics, for these difficulties in large part are inherited from classical electrodynamics. Nevertheless, it can at the very least sharpen our insights into those phenomena which may require field quantization for a satisfactory explanation.

Certainly some readers will feel that any attempts to compare semiclassical and quantum-electrodynamical theories should begin with more "profound" problems. For this reason we emphasize again that quantum electrodynamics developed *inductively* from "simple" problems involving the interaction of light with atoms. Conceptually, little change in the theory established by 1927 has been necessary. Born's remark at the Fifth Solvay Conference, held in 1927, is remarkable in both its boldness and accuracy: "We consider that quantum mechanics is a complete theory, and that its fundamental hypotheses, both physical and mathematical, are not susceptible to further modification" [87].

The neoclassical theory represents the first serious attempt to critically examine Dirac's application of the Born–Jordan commutation rules to the radiation field. Planck's work on the blackbody spectrum obviously suggested a need for a *quantum* electrodynamics, just as the later work of Bohr on the hydrogen spectrum pointed to a quantum mechanics. Historically, then, field quantization actually preceded the quantization of any atomic attributes. But a quantum mechanics developed as a working formalism before a quantum electrodynamics did. The marriage of field and atomic quantization ideas began with Einstein's derivation [2] of the Planck law; Einstein dealt simultaneously with both field quanta and discrete atomic energy levels. Implicit in Einstein's derivation is the assumption that an atom in an excited "stationary" state will drop to a lower state by spontaneous emission. In the very first paper "On the Quantum Theory of Radiation" interacting with quantized matter, therefore, we find a conflict with the predictions of the neoclassical theory of spontaneous emission. The task before us is to critically examine conflicts of this kind. The task is twofold. First, we must find where the predictions of the quantum-electrodynamical and neoclassical theories differ. But given the fact that the neoclassical theory is perhaps susceptible to alteration, we must also seek to determine whether any necessary alterations can be made without destroying the logical content of the theory. Regarding the blackbody spectrum, for example, Nesbet [88] has noted a difference between the predictions of the two theories, and has suggested a modification of the description of the classical radiation field which renders the neoclassical predictions on the blackbody spectrum in agreement, at least superficially, with Planck's law.

4.2. *The Born probabilistic interpretation of the wave function*

Differences in the predictions of quantum-electrodynamical and semiclassical radiation theories stem ultimately from the different ways in which the theories interpret the atomic wave function. Before examining any experimental evidence for the quantum-electrodynamical theory of spontaneous emission, it is appropriate to review the line of reasoning which led Born to the probabilistic interpretation of the wave function. The feeling has been expressed [5] that the actual basis for the abandonment of Schrödinger's interpretation in favor of Born's is not entirely clear today. This feeling may result from the fact that there is no single, classic experimental or theoretical result which stands as the definitive counter-example to the Schrödinger hypothesis. Remarkably, the early quantum theory was successfully applied to many problems before the significance of the wave function was established.

Born, of course, played an important role in the development of Heisenberg's matrix mechanics. The *leitmotiv* of Heisenberg's early work was the acceptance of discrete atomic energy states without attempting to explain their origin; the enumeration of these states led to the idea of coefficients (matrix elements) connecting them. It was Born who recognized in these tables of numbers the representation of dynamical variables as matrices. While Heisenberg, Born and Jordan were laying the foundations of matrix mechanics, Schrödinger's wave mechanics appeared. For the study of scattering, Born actually preferred Schrödinger's formalism over Heisenberg's. But he did not accept Schrödinger's semiclassical interpretation of the wave function:

> On this point I could not follow him. This was connected with the fact that my Institute and that of James Franck were housed in the same building of the Gottingen University. Every experiment by Franck and his assistants on electron collisions (of the first and second kind) appeared to me as a new proof of the corpuscular nature of the electron [89].

The origin of the probabilistic interpretation of the wave function is clear. In the first of a series of papers on the quantum-mechanical theory of scattering [90], Born studied the scattering of an electron by a spherically symmetric potential, and reached the conclusion that $|\psi(\boldsymbol{x})|^2\, d^3x$ must represent the probability of finding the (corpuscular) electron in the volume element d^3x. In this way he reconciled the wave equation with the experimental results of Franck.

Born's interpretation eliminated certain difficulties which confronted the Schrödinger interpretation. These difficulties are well known, and need not be discussed in detail here; the reader unfamiliar with all of these difficulties is referred to the discussion in Jammer's recent treatise [91]. Three of these difficulties bearing on the problem of spontaneous emission will be mentioned.

First, there is the apparent inconsistency, in Schrödinger's approach to the hydrogen-atom problem, of using the Coulomb interaction between point charges, while at the same time interpreting $|\psi(\boldsymbol{r})|^2$ as a continuous charge distribution. Born's interpretation overcomes this difficulty by adhering to the notion of a particulate electron from the outset. Secondly, a measurement implies a discontinuous change in ψ; in Born's interpretation this discontinuity is ascribed to a change in our knowledge about the system brought about by the measurement. This "reduction of the wave packet" issue is by no means a closed one. It is still controversial, with quite profound implications for the interpretation of the formalism of quantum mechanics.

Finally, we mention the ambiguity in interpreting the wave function in the Schrödinger sense for the case of a multielectron atom. Schrödinger himself recognized this problem in his paper on the equivalence of wave and matrix mechanics [92]: "The difficulty encountered in the *poly*-electron problem, in which ψ is actually a function in *configuration* space and not in the real space, should not remain unmentioned". Jaynes and Stroud [5] have suggested that this apparent drawback of the Schrödinger semiclassical interpretation may be overcome by properly interpreting the charge density of a multielectron system. If $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \dots \boldsymbol{r}_N, t)$ is the wave function for an N-electron system, the charge density due to electron 1, for example, may be taken to be [5]

$$\rho_1(\boldsymbol{r}, t) = e\int |\psi(\boldsymbol{r}, \boldsymbol{r}_2, \dots \boldsymbol{r}_N, t)|^2 \, \mathrm{d}^3 r_2 \dots \mathrm{d}^3 r_N \tag{4.1}$$

with analogous expressions for the contributions to the charge density from electrons 2, 3, ... N. The total charge density of the N-electron system is then the sum of the charge densities of the individual electrons:

$$\rho(\boldsymbol{r}, t) = \sum_{i=1}^{N} \rho_i(\boldsymbol{r}, t). \tag{4.2}$$

While this appears at first glance to be a plausible extension of the Schrödinger interpretation to an N-electron system, it is not without probabilistic connotations suggested by the integration over the coordinates of $N-1$ other electrons. In other words, it does not seem to be entirely consistent with the Schrödinger semiclassical interpretation of the *single*-electron wave function.

Bohr viewed the Born probabilistic interpretation of the wave function in terms of single systems. According to this "Copenhagen interpretation", quantum mechanics provides a description of the *individual* system. Einstein, at the Fifth Solvay Congress, advanced a "statistical interpretation". According to this interpretation, the probabilities calculated in quantum mechanics refer to ensembles of identical experiments. The probabilities represent relative frequencies over this ensemble, in the usual sense of probability theory. Jammer [93] notes that

> Although it does not lead to *experimental consequences* – for in both cases the confirmation of predictions requires the performance of an ensemble of identical experiments – it does lead to *interpretative consequences*: The Einstein frequency interpretation opens the way to a hidden-parameter theory, which reduces quantum mechanics to a branch of statistical mechanics; the Bohr–Born probabilistic interpretation precludes this possibility.

It does not seem entirely appropriate to associate Born with the Copenhagen interpretation of quantum mechanics. Commenting on a letter from Einstein, Born proclaimed his agreement with the statistical interpretation [94]:

> Einstein admits that one can regard the "probabilistic" quantum theory as final if one assumes that the ψ-function relates to the ensemble and not to an individual case. This has always been my assumption as well, and I consider the frequent repetition of an experiment as the realization of the ensemble. This coincides exactly with the actual procedure of the experimental physicists, who obtain their data in the atomic and sub-atomic area by accumulating data from similar measurements.

Among the physicists who have voiced support of Einstein's statistical (i.e., ensemble) interpretation are Kemble, Slater, Langevin, and more recently, Ballentine [95]. Many physicists do not recognize a need for a distinction between statistical and Copenhagen interpretations, since such a distinction has no immediate consequences for the comparison of theory and experiment. Because the problem of interpretation is so crucial to an understanding of the differences in the semiclassical and quantum-electrodynamical theories, however, we shall briefly review Einstein's arguments in favor of the statistical interpretation.

At the Fifth Solvay Congress, Einstein considered the following "thought" experiment. Let a beam of electrons impinge on a screen containing an aperture so small that the de Broglie–Schrödinger waves suffer sizeable diffractive effects. Behind the aperture is a large hemisphere which records electron impacts. Consider each elementary process of an electron passing through the aperture and being detected somewhere on the hemisphere. If quantum mechanics "has the pretention to be a complete theory of individual processes", according to Einstein, $|\psi(\boldsymbol{r})|^2$ must be the probability of finding the electron at $\boldsymbol{r}$; that electron must be regarded as potentially detectable at many points on the hemisphere, until a measurement is performed, at which point some peculiar action-at-a-distance must prevent it from having an effect on more than one point on the hemisphere. In the statistical interpretation, however, $|\psi(\boldsymbol{r})|^2$ represents the probability density that some member of an *ensemble* be found at $\boldsymbol{r}$. $\psi(\boldsymbol{r})$ does not describe the individual electron, and no peculiar "reduction of the wave packet" is required. Einstein's misgivings in this regard apply also to the Schrödinger interpretation of $|\psi(\boldsymbol{r})|^2$. Indeed, it seems that his argument was aimed at Schrödinger.

In fact, one of Einstein's arguments [96] is directly applicable to the neoclassical theory. We may phrase it simply in terms of the two-state-atom model. Suppose the atom is initially in the lower energy state, so that $\langle\sigma_z\rangle = -1$, and let a "small disturbing force" act on the atom. Then $-1 < \langle\sigma_z\rangle < 1$. If we have a theory of an *individual system,* we must conclude that after the disturbance the system is characterized by an energy intermediate between those of the upper and lower states, a conclusion at variance with the Franck–Hertz experiment. Therefore, only a statistical interpretation seems tenable: "The ψ function does not in any way describe a condition which could be that of a single system; it relates rather to many systems, to an ensemble of systems in the sense of statistical mechanics" [96]. The neoclassical theory, being – at least at the present – a theory of the individual system, cannot meet this objection. In the neoclassical theory $\langle\sigma_z(t)\rangle$ (or rather $z(t)$) is a direct measure of the atomic energy, and not an "expectation value" in the sense of the fully quantum-mechanical theory.

Whether the distinction between Copenhagen and statistical interpretations is a "legitimate" one is a question that lies quite beyond the scope of this article. Preceding Ballentine's recent article [95] on the subject was an editorial comment which began with the sentence, "The subject of the following paper lies in the border area between physics, semantics and other humanities". In any case, it is clear that the philosophy of the neoclassical theory is more closely allied to a "Copenhagen philosophy": It is a theory of the individual system, and its predictions must be applied to individual systems. In comparing these predictions to those of the quantum theory, however, we need not distinguish between Copenhagen and statistical interpretations of the latter.

4.3. *Photon polarization correlations in a cascade decay: The Kocher –Commins experiment*

Undoubtedly the most serious objection to semiclassical radiation theories has been raised by Clauser [31]. He showed that semiclassical and quantum-electrodynamical theories predict different results for the polarization correlations when photons are emitted successively in an atomic cascade. Such correlations were measured by Kocher and Commins [30], and the experimental result agrees with the predictions of quantum electrodynamics, "but excludes semiclassical theories in general" [31].

We consider here the following idealization of the Kocher–Commins experiment. Our one-electron atom is initially in the state characterized by the quantum numbers $n = 3, l = 0$. It decays to one of the states with $n = 2, l = 1$, and then to the ground state $n = 1, l = 0$. Observer A is equipped with a narrow-bandwidth filter that passes only photons of frequency $\omega_A = (E_3 - E_2)/\hbar$, and observer B's filter passes only photons of frequency $\omega_B = (E_2 - E_1)/\hbar$. A and B are also equipped with perfect-efficiency linear polarizers and are situated diametrically opposed to one another along the z-axis. If A sets his linear polarizer at angle ϕ_A with respect to some arbitrary axis normal to the line joining A and B, and B sets his polarizer at angle ϕ_B with respect to the same axis, what is the probability as a function of $|\phi_A - \phi_B|$ that A and B both observe a photon passing through their respective filters and polarizers when the atom cascades spontaneously from $|n = 3, l = 0\rangle$ to $|n = 2, l = 1\rangle$ to $|n = 1, l = 0\rangle$? This is the quantity to be considered in our idealized Kocher–Commins experiment. We now calculate this quantity, first using quantum electrodynamics and then using neoclassical theory.

To find the coincidence rate of finding a photon $(\omega_A, \hat{z}, \lambda)$ (frequency ω_A, propagating in the z direction, polarization index λ) and a photon $(\omega_B, -\hat{z}, \mu)$, we consider the quantity [77]

$$\langle a^+_{A\lambda}(t)\, a^+_{B\mu}(t)\, a_{B\mu}(t)\, a_{A\lambda}(t)\rangle ,$$

the expectation value taken in the state

$$|\psi(0)\rangle = |n = 3, l = 0\rangle \otimes |\{0\}\rangle . \tag{4.3}$$

The subscript Aλ refers to the mode $(\omega_A, \hat{z}, \lambda)$, and the subscript B$\mu$ to the mode $(\omega_B, -\hat{z}, \mu)$. Labeling the atomic states as follows:

$|1\rangle = |n = 1, l = 0, m = 0\rangle$, $\quad |4\rangle = |n = 3, l = 0, m = 0\rangle$,

$|2\rangle = |n = 2, l = 1, m = 1\rangle$, $\quad |5\rangle = |n = 2, l = 1, m = 0\rangle$,

$|3\rangle = |n = 2, l = 1, m = -1\rangle$,

we have from eq. (2.38),

$$a_{k\lambda}(t) = a_{k\lambda}(0) \exp(-i\omega_k t) + \sum_{i=1}^{5} \sum_{j=1}^{5} C_{k\lambda ij} \int_0^t dt_1\, \sigma_{ij}(t_1) \exp\{i\omega_k(t_1 - t)\} , \tag{4.4}$$

for our "five-state" atom. The state $|5\rangle$ may be excluded from consideration with the z-axis as the axis of quantization, since the cascade $|4\rangle \to |5\rangle \to |1\rangle$ would correspond to longitudinal polarization. In the rotating-wave approximation we drop terms in eq. (4.4) with $j < i$. Then we have

$$a_{A\lambda}(t) = a_{A\lambda}\exp(-i\omega_A t) + C_{A\lambda 12}\int_0^t dt_1\,\sigma_{12}(t_1)\exp\{i\omega_A(t_1-t)\} + C_{A\lambda 13}\int_0^t dt_1\,\sigma_{13}(t_1)\exp\{i\omega_A(t_1-t)\}$$

$$+\,C_{A\lambda 24}\int_0^t dt_1\,\sigma_{24}(t_1)\exp\{i\omega_A(t_1-t)\} + C_{A\lambda 34}\int_0^t dt_1\,\sigma_{34}(t_1)\exp\{i\omega_A(t_1-t)\}$$

$$\cong a_{A\lambda}(0)\exp(-i\omega_A t) + (C_{A\lambda 12}\sigma_{12}(0) + C_{A\lambda 13}\sigma_{13}(0))\exp(-i\omega_A t)\int_0^t dt_1\exp\{i(\omega_A-\omega_B)t_1\}$$

$$+\,(C_{A\lambda 24}\sigma_{24}(0) + C_{A\lambda 34}\sigma_{34}(0))\,t\exp(-i\omega_A t), \tag{4.5}$$

in the approximation $\sigma_{12}(t_1) \cong \sigma_{12}(0)\exp(-i\omega_{21}t_1)$, etc., equivalent to standard short-time perturbation theory. We have defined

$$C_{A\lambda ij} = \frac{1}{\hbar}\left(\frac{2\pi\hbar\omega_A}{V}\right)^{1/2}(\boldsymbol{\mu}_{ij}\cdot\hat{e}_{kA\lambda}), \tag{4.6}$$

where $k_A = (\omega_A/c)\hat{z}$ and $\hat{e}_{k_A\lambda} = \hat{x}\cos\phi_A + \hat{y}\sin\phi_A$, $(\hat{x},\hat{y},\hat{z})$ defining a right-handed triad. For times $t \gg |\omega_A - \omega_B|^{-1}$, eq. (4.5) yields

$$a_{A\lambda}(t) \cong a_{A\lambda}(0)\exp(-i\omega_A t) + (C_{A\lambda 24}\sigma_{24}(0) + C_{A\lambda 34}\sigma_{34}(0))\,t\exp(-i\omega_A t). \tag{4.7}$$

In a similar fashion we obtain

$$a_{B\mu}(t) \cong a_{B\mu}(0)\exp(-i\omega_B t) + (C_{B\mu 12}\sigma_{12}(0) + C_{B\mu 13}\sigma_{13}(0))\,t\exp(-i\omega_B t), \tag{4.8}$$

where

$$C_{B\mu ij} = \frac{1}{\hbar}\left(\frac{2\pi\hbar\omega_B}{V}\right)^{1/2}(\boldsymbol{\mu}_{ij}\cdot\hat{e}_{kB\mu}), \qquad k_B = -\frac{\omega_B}{c}\hat{z}, \qquad \text{and} \qquad \hat{e}_{kB\mu} = \hat{x}\cos\phi_B + \hat{y}\sin\phi_B. \tag{4.9}$$

Using eqs. (4.7) and (4.8) with the initial state (4.3), we obtain

$$\langle a^+_{A\lambda}(t)\,a^+_{B\mu}(t)\,a_{B\mu}(t)\,a_{A\lambda}(t)\rangle \cong (|C_{A24}|^2|C_{B12}|^2\langle\sigma^+_{24}\sigma^+_{12}\sigma_{12}\sigma_{24}\rangle + C^*_{A24}C_{A34}C^*_{B12}C_{B13}\langle\sigma^+_{24}\sigma^+_{12}\sigma_{13}\sigma_{34}\rangle$$

$$+\,C^*_{B13}C^*_{A34}C_{A24}C_{B12}\langle\sigma^+_{34}\sigma^+_{13}\sigma_{12}\sigma_{24}\rangle + |C_{A34}|^2|C_{B13}|^2\langle\sigma^+_{34}\sigma^+_{13}\sigma_{13}\sigma_{34}\rangle + \ldots)t^2, \tag{4.10}$$

where we have only indicated explicitly the contributing terms. All operators are to be evaluated at $t = 0$ and it is easily seen that all expectation values above are equal to one. For simplicity the unnecessary labels λ, μ have now been dropped.

The coincidence rate is therefore

$$R \propto |C_{A24}C_{B12} + C_{A34}C_{B13}|^2 . \tag{4.11}$$

Using $\boldsymbol{\mu}_{12} = (\hat{x} - i\hat{y})\mu_{12}/\sqrt{2}$, etc., we have

$$C_{A24}C_{B12} \propto \exp\{i(\phi_B - \phi_A)\} \quad \text{and} \quad C_{A34}C_{B13} \propto \exp\{-i(\phi_B - \phi_A)\},$$

so that

$$R \propto \cos^2(\phi_A - \phi_B). \tag{4.12}$$

This result indicates that $R_{\min} = 0$: When A and B have their linear polarizers orthogonal they will *never* record a coincidence.

The result (4.11) has the following significance:

(a) C_{B12} is the probability amplitude for the process $|2\rangle \to |1\rangle$ with the emission of a B photon.

(b) C_{A24} is the probability amplitude for the process $|4\rangle \to |2\rangle$ with the emission of an A photon.

(c) Thus $C_{A24}C_{B12}$ is the probability amplitude for the process $|4\rangle \to |2\rangle \to |1\rangle$ with the emission of A and B photons.

(d) C_{B13} is the probability amplitude for the process $|3\rangle \to |1\rangle$ with the emission of a B photon.

(e) C_{A34} is the probability amplitude for the process $|4\rangle \to |3\rangle$ with the emission of an A photon.

(f) Thus $C_{A34}C_{B13}$ is the probability amplitude for the process $|4\rangle \to |3\rangle \to |1\rangle$ with the emission of A and B photons.

(g) $C_{A24}C_{B12} + C_{A34}C_{B13}$ is the total probability amplitude for the process $|4\rangle \to |1\rangle$ with the emission of A and B photons, *when we cannot distinguish whether the process corresponding to $C_{A24}C_{B12}$ or the process corresponding to $C_{A34}C_{B13}$ has occurred.* In our case the processes are indistinguishable because a measurement of linear polarization does not distinguish between left- and right-handed circular polarization.

Hence a more elegant way to derive the coincidence rate is to use the rules [97]

I. "The probability amplitude for two successive possibilities is the product of the amplitudes for the individual possibilities",

II. "The amplitude for a process that can take place in one of several indistinguishable ways is the sum of the amplitudes for each of the individual possibilities".

Before discussing the result (4.12) further, we shall calculate the coincidence rate predicted by the neoclassical theory. The neoclassical coincidence rate must be proportional to

$$\langle a^{+}_{A\lambda}(t)\rangle \langle a^{+}_{B\mu}(t)\rangle \langle a_{B\mu}(t)\rangle \langle a_{A\lambda}(t)\rangle,$$

rather than (4.10). We have instead of eq. (4.10),

$$\begin{aligned} \langle a^{+}_{A\lambda}(t)\rangle \langle a^{+}_{B\mu}(t)\rangle \langle a_{B\mu}(t)\rangle \langle a_{A\lambda}(t)\rangle \cong {} & (|C_{A24}|^2 |C_{B12}|^2 \langle \sigma^{+}_{24}\rangle \langle \sigma^{+}_{12}\rangle \langle \sigma_{12}\rangle \langle \sigma_{24}\rangle \\ & + C^{*}_{A24}C_{A34}C^{*}_{B12}C_{B13}\langle \sigma^{+}_{24}\rangle \langle \sigma^{+}_{12}\rangle \langle \sigma_{13}\rangle \langle \sigma_{34}\rangle \\ & + C^{*}_{B13}C^{*}_{A34}C_{A24}C_{B12}\langle \sigma^{+}_{34}\rangle \langle \sigma^{+}_{13}\rangle \langle \sigma_{12}\rangle \langle \sigma_{24}\rangle + |C_{A34}|^2 |C_{B13}|^2 \langle \sigma^{+}_{34}\rangle \langle \sigma^{+}_{13}\rangle \langle \sigma_{13}\rangle \langle \sigma_{34}\rangle + \dots)t^2 . \end{aligned} \tag{4.13}$$

The above expression is in principle zero, since an atom in an energy eigenstate has vanishing dipole moment (expectation value). In order to have spontaneous radiation in the neoclassical model, we assume that the dipole moments are nonvanishing. Defining

$$\langle \sigma_{lm} \rangle = r_{lm} \exp(\mathrm{i}\theta_{lm}), \tag{4.14}$$

and averaging over ϕ_{A} and ϕ_β with $\phi_{\mathrm{A}} - \phi_\beta$ fixed (in accordance with the experimental procedure of Kocher and Commins), we obtain from eq. (4.13) the coincidence rate

$$R \propto r_{24}^2 r_{12}^2 + r_{13}^2 r_{24}^2 + r_{12}^2 r_{34}^2 + r_{13}^2 r_{34}^2 + 2 r_{24} r_{34} r_{12} r_{13} \cos 2(\phi_{\mathrm{A}} - \phi_{\mathrm{B}}). \tag{4.15}$$

From this result it is seen that

$$R_{\min}/R_{\max} \geqslant \tfrac{1}{3}, \tag{4.16}$$

depending on the r_{lm}. Thus the neoclassical result is different from the quantum-electrodynamical. In particular, the neoclassical theory does not give the noncoincidence predicted by quantum electrodynamics when observers A and B have their polarizers aligned orthogonally. Clauser [31] shows that when the polarizer efficiencies and finite angles subtended by the photodetectors are accounted for in the Kocher–Commins experiment, the quantum-electrodynamical result becomes

$$R_{\min}/R_{\max} \geqslant 0.15, \tag{4.17}$$

rather than zero. The neoclassical result (4.16) is replaced by

$$R_{\min}/R_{\max} \geqslant 0.45. \tag{4.18}$$

The experimental result of Kocher and Commins is

$$R_{\min}/R_{\max} = 0.15 \pm 0.02, \tag{4.19}$$

in striking agreement with (4.17) rather than (4.18).

To understand better the neoclassical prediction, suppose the four transition dipole moments involved are in phase. The "A photon" is a pulse of radiation which is the superposition of the fields from two dipole moments of equal magnitude oscillating in phase with opposite circular polarizations. Therefore, it is linearly polarized. A similar result holds for the "B photon". If we assume equal phases, the A and B pulses are linearly polarized in the same direction. If it happens that the polarizer of observer A is aligned along the direction of the "A photon's" polarization, and if B's polarizer is at angle $\phi_{\mathrm{B}} - \phi_{\mathrm{A}}$ with respect to A's, then the law of Malus gives the fraction of electromagnetic energy that will pass through B's polarizer as $\cos^2(\phi_{\mathrm{B}} - \phi_{\mathrm{A}})$. This appears to give the quantum electrodynamical result. But note that the fact that A observed a pulse does not mean classically that the "A photon" was linearly polarized along the A polarizer direction; if it did, then there would indeed be a noncoincidence effect for $|\phi_{\mathrm{A}} - \phi_{\mathrm{B}}| = \pi/2$. The key effect is that quantum mechanically when A observes a photon he *can* conclude that the "entire" photon is now polarized along the direction at which he set his polarizer. Classically, some *fraction* of energy came through in accordance with Malus' law, and a noncoincidence would be essentially impossible. The noncoincidence effect therefore hinges on this "all or nothing" nature of photon polarization discussed in the first chapter of Dirac's treatise [98].

Einstein, Podolsky and Rosen [7] first noted that quantum theory predicts correlation effects of this type, and they argued that such correlations imply that quantum mechanics does not provide a complete description of an individual system. In our idealized Kocher–Commins experiment, observer A can predict with certitude the linear or circular polarization of the photon detected by B, depending on whether he chooses to measure circular or linear polarization of the photon he detects. Thus the type of measurement performed by A determines what polarization can be predicted for photon B—A has the choice of predicting with certainty either the linear or circular polarization of photon B. The EPR "paradox" in this context is that this circumstance obtains in spite of the obvious fact that the B photon cannot be simultaneously endowed with definite values of the two types of polarization.

The EPR problem has been a subject of intense debate among physicists interested in the conceptual foundations of quantum mechanics. There is not even a consensus of agreement on whether EPR-type correlation effects should be regarded as subtle and intriguing. Feynman [99] has remarked that "we can all agree" that "[the correlation] is *not* a 'paradox'", but "still very peculiar". But another Nobel laureate feels that the EPR problem is "about as subtle as saying that if a coin is cut in half and the two pieces are thrown in opposite directions, then an observer who finds one half can say with certainty that a diametrically opposed observer will find the other half". It is not usually recognized [see, however, ref. 95] that Einstein saw no "paradox" if one adopts the statistical interpretation. His position was clearly stated in 1936:

> Consider a mechanical system constituted of two partial systems A and B which have interaction with each other only during limited time. Then the Schrödinger equation will furnish the ψ function after the interaction has taken place. Let us now determine the physical condition of the partial system A as completely as possible by measurements. Then the quantum mechanics allows us to determine the ψ function of the partial system B from the measurements made, and from the ψ function of the total system. This determination, however, gives a result which depends upon *which* of the determining magnitudes specifying the condition of A has been measured (for instance coordinates *or* momenta). Since there can be only *one* physical condition of B after the interaction and which can reasonably not be considered as dependent on the particular measurement we perform on the system A separated from B it may be concluded that the ψ function is *not* unambiguously coordinated with the physical condition. This coordination of several ψ functions with the same physical condition of system B shows again that the ψ function cannot be interpreted as a (complete) description of a physical condition of a unit system. Here also the coordination of the ψ function to an ensemble of systems eliminates every difficulty [96].

In a footnote he adds that "The operation of measuring A, for example, thus involves a transition to a narrower ensemble of systems. The latter (hence also its ψ function) depends upon the point of view according to which this narrowing of the ensemble of systems is made".

Feynman [100] has expressed the opinion that noncommutativity was not the fundamentally new innovation of quantum mechanics, that "far more fundamental was the discovery that in nature the laws of combining probabilities were *not* those of the classical probability theory of Laplace". According to this view, the most "nonclassical" aspect of quantum mechanics is the existence of a probability amplitude manipulated according to the rules I and II. Dirac [101] now shares the same view. The experimental result of Kocher and Commins is a striking demonstration of the failure of classical probabilistic notions. According to such notions, once a photon passes through A's filter, it has a probability $\cos^2\phi_A$ of passing through his polarizer (given that its polarization is oriented at an angle ϕ_A with respect to that polarizer). Similarly there is a probability $\cos^2\phi_B$ that a photon passing through B's filter will be passed through his polarizer. But, as opposed to the quantum-mechanical treatment, a classical probabilistic approach would associate the quantity $\cos^2\phi_A\cos^2\phi_B$ with the probability of coincidence – a simple multiplication of prob-

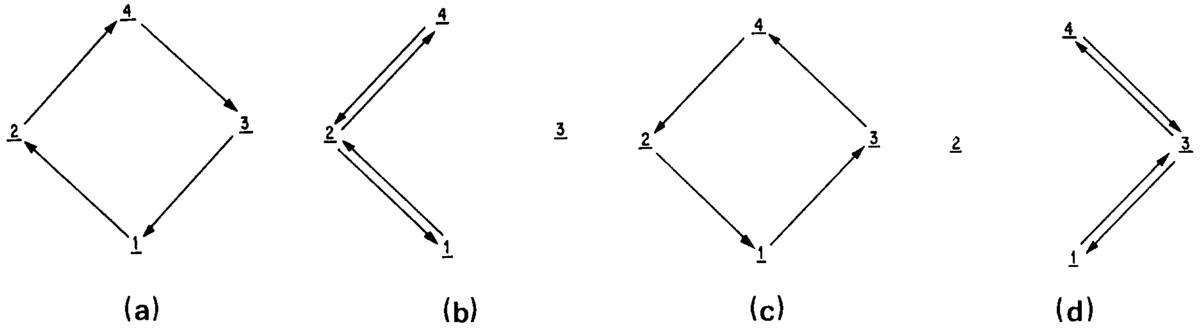

Fig. 1.

abilities with no reference to probability amplitudes. When $\cos^2\phi_A \cos^2\phi_B$ is averaged over ϕ_A and ϕ_B, keeping $\phi_A - \phi_B$ fixed, we find the coincidence rate

$$R \propto \tfrac{1}{2} + \cos^2(\phi_A - \phi_B), \tag{4.20}$$

so that

$$R_{\min}/R_{\max} = \tfrac{1}{3}, \tag{4.21}$$

which is just the prediction of the neoclassical theory.

Photon polarization correlation phenomena have been important in other contexts, as is well known. Wheeler [102] proposed such an experiment as a test of the theory that in the annihilation of singlet positronium the emitted photons should have orthogonal linear polarizations. This prediction results from the use of rules I and II, but by subtracting rather than adding the corresponding probability amplitudes; the subtraction is required because of the odd parity of singlet positronium. Yang [103] considered both types of parity in suggesting that the different polarization correlations in the two cases could provide a means for deciding the symmetry properties of mesons which decay into two photons.

Differences in the predictions of quantum-electrodynamical and semiclassical theories are most significant when they result from the different ways in which the two theories treat probability amplitudes. This is certainly the case in the example discussed in this section. The neoclassical approach fails because it does not treat properly the sequential aspect of the multichannel process involved in the cascade decay. All possible transition probability amplitudes (dipole moments) in the neoclassical approach were treated additively, and they all contributed to the resultant probability. In the quantum-electrodynamical approach, only those terms in the total probability amplitude are included which are consistent with the orthogonality of states. Consider for example the term $\langle\sigma_{24}^{+}\sigma_{12}^{+}\sigma_{13}\sigma_{34}\rangle$ in the quantum-electrodynamical expression (4.10). In a loose sense, we may say that this term takes us from the state $|4\rangle$ to $|3\rangle$ to $|1\rangle$ to $|2\rangle$ and finally back to $|4\rangle$, as illustrated in fig. 1a. Similarly the term $\langle\sigma_{24}^{+}\sigma_{12}^{+}\sigma_{12}\sigma_{24}\rangle$ may be represented as in fig. 1b. The remaining contributions to the quantum-electrodynamical expression for the coincidence rate are represented in figs. 1c and 1d. Only "round-trip" diagrams contribute. The neoclassical expression (4.13), on the other hand, contains terms such as $\langle\sigma_{24}^{+}\rangle\langle\sigma_{12}^{+}\rangle\langle\sigma_{13}\rangle\langle\sigma_{24}\rangle$, which is represented in fig. 2. This term is not included in the quantum-electrodynamical expression because of the orthogonality of $|2\rangle$ and $|3\rangle$. This feature is absent from the neoclassical approach, where dipole moments rather than states are the essential entities characterizing the atom.

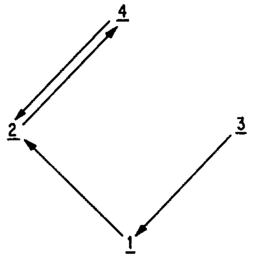

Fig. 2.

4.4. *Quantum beats in spontaneous emission*

With regard to the importance of probability amplitudes, it is of interest to have a difference in neoclassical and quantum-electrodynamical predictions for a multichannel process in which there are no sequential aspects requiring rule I. Such a situation has in effect been discussed recently by two groups of authors [104, 105]. The rule for treating probability amplitudes which is violated by the neoclassical approach in this case is a corollary of rule II: probability amplitudes must never be added for different and distinct final states [106]. Before discussing this example, it may be useful to review briefly the correspondence between an electromagnetic field amplitude in a semiclassical theory and a probability amplitude in the quantum-electrodynamical theory.

The most familiar example of this correspondence is probably the Young two-slit experiment. A monochromatic wave impinging on a plate with two narrow slits produces an interference pattern on a screen behind the plate. Classically, the interference pattern results from the superposition of the fields reaching a point on the screen from the individual slits. Quantum mechanically, the interference pattern results from the addition of probability amplitudes for the (indistinguishable) routes available to the photon for reaching a given point on the screen. In each case a quantity is calculated and then squared to yield a quantity directly related to experiment. In the classical description of the Young experiment the electromagnetic field amplitude is found, and its square is related to the observed intensity distribution. In the quantum-mechanical description a probability amplitude must be found and then squared to obtain the observed photon probability distribution at the screen. This distribution is obtained by repeating the "one-photon" experiment many times and recording the resulting spots on the screen. The resulting distribution represents the photon probability distribution in the sense of the quantum-mechanical ensemble, and is identical to the classical intensity distribution.

Another such example has been given recently, involving the spontaneous radiation from an atom between two parallel mirrors. Milonni and Knight [14] obtained the emission rate from the point of view of quantum-mechanical probability amplitudes, while Chance et al. [107] found the same result from a classical viewpoint dealing directly with the electromagnetic field and the boundary conditions imposed by the mirrors. It is interesting to note that such modification of radiative lifetimes by the presence of mirrors can be studied experimentally [13].

As another example, consider a system of two identical two-state atoms, one of which is initially excited and the other in the ground state. If there is complete uncertainty as to which atom is excited, the initial atomic state vector may be written

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}[\,|+\rangle_1\,|-\rangle_2 \pm |-\rangle_1\,|+\rangle_2]\,. \tag{4.22}$$

Suppose the atoms are located within a transition wavelength of each other and that the excitation has been effected by means of a resonant plane-wave field. Then only the state $|\psi_+\rangle$ requires consideration [108]. When the system emits radiation, the final state is $|-\rangle_1\,|-\rangle_2$, with one photon in the field. There is no way to determine in principle which atom emitted; the two probability amplitudes for single-atom emission are therefore added. It is found therefore that the process $|\psi_+\rangle \to |-\rangle_1\,|-\rangle_2$ has a transition probability which is twice that for the single-atom process $|+\rangle \to |-\rangle$, i.e., the system radiates cooperatively, or "superradiantly". The classical explanation of this cooperation is simply that the state $|\psi_+\rangle$ corresponds classically to a system of two dipoles oscillating in phase (the absolute phase being random), and the fields from these dipoles add coherently to produce the interference effect. Such interference effects are of course of prime importance in antenna theory.

Chow et al. [104] and Herman et al. [105] have noted the differing predictions of neoclassical theory and quantum electrodynamics for certain resonance fluorescence experiments. The essential features of their arguments can be discussed with reference to the two three-state schemes illustrated in figs. 3 and 4. In case A, represented in fig. 3, an atom is supposed to be excited to a coherent superposition of states 2 and 3, both of which are optically connected to a ground state 1. In case B, represented in fig. 4, the atom is supposed to be excited to state 3, which is optically connected to the lower states 1 and 2. Neoclassical theory predicts quantum beats in the spontaneous emission for both cases A and B, whereas quantum electrodynamics predicts beats only for case A. Rather than performing the calculations in detail for the two cases, we shall merely indicate how the different predictions arise.

A simple short-time perturbation-theory approach yields the result that in case A the quantum-electrodynamical positive-frequency part of the electric field operator is $\sigma_{12} + \sigma_{13}$, omitting unessential multiplicative vector constants. Thus the correlation function $\langle E^{(-)}E^{(+)}\rangle$ relating to the intensity of the spontaneous radiation is

$$\langle(\sigma_{12}^\dagger + \sigma_{13}^\dagger)(\sigma_{12} + \sigma_{13})\rangle = \langle\sigma_{22}\rangle + \langle\sigma_{33}\rangle + \langle\sigma_{23}\rangle + \langle\sigma_{23}\rangle^*, \tag{4.23}$$

the expectation value taken over the initial state $c_2|2\rangle + c_3|3\rangle$. The last two terms correspond to the radiation at the beat frequency $(E_3 - E_2)/\hbar$. The corresponding neoclassical intensity function would be effectively

$$\langle\sigma_{12}^\dagger + \sigma_{13}^\dagger\rangle\langle\sigma_{12} + \sigma_{13}\rangle,$$

the term $\langle\sigma_{13}^\dagger\rangle\langle\sigma_{12}\rangle + \langle\sigma_{12}^\dagger\rangle\langle\sigma_{13}\rangle$ corresponding to the beating radiation. The neoclassical approach

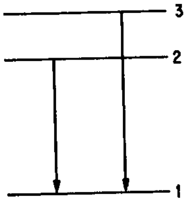

Fig. 3.

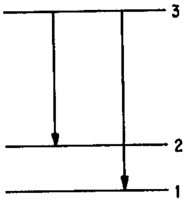

Fig. 4.

gives the beat effect as a result of the interference in the radiation from the two transition dipoles oscillating at the different frequencies $(E_3 - E_1)/\hbar$ and $(E_2 - E_1)/\hbar$. The quantum-electrodynamical result may be interpreted formally in the same way, remembering of course that the fields are Hilbert-space operators. Alternatively, the quantum-electrodynamical result may be interpreted in terms of the addition of probability amplitudes: The amplitudes for the processes $|3\rangle \to |1\rangle$ and $|2\rangle \to |1\rangle$ must be added because the final state does not provide any information as to which of the two processes occurred. This type of interference effect is well known: For the special case where the states $|2\rangle$ and $|3\rangle$ are degenerate Zeeman states of an excited level, for example, the zero-field crossing leads to a Hanle effect.

For the case B, the corresponding quantum-electrodynamical correlation function $\langle E^{(-)}E^{(+)}\rangle$ may be taken for the purpose of this discussion to be

$$\langle(\sigma_{23}^{+} + \sigma_{13}^{+})(\sigma_{23} + \sigma_{13})\rangle = \langle\sigma_{33}\rangle + \langle\sigma_{33}\rangle + \langle\sigma_{31}\sigma_{23}\rangle + \langle\sigma_{32}\sigma_{13}\rangle = 2\langle\sigma_{33}\rangle . \tag{4.24}$$

The term $\langle\sigma_{31}\sigma_{23}\rangle = \langle 3|3\rangle\langle 1|2\rangle\langle 3|3\rangle = \langle 1|2\rangle = 0$ because of the orthogonality of $|1\rangle$ and $|2\rangle$; similarly $\langle\sigma_{32}\sigma_{13}\rangle = 0$. Eq. (4.24) corresponds to the addition of probabilities. The corresponding probability amplitudes do not enter additively because the final state ($|2\rangle$ or $|1\rangle$) provides the information necessary to determine which process ($|3\rangle \to |2\rangle$ or $|3\rangle \to |1\rangle$) took place. Probability amplitudes are not added for distinct final states. The neoclassical approach, however, deals with an intensity function related to

$$\langle\sigma_{23}^{+} + \sigma_{13}^{+}\rangle\langle\sigma_{23} + \sigma_{13}\rangle = \langle\sigma_{23}^{+}\rangle\langle\sigma_{23}\rangle + \langle\sigma_{13}^{+}\rangle\langle\sigma_{13}\rangle + \langle\sigma_{13}^{+}\rangle\langle\sigma_{23}\rangle + \langle\sigma_{32}\rangle\langle\sigma_{13}\rangle \tag{4.25}$$

rather than (4.24). The orthogonality of $|1\rangle$ and $|2\rangle$ is not accounted for (or equivalently the probability amplitudes are added even for distinct final states) and so a beating at frequency $(E_2 - E_1)/\hbar$ is predicted which is not predicted by the quantum-electrodynamical approach.

Interference effects in the resonance fluorescence of crossed excited atomic states were observed experimentally by Colegrove et al. [109] and discussed further by Franken [110]. The effect was discussed also by Breit [111] much earlier; Breit noted the absence of interference effects for the case B. Chow et al. [104] have suggested an experiment to determine whether such interference effects are actually absent in case B. Such an experimental verification would show, as does Clauser's analysis of the Kocher–Commins experiment, that the neoclassical theory fails to treat properly the quantum-mechanical probability amplitude, i.e., it would provide another example where the difference between an electromagnetic field amplitude in the classical sense and a quantum-mechanical probability amplitude is of direct experimental consequence.

4.5. *Neoclassical theory and the blackbody spectrum*

Nesbet [88] has studied the implications of the neoclassical theory of spontaneous emission for the blackbody spectrum, using a neoclassical version of Einstein's original rate-equation approach. For simplicity, we consider here an ensemble of two-state atoms and ignore the possibility of degeneracy. The condition for thermal equilibrium is that the rate at which atoms are removed from the lower state due to absorption of radiation be equal to the rate at which they are removed from the upper state because of spontaneous and stimulated emission of radiation. Thus, in thermal equilibrium,

$$|c_1|^2 B\,\rho(\omega_0) = |c_2|^2 B\,\rho(\omega_0) + A\,|c_1|^2\,|c_2|^2\,. \tag{4.26}$$

The notation is standard. B and A are the Einstein coefficients, with $A/B = \hbar\omega_0^3/\pi^2 c^3$, and $\rho(\omega_0)$ is the radiation density at frequency ω_0. Eq. (4.26) differs from the Einstein rate equation only in the second term on the right-hand side: Einstein assumed the rate of spontaneous emission to be proportional to the upper-state probability $|c_2|^2$. In the neoclassical theory the corresponding spontaneous emission rate is instead proportional to $|c_1|^2\,|c_2|^2$, the product of lower- and upper-state probabilities. Thus $A\,|c_1|^2\,|c_2|^2$ replaces the usual $A\,|c_2|^2$.

Equation (4.26) may not seem entirely in accord with the neoclassical description of an atomic transition. The stimulated emission and absorption terms in this equation depend respectively on upper- and lower-state probabilities. In the neoclassical theory the energy levels are superfluous and it is the transition dipole moment – treated as a classically radiating dipole – that really matters. This conceptual difficulty may be overcome by assuming that the state populations are altered predominantly by collisional rather than radiative processes. In this case the neoclassical equations for a two-state atom yield predictions identical to those based on the optical Bloch equations insofar as stimulated emission and absorption are concerned. This assumption incidently justifies the use of the Boltzmann distribution $|c_2|^2/|c_1|^2 = \exp(-\hbar\omega_0/kT)$ below [112].

But a serious implication for the thermal distribution of atomic energies arises owing to the fact that the neoclassical theory describes the individual atom. The quantity $z(t)$ defining the atomic inversion does not have the probabilistic interpretation in the neoclassical theory that it does in the optical Bloch equations. The *individual atom* is therefore characterized by the energy $\frac{1}{2}\hbar\omega_0 z(t)$, and this of course need not be either $\pm\frac{1}{2}\hbar\omega_0$. The neoclassical atom interacting with an electromagnetic field must be considered to have a continuous distribution of energies if this interpretation is adhered to. We have already noted Einstein's objection that such a theory of the individual system is inconsistent with the results of the Franck–Hertz experiment. It will also be noted that such a theory does not account properly for the specific heats of diatomic gases. Barring unforeseen developments, these two observations alone seem to require a modification – or re-interpretation – of the neoclassical theory along the lines of an approach which does not describe the individual system. We shall suppose, nevertheless, that Nesbet's rate equation (4.26) is relevant to the neoclassical theory, or a modification thereof.

The assumption of the Boltzmann distribution

$$|c_2|^2/|c_1|^2 = \exp(-\hbar\omega_0/kT) \tag{4.27}$$

at thermal equilibrium defines the radiation density

$$\rho(\omega_0) = \frac{(A/B)\,|c_1|^2}{\cancel{|c_1|^2/|c_2|^2} - 1} = \rho_0(\omega_0)\,|c_1|^2\,, \tag{4.28}$$

where $\rho_0(\omega_0)$ is the Planck spectral density for the assumed blackbody temperature T. The neoclassical approach therefore does not yield the Planck spectrum, but rather a spectrum which depends upon the population of the lower state of each transition, i.e., with each level i is associated a spectral density

$$\rho_i(\omega) = \rho_0(\omega)\,|c_i|^2\,. \tag{4.29}$$

Nesbet [88] has noted that the Planck spectrum can be obtained from the neoclassical approach if it is hypothesized that the "total" equilibrium radiation density is

$$\rho_{\mathrm{T}}(\omega)=\sum_i \rho_i(\omega), \tag{4.30}$$

i.e., the incoherent sum of the "partial" densities $\rho_i(\omega)$. The Planck spectrum then results from the fact that

$$\sum_i |c_i|^2 = 1. \tag{4.31}$$

It is difficult to imagine a test of this hypothesis, since only $\rho_{\mathrm{T}}(\omega)=\rho_0(\omega)$, and not $\rho_i(\omega)$ for any level i, would be measurable. This circumstance, together with the conceptual difficulties noted above, forces us to conclude that a full understanding of the implications of the neoclassical theory for the blackbody spectrum requires further study. No very conclusive evidence for or against the validity of a neoclassical theory of blackbody radiation is available at this time.

4.6. *Further evidence and concluding remarks*

The most immediately obvious feature of the neoclassical theory is the prediction that an atom in a pure excited state will not radiate spontaneously. The experimental test of this prediction is complicated by the neoclassical interpretation of the atomic inversion. In the neoclassical model of the two-state atom, for example, the *individual atom* is characterized by an inversion $z(t)$. The statement, for example, that $z(t)$ is zero is not interpreted as meaning that the atom is with equal probability either in the upper or lower state; it can only be interpreted to mean that during a transition the atom must be characterized by an energy intermediate between those of the two states involved.

The experimental test of the neoclassical prediction of decreasing spontaneous emission rates as z approaches 1 requires, therefore, a highly effective pumping scheme. For a sample of atoms characterized by z close to -1, the quantum-mechanical interpretation is that each atom is either in the upper state or the lower state, with the majority of atoms in the lower state. Neoclassically, *each* atom is characterized by an energy close to that of the ground state. A measurement of the rate of radiation from such a sample cannot test our interpretation; the two theories predict very similar results. Only when z is very close to 1 can a measurement of the radiation rate unequivocally decide between the two theories, for in this case they both describe effectively every atom of the sample as occupying the upper state. Jaynes [113] has suggested that the test might be accomplished by pumping with a π-pulse and observing the resulting spontaneous emission. The results of such an experiment have been described by Gibbs [114] and lend support to the quantum-electrodynamical theory. Jaynes [6] has questioned the relevance of the experiment on the grounds that atoms were not pumped far enough from the ground state to make the differences between the two theories readily observable. A different experiment to test the neoclassical prediction of excited-state metastability has been performed by Wessner et al. [115]. Again results contrary to neoclassical predictions were reported.

Nesbet [116] has noted the apparent failure of the neoclassical theory to explain the differing intensities of Stokes and anti-Stokes lines in Raman scattering. The neoclassical theory predicts equal cross sections for Stokes and anti-Stokes processes, and therefore does not account for the higher intensity of the Stokes line. The neoclassical prediction can be understood in terms of the corresponding prediction for spontaneous emission; there the emission rate depends symmetrically on the probabilities of the initial and final states, and this is basically the reason for the neoclassical prediction of equal intensities for the Stokes and anti-Stokes lines in Raman scattering.

The most serious objection to neoclassical theory, and semiclassical theories in general, remains that of Clauser [31]. It seems unlikely that *any* theory describing the electromagnetic field classically within the framework of the Maxwell equations can account for the polarization correlations discussed by Clauser. For this reason Clauser's argument, and those of Chow et al. [104] and Herman et al. [105], which are of similar generality, have been discussed in some detail. These objections can be understood in terms of the differing predictions of calculations based on quantum-mechanical probability amplitudes on the one hand, and classical electromagnetic field amplitudes on the other. Moreover, they can be unequivocally tested experimentally.

Aside from the failure to explain the polarization correlations observed by Kocher and Commins, the most serious shortcoming of the neoclassical theory is probably conceptual. As a theory of the individual system it suffers from the conceptual difficulties best illustrated by Schrödinger's "cat experiment". Indeed, it would seem to grant to Schrödinger's cat during the radioactive decay of the "trigger" a status intermediate between life and death! It is perhaps worthwhile here to quote one of Einstein's objections to the interpretation of the wave function as a description of the individual system. The objection was briefly mentioned earlier.

> I ask first: How far does the ψ function describe a real condition of a mechanical system? Let us assume the ψ_r to be the periodic solutions (put in the order of increasing energy values) of the Schrödinger equation. I shall leave open, for the time being, the question as to how far the individual ψ_r are *complete* descriptions of physical conditions. A system is first in the condition ψ_1 of lowest energy E_1. Then during a finite time a small disturbing force acts upon the system. At a later instant one obtains then from the Schrödinger equation a ψ function of the form
>
> $$\psi = \Sigma\, C_r\, \psi_r,$$
>
> where the C_r are (complex) constants. If the ψ_r are "normalized", then $|C_1|$ is nearly equal to 1, $|C_2|$ etc. is small compared with 1. One may now ask: Does ψ describe a real condition of the system? If the answer is yes, then we can hardly do otherwise than ascribe to this condition a definite energy E, and, in particular, such an energy as exceeds E_1 by a small amount (in any case $E_1 < E < E_2$). Such an assumption is, however, at variance with the experiments on electron impact such as have been made by J. Franck and G. Hertz ... the ψ function does not, in any sense, describe the condition of *one* single system [96].

While it is true that Einstein's argument was made to refute the Copenhagen interpretation of quantum mechanics, it is clear that the argument is even stronger when applied against the neoclassical theory. Here, it seems, there can be no escape into "reduction-of-the wave-packet" defenses.

Another inconsistency of the neoclassical theory is the interpretation of the wave function in the Schrödinger sense while at the same time accepting the energy levels calculated from a Schrödinger equation for *point* particles. This inconsistency was recognized by Schrödinger himself. We emphasize again that, physically, the neoclassical theory is identical to Schrödinger's semiclassical theory. It was abandoned, not by fiat, but on solid physical grounds, by quantum theorists of nearly half a century ago. An examination of the plethora of published papers pertaining to the neoclassical theory reveals that this fact goes practically unnoticed, or at least unmentioned.

We must conclude that the neoclassical theory in its present form is not a viable one. It has, however, served one very laudable purpose: It has encouraged researchers in quantum optics and electronics to examine more critically the foundations of their subject insofar as it involves the interaction between light and atomic matter. The result has undeniably been a deeper appreciation for semiclassical approaches.

For a detailed analysis of the neoclassical theory, the reader is referred to the review by Jaynes [6]; that article gives the "other side of the story". Our discussion has centered predominantly on the neoclassical approach to spontaneous emission. Spatial limitations would have rendered a discussion of broader scope essentially bibliographical.

5. The vacuum field in nonrelativistic radiation theory

5.1. *Introduction*

The concept of a fluctuating radiation field existing even in the absence of any radiating matter, briefly referred to as the "vacuum field", has played an important heuristic role in the theory of the interaction of light with bound charges. Formally, this field appears as the homogeneous solution of the operator Maxwell equations for the electromagnetic field. In the model of the one-electron atom in the absence of any externally applied fields, the vacuum field prevents the electron from spiralling into the nucleus under the dissipative influence of the radiation reaction field; formally, it prevents the collapse of unitarity. But in the Lamb-shift problem, the vacuum field is of more direct significance, and from one point of view, is *responsible* for the Lamb shift. The vacuum field also appears directly responsible for such effects as the van der Waals forces between neutral polarizable particles [117] and the forces between conducting parallel plates [118].

The vacuum field has the zero-point energy density $\frac{1}{2}\hbar\omega$ per normal mode. The consequent divergences in the field Hamiltonian can be avoided by normally ordering and subtracting away the infinite zero-point energy of the field on the grounds that it is a constant and therefore makes no contribution to the Heisenberg equations of motion. But the vacuum field nevertheless reappears as the homogeneous solution of the Maxwell equations. In some problems this is inconsequential, since it can be argued that only normally ordered expectation values of field operators are of any physical significance. This circumstance usually obtains, for example, when field correlation functions measured with photodetectors are to be calculated. The argument that only normally ordered field correlation functions are physically significant is in general unfounded, however, and we have seen for example that the calculation of the nonrelativistic part of the Lamb shift can be cast into a form where non-normally ordered vacuum field correlation functions are physically significant.

The question arises then whether the vacuum field is of more than formal significance. Is there really a physical significance to the vacuum field, or is it merely an artifact resulting from the quantization of the field, allowing us, if we choose, to interpret some of the more peculiar (nonclassical) predictions of quantum electrodynamics in the language of classical physics? If there is a "physical" vacuum field, can we perhaps construct a theory of classical fluctuations which might incorporate in a classical way the successful predictions of quantum electrodynamics? This is evidently fertile ground for speculation; but little is gained unless the speculation is founded upon actual calculations. In this section is reviewed some of the recent work on a classical theory of the "vacuum field". As will be seen, there is some calculational support for such a theory. There are only a few problems which have been treated in detail in this area, but the results of these calculations are rather remarkable.

5.2. *Random electrodynamics*

The solution of the operator Maxwell equations of quantum electrodynamics is as in classical electrodynamics the sum of the solutions of the homogeneous and inhomogeneous equations. When there are no externally applied fields the solution of the homogeneous equation plays the role, in the sense of expectation values, of the vacuum field. We have already examined a theory in which the source field is described classically. It is natural now to focus our attention on a classical theory of the vacuum field. But we must approach such a theory in a rather different spirit. Our examination of the classical treatment of the source field was restricted to the source field at the source – the field of radiation reaction; few physicists would doubt that classical physics provides for a field of radiation reaction. On the contrary, it is not immediately obvious that a theory of the vacuum field can remain within the bounds of classical physics. This field vanishes, after all, in the "classical limit" $h \to 0$.

As a first step toward a classical theory of the vacuum field, we might simply invent a random classical field to simulate the quantum-electrodynamical vacuum field. The individual Fourier components in the plane-wave expansion of the field might be given random phases, with their amplitudes adjusted to give an energy $\frac{1}{2}\hbar\omega$ per normal mode. Although such a theory leads to some interesting results, as will be evident from the discussion below, it can hardly be said to have an independent *raison d'etre*; it is at best an interesting subterfuge.

Suppose, however, that a fluctuating vacuum field can be shown to be entirely consistent with classical electrodynamics, and that it need not be introduced on an *ad hoc* basis. Suppose further that even the form of the zero-point energy density can be inferred without invoking any quantum-mechanical concepts, and that Planck's constant appears naturally as a required fundamental constant. Then the classical theory of the vacuum field would clearly deserve more careful investigation – if only to deepen our understanding of the quantum-electrodynamical theory. It is precisely such a classical theory of the vacuum field which has been examined in recent years by Boyer [15–21] and others.

Nernst [119] took up the idea of a finite zero-point radiation energy density in 1916, but it has only been in recent years, especially in the work of Marshall [120, 121] and Boyer [15–21], that the notion has been incorporated into a consistent classical theory. Marshall derived the form of the zero-point radiation energy density by requiring the mean-square displacement of a classical charged harmonic oscillator located anywhere in space to be that predicted by quantum mechanics; the fluctuating vacuum field was viewed as the *cause* of this mean-square displacement. Marshall further showed that the zero-point radiation spectrum is Lorentz-invariant. Boyer [15] independently demonstrated the Lorentz invariance of the spectrum, and has viewed this Lorentz invariance as a basic starting point from which a classical theory of the vacuum field derives. Lorentz invariance *requires* a spectral density proportional to frequency; comparison of theory and experiment requires the constant of proportionality to be Planck's constant. Planck's constant thus arises in a way no less "basic" than the way it was introduced by Planck. In fact the derivation of the blackbody spectrum given by Boyer [15] on the basis of this new classical theory would probably have seemed more reasonable to physicists in 1900 than Planck's.

Boyer [122] has recently discussed the current status of this classical theory of the vacuum field. Because it is a classical theory, the atomic matter with which the field interacts has been described according to the Lorentz model. The resulting theory of interacting zero-point radiation and matter has been named "random electrodynamics". We shall now review briefly Boyer's arguments for the construction of the theory of random electrodynamics from classical electrodynamics.

In Coulomb-gauge electrodynamics the transverse fields may be obtained by first solving the wave equation for the (transverse) vector potential:

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\ddot{\boldsymbol{A}} = -\frac{4\pi}{c}\boldsymbol{J}^{\perp}, \tag{5.1}$$

where $\boldsymbol{J}$ is the current density of the source of radiation. The solution of eq. (5.1) is of the form

$$\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{A}_0(\boldsymbol{r}, t) + \boldsymbol{A}_{\mathrm{s}}(\boldsymbol{r}, t), \tag{5.2}$$

where $\boldsymbol{A}_{\mathrm{s}}(\boldsymbol{r}, t)$ is a solution of the inhomogeneous wave equation and $\boldsymbol{A}_0(\boldsymbol{r}, t)$ is a transverse solution of the homogeneous equation

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\ddot{\boldsymbol{A}} = 0. \tag{5.3}$$

The solution $\boldsymbol{A}_{\mathrm{s}}(\boldsymbol{r}, t)$ may be written formally in terms of a Green function for the scalar wave equation. The obvious, or rather traditional, approach is to use the retarded Green function and take $\boldsymbol{A}_0$ to be identically zero. This is the approach used in the first theory of interacting radiation and matter, advanced by Lorentz [81]. The use of the retarded Green function satisfies our intuitive notion of causality, while the *choice* $\boldsymbol{A}_0 \equiv 0$ is reasonable since we expect no field when there is no source. But we can choose any (transverse) solution of the homogeneous wave equation for $\boldsymbol{A}_0$ and still satisfy the wave equation for the total vector potential. The choice $\boldsymbol{A}_0 \equiv 0$ defines "traditional electrodynamics". The choice of a random field defines random electrodynamics. At this stage we need not inquire into the origin of this random field, but we emphasize that its inclusion clearly keeps us within the framework of classical electrodynamics.

Now the random electric and magnetic fields may be expanded in transverse plane waves as

$$\boldsymbol{E}_0(\boldsymbol{r}, t) = \mathrm{i}\sum_{\boldsymbol{k},\lambda} \left(C_{\boldsymbol{k}\lambda}\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t + \theta_{\boldsymbol{k}\lambda})\} - C^*_{\boldsymbol{k}\lambda}\hat{\boldsymbol{e}}^*_{\boldsymbol{k}\lambda}\exp\{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t + \theta_{\boldsymbol{k}\lambda})\}\right), \tag{5.4}$$

with $\boldsymbol{B}_0$ obtained from the equation

$$-\frac{1}{c}\dot{\boldsymbol{B}}_0(\boldsymbol{r}, t) = \nabla\times \boldsymbol{E}_0(\boldsymbol{r}, t). \tag{5.5}$$

The $\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}$ are unit polarization vectors, with $\boldsymbol{k}\cdot\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda} = 0$. The "randomness" of the field is contained in the phases $\theta_{\boldsymbol{k}\lambda}$. The $\theta_{\boldsymbol{k}\lambda}$ are assumed to be uniformly and independently distributed over $[0, 2\pi]$. Similar random classical fields were used by Planck [123] and by Einstein and Hopf [124] to represent fields of *thermal* radiation. Einstein and Hopf took the amplitudes of the different plane waves in the expansion to be independent random variables normally distributed about zero mean. A long controversy ensued between Einstein and von Laue [125] over the assumption of independence. Many years passed before the theory of random processes was approached with sufficient mathematical rigor to justify the assumption of Einstein and Hopf.

The mean energy density of the random field is

$$\frac{1}{8\pi}\langle \boldsymbol{E}_0^2(\boldsymbol{r},t)+\boldsymbol{B}_0^2)\rangle_\theta$$

where $\langle\ \rangle_\theta$ denotes an average with respect to the random variables $\{\theta_{k\lambda}\}$. It is easily found that

$$\frac{1}{8\pi}\langle \boldsymbol{E}_0^2+\boldsymbol{B}_0^2\rangle_\theta=\frac{1}{2\pi}\sum_{\boldsymbol{k},\lambda}|C_{\boldsymbol{k}\lambda}|^2. \tag{5.6}$$

Now if the field is to be isotropic we must have $C_{\boldsymbol{k}\lambda}$ depend on $\boldsymbol{k}$ only through its magnitude k; therefore, we have

$$\frac{1}{8\pi}\langle \boldsymbol{E}_0^2+\boldsymbol{B}_0^2\rangle_\theta=\frac{1}{\pi}\sum_{k}|C_k|^2. \tag{5.7}$$

To be consistent with our earlier convention we have represented the random field as a discrete sum over plane-wave modes. If we use a continuous, integral summation from the start we obtain

$$\frac{1}{8\pi}\langle \boldsymbol{E}_0^2+\boldsymbol{B}_0^2\rangle_\theta=\int_0^\infty \mathrm{d}\omega\,\frac{\omega^2}{c^3}\,h^2(\omega) \tag{5.8}$$

in Boyer's notation [15] in which

$$\boldsymbol{E}_0(\boldsymbol{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int \mathrm{d}^3k\,\hat{\boldsymbol{e}}(\boldsymbol{k},\lambda)\,h(\omega_k)\exp\{\mathrm{i}(kct-\boldsymbol{k}\cdot\boldsymbol{r}-\theta(\boldsymbol{k},\lambda))\} \tag{5.9}$$

instead of (5.4). $|C_k|^2$ is related to Boyer's $h^2(\omega_k)$ by

$$|C_k|^2=(2\pi^3/V)h^2(\omega_k), \tag{5.10}$$

where V is the normalization volume implicit in (5.4).

The function $h^2(\omega)$ may be determined from the requirement that the spectral density be isotropic in all inertial frames. The radiation energy density in the frequency interval $[a,b]$ measured by an observer should be the same as that measured when he is moving with a constant velocity with respect to the original frame of measurement. Carrying out the Lorentz transformations on the electric and magnetic fields, and taking into account the Doppler shift in frequencies, it is found that to satisfy the above requirement we must have [15]

$$h^2(\omega)=\text{const.}\times\omega. \tag{5.11}$$

This result can be equivalently obtained by requiring that the zero-point field does not give rise to any velocity-dependent forces (see eq. (5.31)). Thus the zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode follows by a suitable choice of a constant of proportionality. The constant may be left undetermined at this point, but from the consideration of different examples, such as the blackbody problem, it is found necessary to take

$$h^2(\omega) = (\hbar/2\pi^2)\omega. \tag{5.12}$$

From eqs. (5.7) and (5.10), it is seen that this choice of constants corresponds to

$$\langle E_0^2 + B_0^2 \rangle_\theta = \frac{1}{V} \sum_{k,\lambda} \tfrac{1}{2}\hbar\omega_k . \tag{5.13}$$

This is of course a familiar result of quantum electrodynamics, if $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are replaced by the corresponding free-field operators and $\langle\ \rangle_\theta$ is replaced by the quantum-mechanical ensemble average (expectation value). But in random electrodynamics there are no operators acting in a Hilbert space of state vectors; it is a purely classical electrodynamical theory of the "vacuum field".

One might question the validity of assigning a zero-point radiation energy density to the classical vacuum field; the apparent failure to measure such temperature-independent radiation seems reason enough for Lorentz's choice $\boldsymbol{A}_0 \equiv 0$. But random electrodynamics is intended to be a theory of microscopic, atomic matter. Just as in quantum electrodynamics the influence of the fluctuating vacuum field can be considered to maintain the stability of the atom, so too in random electrodynamics it is hoped that a similar result can be established. We must not expect to be able to detect the zero-point radiation which maintains the stability of the molecules of a detector, neither in the quantum-electrodynamical nor the random-electrodynamical description of the zero-point radiation. Only the radiation above the zero-point background is measurable [122]. Observation of *changes* in the density of random radiation at different points in space is precluded, or at least complicated, by its homogeneity, isotropy, and Lorentz invariance [122].

Random electrodynamics seems to be only in an initial stage of its development. Much of what has been done toward a classical theory of the vacuum field is in the realm of speculative physics and as such will not be discussed in detail here. A very interesting derivation of the blackbody spectrum has been given by Boyer [15], however, and the calculation is reviewed in the next subsection. The subsection following the next is devoted to Boyer's random-electrodynamical considerations on the van der Waals forces. The corresponding Heisenberg-picture quantum-electrodynamical calculations are shown to suggest a possible correspondence between random and quantum electrodynamics.

5.3. *Random electrodynamics and the blackbody spectrum*

Einstein's rate-equation derivation of the blackbody spectrum [2] was quantum-mechanical in that it dealt with discrete atomic energy levels. The fundamentally new innovation in the derivation was the introduction of the concept of stimulated emission. Without the stimulated-emission term, the rate-equation approach yields Wien's law. Moreover, all attempts to derive the Planck spectrum on the basis of Lorentz-model oscillators in equilibrium with radiation failed, and led inevitably to the Rayleigh–Jeans law. The classical oscillator absorbs energy from the field at a rate proportional to the radiation energy density at the oscillator frequency; it radiates energy back into the field at a rate proportional to its energy, which according to classical statistical mechanics is proportional to the (equilibrium) temperature T. The equilibrium condition is that these two rates be equal, and therefore that the radiation spectral density be proportional to T. Classical physics thus predicts the Rayleigh–Jeans spectrum

$$\rho(\omega) = \omega^2 kT/\pi^2 c^3 . \tag{5.14}$$

Einstein and Hopf [124] presented a detailed analysis of such a classical model of the blackbody problem in 1910, and obtained the Rayleigh–Jeans spectrum. Boyer [15] has analyzed the Einstein–Hopf model from the viewpoint of random electrodynamics. The analysis differs from that of Einstein and Hopf by the inclusion of the zero-point field and a consequent need to consider more carefully the effect of the container walls. It seems quite remarkable that Boyer's analysis should lead to the Planck radiation law. Random electrodynamics, a classical electrodynamical theory, explains the phenomenon often regarded as the paradigm for the failure of classical physics. There are no discrete atomic energy levels in Boyer's analysis, and the concept of stimulated emission does not enter.

Einstein regarded his rate-equation derivation of the Planck radiation law as "astonishingly simple", and indeed the derivation is familiar to every student of elementary modern physics. But it is not so well known that Einstein considered more important the part of his paper dealing with the change in momentum of the atoms due to the absorption and emission of radiation. In the last sentence of his paper, he remarks that "a theory [of thermal radiation] can only be regarded as justified when it is able to show that the impulses transmitted by the radiation field to matter lead to motions that are in accordance with the theory of heat". Einstein showed that the momentum transfers are consistent with classical statistical mechanics if the thermal radiation satisfies Planck's law. It was with such an approach, based on considerations of particle momentum changes due to radiation, that Einstein and Hopf [124] had earlier obtained the Rayleigh–Jeans law. The consideration of discrete atomic processes, suggested by the Bohr model, was the key to the success of Einstein's second attempt. Because it is apparently not so well known, we shall review here the second part of Einstein's famous 1917 paper, and show that the Planck law follows from the consideration of momentum transfers. Then we shall discuss the Einstein–Hopf approach, and why it failed to yield the Planck law. Finally we discuss the modifications of the Einstein–Hopf approach suggested by random electrodynamics, and how Boyer [15] thus derives the blackbody spectrum.

Einstein [2] considered the changes in momentum of a gas molecule due to the interaction with thermal radiation. At equilibrium the corresponding changes in kinetic energy must balance, the mean kinetic energy being stationary in time. If the model only allows the molecules to move in one dimension, this mean kinetic energy must be

$$\tfrac{1}{2} m \overline{V^2} = \tfrac{1}{2} kT. \tag{5.15}$$

It is assumed that each molecule is a two-level system, with energy levels E_1 and $E_2 > E_1$. The molecules interact with radiation by spontaneous and stimulated emission and by absorption. Molecular collisions are assumed to be perfectly elastic.

We first consider a molecule initially at rest in the laboratory frame, and consider its interaction with radiation during some time interval τ. After this time the molecule has acquired some linear momentum Δ due to emission and absorption of radiation. Each emission or absorption process contributes a linear momentum λ_i to the molecule; λ_i may of course be positive or negative. If n

emission and absorption processes occur during the time τ, we have

$$\Delta = \sum_{i=1}^{n} \lambda_i . \tag{5.16}$$

The λ_i may be taken to be independent random variables of zero mean, so that

$$\overline{\Delta^2} = \sum_{i=1}^{n} \overline{\lambda_i^2} . \tag{5.17}$$

Now each process of emission or absorption involves a momentum transfer of magnitude $\hbar\omega/c$, where $\hbar\omega = E_2 - E_1$. Thus we take

$$\overline{\lambda_i^2} = \tfrac{1}{3}(\hbar\omega/c)^2 . \tag{5.18}$$

The factor $\frac{1}{3}$ is included because by assumption the molecule is only allowed to move in one of three possible field directions. Combining eqs. (5.17) and (5.18), we have

$$\overline{\Delta^2} = \tfrac{1}{3} n(\hbar\omega/c)^2 . \tag{5.19}$$

It remains to determine the number n of emission and absorption processes occurring during the time interval τ. But this number follows from the first part of Einstein's paper, where the A and B coefficients are introduced. A is the rate of spontaneous transitions from level 2 to level 1. $B\rho(\omega)$, where $\rho(\omega)$ is the radiation energy density at frequency ω, is the rate for stimulated transitions from level 2 to level 1. By assuming that $\rho(\omega)$ approaches infinity as T does, Einstein showed in the first part of his paper that $B\rho(\omega)$ is also the rate of absorption of radiation associated with the transition from level 1 to level 2. If there are N_1 molecules in the lower state 1 and N_2 in the upper state 2, therefore,

$$n = N_2 A\tau + (N_2 + N_1) B \rho(\omega) \tau . \tag{5.20}$$

For simplicity we have ignored the possibility of degeneracy.

Using the result (5.20) in eq. (5.19), we have finally

$$\tau^{-1} \overline{\Delta^2} = \tfrac{1}{3}(\hbar\omega/c)^2 [N_2 A + (N_1 + N_2) B \rho(\omega)] . \tag{5.21}$$

This equation indicates that a molecule interacting with thermal radiation will continually gain kinetic energy. There must evidently be some velocity-dependent force which tends to slow down the molecule. The hypotheses concerning emission and absorption were put forth by Einstein for a molecule at rest, so we must investigate how the thermal radiation appears in the frame of a moving molecule.

In the laboratory frame the radiation is isotropic, so that the radiation energy per unit volume in the frequency interval ω to $\omega + \mathrm{d}\omega$, and within the infinitesimal solid angle $\mathrm{d}\Omega$ about some arbitrary direction of propagation, is

$$\frac{1}{4\pi} \rho(\omega)\, \mathrm{d}\omega\, \mathrm{d}\Omega ,$$

where $\rho(\omega)$ is independent of direction. For definiteness at this point we consider some propagation direction defined by the angle θ with respect to the single axis along which molecules are allowed to move; the azimuthal angle is irrelevant to our considerations. A molecule with velocity v sees this "radiation bundle" as having a Doppler-shifted frequency

$$\omega' \cong \omega\left(1 - \frac{v}{c}\cos\theta\right) \tag{5.22}$$

for $v/c \ll 1$. Because of the aberration effect, furthermore, the bundle appears to the molecule to be directed at an angle θ' given by

$$\cos\theta' \cong \cos\theta - \frac{v}{c}\sin^2\theta. \tag{5.23}$$

The radiation density at frequency ω transforms to

$$\frac{1}{4\pi}\rho'(\omega', \theta')\,\mathrm{d}\omega'\,\mathrm{d}\Omega',$$

and does not appear isotropic to the moving molecule. This quantity must transform in the same way as the squared amplitude of a plane wave having direction defined by the same angle θ in the laboratory frame. From the transformation properties of the electromagnetic field, therefore, we have

$$\rho'(\omega', \theta')\,\mathrm{d}\omega'\,\mathrm{d}\Omega' \cong \left(1 - \frac{2v}{c}\cos\theta\right)\rho(\omega)\,\mathrm{d}\omega\,\mathrm{d}\Omega \tag{5.24}$$

to first order in v/c. In the same first-order approximation we note furthermore that

$$\rho'(\omega', \theta') \cong \left(1 - \frac{2v}{c}\cos\theta\right)\rho(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}(\cos\theta')} \cong \left(1 - \frac{3v}{c}\cos\theta'\right)\rho(\omega) \tag{5.25}$$

where we have used the results (5.22) and (5.23). Einstein now writes

$$\rho(\omega) \cong \rho\left(\omega' + \frac{v}{c}\omega'\cos\theta'\right) \cong \rho(\omega') + \frac{\mathrm{d}\rho(\omega')}{\mathrm{d}\omega}\left(\frac{v}{c}\right)\omega'\cos\theta', \tag{5.26}$$

so that, from eq. (5.25),

$$\rho'(\omega', \theta') \cong \left[\rho(\omega') + \frac{v}{c}\frac{\mathrm{d}\rho(\omega')}{\mathrm{d}\omega}\omega'\cos\theta'\right]\left[1 - \frac{3v}{c}\cos\theta'\right]. \tag{5.27}$$

For the moving molecule the radiation bundle associated with the solid angle $\mathrm{d}\Omega'$ induces

$$n_2 = \frac{1}{4\pi}B\,\rho'(\omega', \theta')\,\mathrm{d}\Omega'\,N_2 \tag{5.28}$$

emission processes and

$$n_1 = \frac{1}{4\pi} B\,\rho'(\omega', \theta')\,\mathrm{d}\Omega'\,N_1 \tag{5.29}$$

absorption processes per second. The net linear momentum per unit time added to the molecule due to these processes is therefore

$$\frac{\mathrm{d}p}{\mathrm{d}t} = (n_1 - n_2)\frac{\hbar\omega'}{c}\cos\theta' = \frac{\hbar\omega'}{c}\left(\frac{B}{4\pi}\right)(N_1 - N_2)\,\rho'(\omega', \theta')\,\mathrm{d}\Omega'\cos\theta'. \tag{5.30}$$

Spontaneous emission is equally likely in any direction and hence adds no net momentum to the molecule.

The net rate of momentum transfer to the molecule due to all such "radiation bundles" is therefore

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\hbar\omega'}{2c} B(N_1 - N_2)\int \mathrm{d}\Omega' \cos\theta'\,\rho(\omega', \theta') \cong -\frac{\hbar\omega v}{c^2}(N_1 - N_2)\,B\left[\rho(\omega) - \frac{\omega}{3}\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega}\right] = -Rv, \tag{5.31}$$

where to lowest order we have dropped the primes. Eq. (5.31) is the desired result. It shows that as a result of the molecular motion through the radiation field the molecule is subjected to a velocity-dependent force tending to slow it down.

Einstein now considers the change in momentum of a molecule during a short time interval τ. If the molecule initially has momentum mv, then after a very short time interval τ it has momentum

$$mv' = mv + \Delta - Rv\tau, \tag{5.32}$$

so that

$$m^2\overline{v'^2} = m^2\overline{v^2} + \overline{\Delta^2} + R^2\tau^2\overline{v^2} + 2m\overline{v\Delta} - 2mR\overline{v^2}\tau - 2R\overline{\Delta v}\tau = m^2\overline{v} + \overline{\Delta^2} + R^2\tau^2\overline{v^2} - 2mR\tau\overline{v^2}, \tag{5.33}$$

since $\overline{\Delta v} = \overline{v} = 0$ due to the fact that the two types of force acting on the molecules are random and independent. Now for τ sufficiently small (or m sufficiently large) we have

$$m^2\overline{v'^2} = m^2\overline{v^2} + \overline{\Delta^2} - 2mR\tau\overline{v^2}\,. \tag{5.34}$$

But under the condition of equilibrium we have

$$m^2\overline{v'^2} = m^2\overline{v^2} = 2m(\tfrac{1}{2}kT) \tag{5.35}$$

from eq. (5.15). Therefore

$$\overline{\Delta^2}/\tau = 2RkT, \tag{5.36}$$

which may be written, from eqs. (5.21) and (5.31), as

$$B(N_1 - N_2)\left[\rho(\omega) - \frac{\omega}{3}\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega}\right] = \frac{\hbar\omega}{6kT}\,[N_2A + (N_1 + N_2)B\rho(\omega)]\,. \tag{5.37}$$

Now from the assumption of detailed balance in the first part of Einstein's paper,

$$N_2A + (N_2 + N_1)B\,\rho(\omega) = 2N_1B\,\rho(\omega). \tag{5.38}$$

This result of course already implies the Planck spectrum when we note that

$$N_2/N_1 = \exp(-\hbar\omega/kT) \tag{5.39}$$

at thermal equilibrium. Substituting eqs. (5.38) and (5.39) into (5.37), we obtain the differential equation

$$\rho(\omega) - \frac{\omega}{3}\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} = \frac{\hbar\omega}{3kT}(\exp(\hbar\omega/kT) - 1)^{-1}\rho(\omega)\exp(\hbar\omega/kT), \tag{5.40}$$

with the solution

$$\rho(\omega) = \frac{\text{const.} \times \omega^3}{\exp(\hbar\omega/kT) - 1}. \tag{5.41}$$

Thus the Planck spectrum is consistent with the assumption that the momentum changes in molecules interacting with thermal radiation are in accord with the classical statistical mechanics of the gas in thermal equilibrium with the radiation.

To understand the importance which Einstein attached to this result, we must consider his earlier investigation with Hopf [124] into the spectrum of thermal radiation. Their approach was based on the above assumption. But there are no discrete, quantum processes in the Einstein–Hopf approach, nor are there discrete atomic energy levels. The molecules instead are taken to be point masses to which are appended simple dipole harmonic oscillators, in the spirit of the Lorentz model. The mass of a molecule allows the assignment of a mean kinetic energy

$$\overline{K \cdot E} = \tfrac{3}{2}kT \tag{5.42}$$

to each molecule of a gas at thermal equilibrium. The dipole oscillator allows the molecule to interact with the radiation in equilibrium with the gas. For simplicity the molecule is again allowed to move in only one direction, so that the right-hand side of eq. (5.42) is multiplied by $\frac{1}{3}$.

If a molecule has a linear momentum mv at some time t, then after a short time τ it has the momentum

$$mv' = mv + \Delta - Rv\tau, \tag{5.43}$$

which is the same as eq. (5.32). Δ is the momentum imparted to a stationary molecule during time τ due to the thermal radiation. $-Rv$ is the force tending to slow down a molecule moving with velocity v through the radiation field. But now Δ and R must be calculated on the basis of this classical model of the molecule. They must be related to $\rho(\omega)$, which is then determined by the equilibrium condition (5.36).

Suppose that the molecules are constrained to move along the x-axis and that their dipoles are oriented along the z-axis. Then the equation of motion for the dipole oscillator has the form

$$\ddot{p} - \Gamma\dddot{p} + \omega^2 p = \tfrac{3}{2}\Gamma c^3 E_z, \tag{5.44}$$

where ω is the oscillator frequency. The second term on the left-hand side of this equation represents the force of radiation reaction; the oscillator mass is assumed to be appropriately renormalized. When the thermal field E_z is expanded into normal modes, the different components have amplitudes essentially proportional to the square root of the radiation density ρ. The corresponding components of the induced dipole moment are accordingly also proportional to $\sqrt{\rho}$. The force on the molecule is

$$F_x = p\,\frac{\partial E_x}{\partial z} - \frac{1}{c}\,B_y\,\frac{\mathrm{d}p}{\mathrm{d}t} \tag{5.45}$$

and it is clear that $\overline{F}_x = 0$. The term $\overline{\Delta^2}$, where

$$\Delta = \int_0^\tau \mathrm{d}t\,F_x , \tag{5.46}$$

is nonvanishing, and clearly will be proportional to the square of the radiation density at the oscillator frequency, $\rho^2(\omega)$, if the oscillator linewidth is very small compared with its oscillation frequency. In fact the detailed calculation [15, 124] gives

$$\overline{\Delta^2} = \frac{4\Gamma\pi^4c^4}{5\omega^2}\,\rho^2(\omega)\,\tau. \tag{5.47}$$

We see already a profound difference between the quantum-mechanical and classical models. In the quantum-mechanical model we deal with discrete events, and $\overline{\Delta^2}$ depends on the *number* of these events during time τ, and hence is proportional to $B\rho(\omega)$ (eqs. (5.21) and (5.38)). Thus the quantum-mechanical model predicts $\overline{\Delta^2} \propto \rho(\omega)$, whereas the classical model predicts $\overline{\Delta^2} \propto \rho^2(\omega)$.

The classical calculation of R, on the other hand, gives the result that R depends on $\rho(\omega)$ in the same way as in the quantum-mechanical case. This results from the fact that the velocity-dependent force is

$$F_x' = \frac{\partial E_x'}{\partial z'}\,p' - \frac{1}{c}\,B_y'\,\frac{\mathrm{d}p'}{\mathrm{d}t'} , \tag{5.48}$$

where the prime indicates the transformation to the rest frame of a moving molecule. Unlike $\overline{F}_x$ (and therefore $\overline{\Delta}$), $\overline{F_x'}$ is nonvanishing, due to the anisotropy of the thermal field as viewed from the moving coordinate system. Thus the classical calculation [15, 124] gives the result

$$R = \tfrac{6}{5}\pi^2 c\Gamma\left[\rho(\omega) - \frac{\omega}{3}\,\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega}\right], \tag{5.49}$$

in close analogy to eq. (5.31).

Substituting the results (5.47) and (5.49) into eq. (5.36), we obtain the differential equation of Einstein and Hopf,

$$\rho(\omega) - \frac{\omega}{3}\,\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} = \frac{\pi^2c^3}{3\omega^2kT}\,\rho^2(\omega), \tag{5.50}$$

the solution of which is the Rayleigh–Jeans spectrum (5.14). Thus the Einstein–Hopf approach leads to the Rayleigh–Jeans law, as would be expected from such a classical approach.

Boyer [15] has considered the implications of random electrodynamics for the classical model of Einstein and Hopf. In addition to thermal radiation characterized by the density function $\rho(\omega)$, there is according to random electrodynamics a zero-point (temperature-independent) fluctuating field (eq. (5.4) or (5.9)), the density function $\rho_0(\omega)$ of which is given by

$$\rho_0(\omega) = \hbar\omega^3/2\pi^2c^3 . \tag{5.51}$$

Its effect is to add to a molecule a mean-square impulse $\overline{\Delta_0^2}$ given by eq. (5.47) with $T = 0$:

$$\overline{\Delta_0^2} = \frac{4\Gamma\pi^4c^4}{5\omega^2}\,\rho_0^2(\omega)\tau. \tag{5.52}$$

This represents a source of kinetic energy for a molecule of a gas at $T = 0$. The velocity of the molecule cannot result in a velocity-dependent force, since the zero-point field is Lorentz-invariant. At a wall of the container, however, the deceleration of the molecule results in a *loss* of kinetic energy due to dipole radiation. Thus Boyer suggests that the classical model be extended to include the effects of the walls of the container.

Consider again the change in linear momentum of a molecule during time τ. Instead of eq. (5.43), Boyer now writes

$$mv' = mv + \Delta - Rv\tau + J, \tag{5.53}$$

where J is the momentum change of the molecule due to a collision with a wall. Since we are interested in the equilibrium situation in which the entire gas is at thermal equilibrium, we may restrict our consideration to molecules which strike a wall during time τ. The equilibrium condition $\overline{v^2} = \overline{v'^2}$ now takes the form

$$\overline{\Delta^2} + 2m\overline{vJ} = 2mR\tau\overline{v^2}, \tag{5.54}$$

following the same arguments that led from eq. (5.32) to (5.33), and noting that $\overline{\Delta J} = 0$.

The term $\overline{vJ}$ in eq. (5.54) is proportional to the kinetic energy change of a molecule on striking a wall. At thermal equilibrium the molecules on the average neither gain nor lose kinetic energy on colliding with a wall, so that $\overline{vJ}$ must be independent of T. For the special case $T = 0$, however, we have seen that a molecule approaching a wall must have a greater velocity before than after striking the wall, so that $\overline{vJ}$ is negative. Therefore $\overline{vJ}$ may be obtained from the equilibrium condition at $T = 0$. At $T = 0$, $R = 0$ because the zero-point field gives rise to no velocity-dependent force (see eqs. (5.51) and (5.49)), so that from eq. (5.54),

$$2m\overline{vJ} = -\overline{\Delta_0^2}. \tag{5.55}$$

The equilibrium condition (5.54) therefore takes the form

$$\overline{\Delta^2} - \overline{\Delta_0^2} = 2mR\tau\overline{v^2}. \tag{5.56}$$

Substituting eqs. (5.47), (5.49) and (5.52) into eq. (5.56), we obtain the differential equation

$$\frac{\pi^2c^3}{3\omega^2kT}\left[\rho^2(\omega) - \left(\frac{\hbar\omega^3}{2\pi^2c^3}\right)^2\right] = \rho(\omega) - \frac{\omega}{3}\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega}, \tag{5.57}$$

where we have used also equations (5.35) and (5.51) for $\overline{v^2}$ and $\rho_0(\omega)$. The solution of Boyer's differential equation (5.57) is the Planck spectrum

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} + \tfrac{1}{2}\hbar\omega\right). \tag{5.58}$$

This result is perhaps the most remarkable achievement of random electrodynamics at this time. The extension of "traditional" classical electrodynamics has removed one of the most significant stumbling blocks of classical physics.

Boyer [16] has discussed in detail some implications of random electrodynamics for classical statistical thermodynamics. Perhaps the reader will not be surprised to learn that the success of random electrodynamics in explaining the blackbody spectrum extends to several other aspects of thermal radiation usually thought to be explanable only through the introduction of radiation quanta. It is certainly a theory worthy of further attention.

5.4. *Random electrodynamics and the van der Waals forces*

There are several well-known physical effects which have heuristic explanations in terms of the quantum-electrodynamical vacuum fluctuations of the radiation field. In assessing the limits of validity of the classical theory of random electrodynamics, it is necessary to determine whether the theory provides an adequate explanation of such effects. One such effect is the van der Waals interaction between two neutral polarizable particles. The usual heuristic explanation of this interaction is that the fluctuating dipole moments induced in the two particles by the vacuum field interact to produce a net force between the two particles.

Suppose we attempt to explain this interaction along the lines of the classical Lorentz model of an atom or molecule. Each atom is pictured as an harmonic-oscillator dipole; the electron may be considered to be bound elastically to an infinitely massive nucleus. The atoms are a fixed distance r apart, and the electron in each atom is acted upon by its own radiation reaction field and the retarded field from the other atom, as well as the elastic restoring force binding the electron to the nucleus. There are no "external" fields involved in the calculation of van der Waals forces.

Let the displacements from equilibrium of the electrons in atoms 1 and 2 be $x_1(t)$ and $x_2(t)$, respectively. Then we have the equations of motion

$$\ddot{x}_1(t) + \omega_0^2 x_1(t) - \left(\frac{2e^2}{3mc^3}\right)\dddot{x}_1(t) = \frac{e^2}{m}\left[-\frac{p}{c^2 r}\ddot{x}_2\left(t - \frac{r}{c}\right) + \frac{q}{r^3}x_2\left(t - \frac{r}{c}\right) + \frac{q}{cr^2}\dot{x}_2\left(t - \frac{r}{c}\right)\right], \tag{5.58}$$

$$\ddot{x}_2(t) + \omega_0^2 x_2(t) - \left(\frac{2e^2}{3mc^3}\right)\dddot{x}_2(t) = \frac{e^2}{m}\left[-\frac{p}{c^2 r}\ddot{x}_1\left(t - \frac{r}{c}\right) + \frac{q}{r^3}x_1\left(t - \frac{r}{c}\right) + \frac{q}{cr^2}\dot{x}_1\left(t - \frac{r}{c}\right)\right]. \tag{5.59}$$

The electrons in atoms 1 and 2 are assumed to move only along the axis defined by the unit vectors $\hat{\boldsymbol{\mu}}_1$, and $\hat{\boldsymbol{\mu}}_2$, respectively, and p and q are defined as

$$p = \hat{\boldsymbol{\mu}}_1 \cdot \hat{\boldsymbol{\mu}}_2 - (\hat{\boldsymbol{\mu}}_1 \cdot \hat{r})(\hat{\boldsymbol{\mu}}_2 \cdot \hat{r}), \tag{5.60}$$

$$q = 3(\hat{\boldsymbol{\mu}}_1 \cdot \hat{r})(\hat{\boldsymbol{\mu}}_2 \cdot \hat{r}) - \hat{\boldsymbol{\mu}}_1 \cdot \hat{\boldsymbol{\mu}}_2, \tag{5.61}$$

where $\hat{r}$ is a unit vector along the line connecting atoms 1 and 2.

Under no influence of external forces, the electrons are supposed to be at rest at their equilibrium positions. Equations (5.58) and (5.59) show that two such initially unexcited atoms, a fixed distance r apart, remain unexcited: $x_1(t) = x_2(t) = 0$, and there is no interaction between the atoms. It is known, however, that two ground-state atoms attract each other with a force that varies as r^{-7}. This force, introduced by van der Waals in order to account for corrections to the ideal gas equation of state, is the dominant binding effect in many inert-gas and organic crystals. We note that this interaction eludes a classical (Lorentz-model) explanation. In fact, London's 1930 derivation [126] based on fourth-order perturbation theory was considered a major accomplishment of the new quantum mechanics.

A simple argument showing the importance of the quantum theory for the explanation of the van der Waals interaction goes as follows. Suppose the atoms are close enough together ($\omega_0 r/c \ll 1$) that retardation may be neglected and only the dominant near-field terms are retained in eqs. (5.58) and (5.59):

$$\ddot{x}_1(t) + \omega_0^2 x_1(t) = K x_2(t), \tag{5.62}$$

$$\ddot{x}_2(t) + \omega_0^2 x_2(t) = K x_1(t), \tag{5.63}$$

where

$$K = \frac{q}{r^3}\left(\frac{e^2}{m}\right). \tag{5.64}$$

The normal-mode frequencies of this system are given by

$$\omega_\pm = (\omega_0^2 \pm K)^{1/2}. \tag{5.65}$$

In a quantum-mechanical treatment we should expect the ground-state energy of the system to become

$$E = \tfrac{1}{2}\hbar(\omega_+ + \omega_-) \tag{5.66}$$

due to the coupling. To lowest order in K/ω_0^2 this is

$$E = \hbar\omega_0 - K^2\hbar/8\omega_0^3, \tag{5.67}$$

which implies the interaction energy

$$U(r) = -\frac{\hbar}{8\omega_0^3}\left(\frac{q^2}{r^6}\right)\left(\frac{e^2}{m}\right)^2. \tag{5.68}$$

The quantum-mechanical expression derived by London [126] is

$$U(r) = -\mu^4 q^2/2\hbar\omega_0 r^6. \tag{5.69}$$

In this expression $\boldsymbol{\mu}$ is the transition dipole moment connecting the ground and first-excited states, only these states being included in a first approximation; q^2 is easily evaluated, assuming closure over these states. The results (5.68) and (5.69) are identical if the mass and charge of the Lorentz-model electron are chosen such that

$$e^2/m = 2\mu^2\omega_0/\hbar. \tag{5.70}$$

The relation (5.70) results in fact from an interesting correspondence between the Lorentz model and the two-state model. This correspondence warrants a brief digression here.

In many natural phenomena involving the interaction of light with atoms, the light is predominantly nonresonant with any transition involving the normal atomic state. When this circumstance obtains, the atom has negligible probability of being "removed" from its normal (ground) state. Under the frequently occurring circumstance that an applied field frequency ω is much less than any of the transition frequencies involving the ground state, the Kramers–Heisenberg formula gives the polarizability $\alpha(\omega)$ as

$$\alpha(\omega) = \frac{2}{\hbar} \sum_j \frac{|\boldsymbol{\mu}_{j0}|^2 \omega_{j0}}{\omega_{j0}^2 - \omega^2} \cong \frac{2}{\hbar} \sum_j \frac{|\boldsymbol{\mu}_{j0}|^2 \omega_{j0}}{\omega_{j0}^2} . \tag{5.71}$$

The sum is over all excited states optically connected to the ground state. ω_{j0} and $\boldsymbol{\mu}_{j0}$ are respectively the transition frequency and dipole moment connecting state $|j\rangle$ to the ground state $|0\rangle$. A perusal of a table of atomic oscillator strengths reveals that the transition connecting the ground and first-excited states usually has an oscillator strength much larger than those of other transitions. Then

$$\alpha(\omega) \cong 2\mu^2/\hbar\omega_0 , \tag{5.72}$$

where ω_0 and $\boldsymbol{\mu}$ are the transition frequency and dipole moment of the transition involving the ground and first-excited states. For the case $\omega \ll \omega_0$, the Lorentz model gives for the static polarizability

$$\alpha(\omega) = e^2/m\omega_0^2 , \tag{5.73}$$

which is the same as the quantum-mechanical result (5.72) if e^2/m is chosen according to (5.70). This choice of e^2/m may be considered to result from the incorrect application of the Thomas–Reiche–Kuhn sum rule to the fictitious two-state atom. This sum rule is essentially a statement of the commutation rule $[x, p_x] = i\hbar$, which cannot be satisfied in the finite-dimensional Hilbert space associated with the two-state atom.

It remains to show how the Lorentz dipole-oscillator model follows as a crude approximation to the quantum-mechanical theory. As already implied, the Lorentz model must be applicable when an atom will most likely remain in its ground state, and when a single oscillator strength involving a transition from the ground state is dominant. Thus we expect that the two-state model will reduce to the Lorentz model when we demand that the atom effectively remain in its ground state, i.e., when we take $\sigma_z = -1$. From equations (2.81) and (2.89), then, we are left with the single equation

$$\mu(\ddot{\sigma}_x(t) + \omega_0^2 \sigma_x(t)) = \frac{2\omega_0\mu^2}{\hbar} E(0, t) \tag{5.74}$$

where we have taken $\boldsymbol{\mu}$ and E to be parallel. $\mu\sigma_x(t)$ thus corresponds to the quantity $ex(t)$ in the Lorentz model, $x(t)$ being the displacement of the electron from its equilibrium position:

$$e(\ddot{x}(t) + \omega_0^2 x(t)) = \frac{e^2}{m} E(0, t). \tag{5.75}$$

Equations (5.74) and (5.75) are formally identical when the transcription (5.70) is made, except for one important difference. Equation (5.74) is an *operator* equation, and the field appearing on the right-hand side is an operator. If expectation values are taken on both sides of this equation for the situation where there is no "external" field applied to the atom, then (5.74) reduces to (5.75) with $E(0, t) = 0$, since the expectation value of the solution of the homogeneous Maxwell equation vanishes. Equation (5.74) also reduces to (5.75) for the case where the field is described by a coherent state such that the electric field expectation value is the same as the classically prescribed field used in the Lorentz-model equation.

But the attempt to calculate the van der Waals interaction reveals a failure of the Lorentz model, even though each atom is considered to remain in its ground state. If the two atoms are initially unexcited, so that their electrons are at rest in their equilibrium positions, then eqs. (5.58) and (5.59) indicate that they will remain in these positions: the atoms do not interact. Evidently the vacuum field must play an essential role in the quantum-electrodynamical theory of the van der Waals interaction.

The quantum-mechanical analogues to eqs. (5.58) and (5.59) are

$$\ddot{x}_1(t) + \omega_0^2 x_1(t) - \left(\frac{4\omega_0\mu^2}{3\hbar c^3}\right)\dddot{x}_1(t) = \frac{2\omega_0\mu^2}{\hbar}\hat{\boldsymbol{\mu}}_1\cdot \boldsymbol{E}_0(\boldsymbol{x}_1, t)$$

$$-\left(\frac{2\omega_0\mu^2}{\hbar c^2 r}\right)p\,\ddot{x}_2\left(t-\frac{r}{c}\right) + \left(\frac{2\omega_0\mu^2}{\hbar r^3}\right)q\,x_2\left(t-\frac{r}{c}\right) + \left(\frac{2\omega_0\mu^2}{\hbar c r^2}\right)q\,\dot{x}_2\left(t-\frac{r}{c}\right), \tag{5.76}$$

$$\ddot{x}_2(t) + \omega_0^2 x_2(t) - \left(\frac{4\omega_0\mu^2}{3\hbar c^3}\right)\dddot{x}_2(t) = \frac{2\omega_0\mu^2}{\hbar}\hat{\boldsymbol{\mu}}_2\cdot \boldsymbol{E}_0(\boldsymbol{x}_2, t)$$

$$-\left(\frac{2\omega_0\mu^2}{\hbar c^2 r}\right)p\,\ddot{x}_1\left(t-\frac{r}{c}\right) + \left(\frac{2\omega_0\mu^2}{\hbar r^3}\right)q\,x_1\left(t-\frac{r}{c}\right) + \left(\frac{2\omega_0\mu^2}{\hbar c r^2}\right)q\,\dot{x}_1\left(t-\frac{r}{c}\right), \tag{5.77}$$

where $\mu\hat{\boldsymbol{\mu}}_i x_i(t)$ is the dipole moment operator associated with atom i. We emphasize that it is *essential* to retain the free-field operator $\boldsymbol{E}_0$ in these equations. Without it, the dipole moment operators themselves decay to zero, and commutation rules are broken. With the free-field term, however, we can show that the commutation rules are preserved. In fact, for the harmonic oscillators under consideration, these commutation rules may be considered to be *defined* by the commutator properties of the free field [127].

Before discussing the derivation of the van der Waals interaction based on eqs. (5.76) and (5.77), let us consider the implications of random electrodynamics for the Lorentz-oscillator model. The random-electrodynamical equations will obviously be formally identical to eqs. (5.76) and (5.77), when the mass and charge of the Lorentz electron are chosen in accordance with eq. (5.70). The $x_i(t)$ will now be c-numbers. $\boldsymbol{E}_0$, rather than being the free-field solution of the Maxwell operator equation for the electric field, will be the zero-point field of the theory of random electrodynamics.

Boyer [19, 20] has given random-electrodynamical derivations of the van der Waals interaction between two neutral polarizable particles. The formal identity between eqs. (5.76) and (5.77) and the corresponding equations of random electrodynamics allows us to apply Boyer's method of solution simultaneously to both pairs of equations.

We consider here only the special case of the asymptotic retarded van der Waals interaction. It was suggested by Verwey and Overbeek [128] that at large separations the van der Waals interaction might fall off more rapidly than r^{-6}, due to the effects of retardation. Wheeler [129] actually had included the effect of retardation in the calculation of the van der Waals force in 1941. Following the suggestion of Verwey and Overbeek, Casimir and Polder [46] presented a detailed analysis of the influence of retardation on the van der Waals forces. They found for large separations the interaction energy

$$U(r) = -23\alpha_1\alpha_2\hbar c/4\pi r^7 \tag{5.78}$$

between two neutral polarizable particles with static polarizabilities α_1 and α_2.

Boyer's random-electrodynamical derivation of (5.78) is conceptually very simple. For r very large, it is expected that the interatomic interaction will be primarily a low-frequency effect; higher frequency contributions will cancel due to small phase shifts building up over the large separation distance. The second- and third-derivative terms on the left-hand sides of eqs. (5.76) and (5.77) may therefore be neglected, and we have simply

$$x_1(t) = \frac{2\mu^2}{\hbar\omega_0}\,\hat{\boldsymbol{\mu}}_1 \cdot [\boldsymbol{E}_0(\boldsymbol{x}_1, t) + \boldsymbol{E}_{12}(\boldsymbol{x}_1, t)]\,, \tag{5.79}$$

$$x_2(t) = \frac{2\mu^2}{\hbar\omega_0}\,\hat{\boldsymbol{\mu}}_2 \cdot [\boldsymbol{E}_0(\boldsymbol{x}_2, t) + \boldsymbol{E}_{21}(\boldsymbol{x}_2, t)]\,, \tag{5.80}$$

where $\boldsymbol{E}_{ij}(\boldsymbol{x}_i, t)$ is the dipole field on atom i due to atom j. In the quantum-mechanical case the polarizabilities, assumed to be the same for the two atoms, are given by eq. (5.72); in the random-electrodynamical case they are given by the classical result (5.73).

The interaction energy is simply

$$U(r) = -\tfrac{1}{2}\alpha\langle[\boldsymbol{E}_0(\boldsymbol{x}_1, t) + \boldsymbol{E}_{12}(\boldsymbol{x}_1, t)]^2\rangle_\theta - U(\infty) = -\alpha\langle \boldsymbol{E}_0(\boldsymbol{x}_1, t)\cdot\boldsymbol{E}_{12}(\boldsymbol{x}_1, t)\rangle_\theta - U(\infty), \tag{5.81}$$

to lowest order, where again $\langle\ \rangle_\theta$ denotes an average over the random phases in the random-electrodynamical calculation. Noting that

$$x_2(t) = \alpha\boldsymbol{E}_0(\boldsymbol{x}_2, t) \tag{5.82}$$

to lowest order, and using this expression for $x_2(t)$ in (5.81) to calculate $\boldsymbol{E}_{12}(\boldsymbol{x}_1, t)$, we finally incur a complicated integral expression for $U(r)$. The evaluation of the integral yields the result (5.78) of Casimir and Polder. The derivation shows explicitly the role played by the zero-point field. In the random-electrodynamical calculation given by Boyer, there are nonvanishing contributions only from terms such as

$$\langle\exp\{\mathrm{i}(\theta_{\boldsymbol{k}\lambda} - \theta_{\boldsymbol{k}'\lambda'})\}\rangle_\theta = \langle\exp\{-\mathrm{i}(\theta_{\boldsymbol{k}\lambda} - \theta_{\boldsymbol{k}'\lambda'})\}\rangle_\theta = \delta^3_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda\lambda'}. \tag{5.83}$$

In the corresponding Heisenberg-picture, quantum-mechanical calculation, which proceeds in the same way except of course that the quantum-mechanical ensemble average $\langle\ \rangle$ replaces $\langle\ \rangle_\theta$, nonvanishing contributions result only from terms such as

$$\langle a_{\boldsymbol{k}',\lambda'}(0)\, a^+_{\boldsymbol{k}\lambda}(0)\rangle = \delta^3_{\boldsymbol{k},\boldsymbol{k}'}\,\delta_{\lambda\lambda'}. \tag{5.84}$$

Boyer [19–21] has given random-electrodynamical calculations for the van der Waals interactions at all separations for particles which can be either electrically or magnetically polarizable, as well as for the forces between conducting plates, dielectric and permeable plates, and polarizable particles and plates. In every case the results are in agreement with the known predictions of quantum electrodynamics.

5.5. *Discussion*

The explanation of the blackbody spectrum and the van der Waals forces within the classical theory of random electrodynamics may be considered as quite striking or merely amusing, depending on one's persuasion.

For the physicist accustomed to thinking of blackbody radiation in terms of radiation quanta, the derivation of Planck's law from random electrodynamics is remarkable indeed. The failure of classical physics to account for the observed blackbody spectrum is often regarded as proof enough for the failure of classical electrodynamics. It was only after great effort and considerable reluctance that Planck introduced the notion of field quanta; there seemed to be no other alternative. Einstein's derivation of Planck's law in his paper "On the Quantum Theory of Radiation" [2] showed convincingly that the field quanta carry a linear momentum $\hbar\omega/c$ in addition to energy $\hbar\omega$. The existence of field quanta seemed indisputable.

But the very phenomenon which signalled the collapse of classical physics has now been shown to be explanable from the *classical* theory of random electrodynamics. The theory is founded on the classical Maxwell equations for the electromagnetic field. Rather than the identically vanishing solution of the homogeneous equations, a fluctuating field is chosen, the energy spectrum of which is determined from the requirement of Lorentz invariance; Planck's constant then enters the theory naturally as a fundamental constant whose value is set by comparison of the theory to experiment. The blackbody spectrum can be obtained by considering the interaction of Lorentz-model dipole oscillators in a container with the fields they produce as well as the zero-point field of random electrodynamics. The derivation given by Boyer [15] shows that Planck's law can be understood in terms of classical zero-point fluctuations. Field quanta, quantized atomic energy levels, and stimulated emission do not enter into the derivation. Physicists in 1900 would probably have been more comfortable with this (fully classical) approach than with Planck's.

The derivation of the van der Waals forces in random electrodynamics is equally striking. The reader familiar with the Casimir–Polder paper [46] on the asymptotic retarded van der Waals interaction, for example, is aware of the lengthy quantum-mechanical calculations required. The physical interpretation of the interaction is not elucidated by the calculations, and one normally relies on heuristic notions involving quantum-electrodynamical vacuum fluctuations. In the calculations based on random electrodynamics, on the other hand, the physical interpretation is completely clear. Perhaps more remarkable than the fact that random electrodynamics predicts an interaction between "unexcited" atoms is the fact that for all atomic separations, and for either electrically or magnetically polarizable particles, it predicts an interaction of precisely the same form as in quantum electrodynamics.

The theory of random electrodynamics has other apparently attractive features. All particles coupled to the zero-point field undergo zero-point fluctuations. The energy balance between emission and absorption by a charged dipole oscillator interacting with the zero-point radiation, for example, requires that the oscillator undergo random oscillations. As the charge of the oscillator approaches zero, these random oscillations persist; the zero-point field has effected a kind of "singular perturbation". In this limit of vanishing charge, moreover, the mean-square position and momentum of the oscillator are such as to satisfy the condition of minimum uncertainty demanded in quantum mechanics by the Heisenberg principle. The limit of vanishing charge defines a theory which Boyer terms "random mechanics". The reader is referred to ref. [122] for a discussion of this point, and for an outline of the current status of random electrodynamics.

Having stressed the attractive features of random electrodynamics, we must now present the point of view from which this classical theory appears as far less than a possible alternative or equivalent to quantum electrodynamics. We first recall that the derivation of the retarded van der Waals force given by Boyer [19] can be followed through in formally the same way in a Heisenberg-picture quantum-mechanical treatment. For the problem under consideration the atom is well described quantum mechanically by the dipole-oscillator model. The retarded dipole fields of the oscillators are of the same form in the classical and quantum theories, the quantum-mechanical field being of course a Hilbert-space operator. Similarly the free fields are of essentially the same form in random and quantum electrodynamics. There is a minor difference in that the zero-point field of random-electrodynamics differs by a factor of $2^{-1/2}$ from the corresponding quantum-electrodynamical equations (cf. eqs. (2.34) and (5.4)). But this only represents the fact that both normally and anti-normally ordered field correlation functions contribute to the zero-point energy spectrum in random electrodynamics, while only the anti-normally ordered correlation function $\langle E_0^{(+)} E_0^{(-)} \rangle$ contributes in the quantum-electrodynamical case; the zero-point energy spectra are the same.

The derivation of the van der Waals forces in random electrodynamics yields the same result as in quantum electrodynamics because the final answer depends ultimately on the zero-point energy spectrum of the field. The equations of motion are formally the same in the two theories. The final step in the derivation involves a term bilinear in the zero-point electric field. In the quantum-electrodynamical case we require an expectation value of this term over the vacuum field state. In random electrodynamics we require an average of formally the same term over the random phases of the zero-point field. The two types of ensemble average yield the same answer, and therefore the same result for the van der Waals interaction. From this viewpoint, we might even contend that the principal merit in Boyer's derivation is the treatment of the problem in the Heisenberg picture, with a consequent ease of physical interpretation. The use of the language of random electrodynamics might be considered a subterfuge.

From the same point of view, the derivation of the Planck radiation law in random electrodynamics might be considered as a disguised Heisenberg-picture treatment of the equilibrium between material oscillators and radiation oscillators. The random-electrodynamical treatment once again is regarded as a subterfuge. It might be expected that all the essential features of the interaction of dipole harmonic oscillators with the radiation field in quantum electrodynamics have an isomorphic explanation in random electrodynamics. The basic equations of motion and final terms to be evaluated are formally the same in the two theories. The linear coupling between the two systems according to this point of view is too simple and too specific to allow any general conclusions about random electrodynamics.

Suppose we perform a quantum-electrodynamical calculation and are led to some final expectation value to be evaluated which involves the free field. It appears that random electrodynamics will give the same result for the problem whenever this final expression is of a particular structure. Namely, there are terms bilinear in the free field and the positive- and negative-frequency parts of the free field enter additively as the total free field. This appears in the linearization approximation in which an atom or molecule is replaced by an harmonic oscillator. In fact Boyer [130] has shown that the two theories give identical results for N-point correlation functions of the free field if the quantum-electrodynamical field operators are symmetrically ordered.

In the problem of spontaneous emission we found in section 2 that for a symmetric ordering of field operators in the Hamiltonian the final expression for the radiative frequency shift involved an expectation value of the above form. Thus if we replace the quantum-electrodynamical free field with the zero-point field of random electrodynamics, we obtain the familiar result which after mass renormalization gives Bethe's result for the nonrelativistic treatment of the Lamb shift. But to conform to the spirit of the classical theory of random electrodynamics we would have to treat the source or radiation reaction field classically. We would then be left with a neoclassical theory modified to include the zero-point field of random electrodynamics.

But the juxtaposition of the neoclassical and random-electrodynamical theories runs counter to the philosophy underlying the theory of random electrodynamics. The theory is only in an initial stage of its development, and it is hoped that eventually the discrete spectra of atoms and molecules will be explanable with this classical theory of fluctuations. This explains why, for example, there has been no attempt to derive the ratio of the intensities of the Stokes and anti-Stokes lines in Raman scattering from random electrodynamics. Problems of this kind cannot yet be addressed, because the theory has not yet been applied to the problem of atomic structure. It should be remembered that a quarter of a century elapsed between Planck's paper on the blackbody spectrum and Schrödinger's paper on the spectrum of atomic hydrogen.

The reader interested in the theory of random electrodynamics is referred to the papers of Marshall and Boyer on the subject, as well as other recent work [131–134] on a classical theory of the vacuum or zero-point field.

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References

[1] P.A.M. Dirac, Proc. Roy. Soc. A114 (1927) 243.
[2] A. Einstein, Phys. Z. 18 (1917) 121.
[3] H.A. Kramers and W. Heisenberg, Z. Physik 31 (1925) 681.
[4] M.D. Crisp and E.T. Jaynes, Phys. Rev. 179 (1969) 1253.
[5] C.R. Stroud Jr. and E.T. Jaynes, Phys. Rev. A1 (1970) 106.
[6] E.T. Jaynes, Coherence and Quantum Optics, ed. L. Mandel and E. Wolf (Plenum, N.Y., 1973) p. 35.
[7] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47 (1935) 777.
[8] R.H. Dicke, Phys. Rev. 93 (1954) 99.

[9] N.E. Rehler and J.H. Eberly, Phys. Rev. A3 (1971) 1735.
[10] N. Skribanowitz, I.P. Herman, J.C. MacGillivray and M.S. Feld, Phys. Rev. Lett. 30 (1973) 309.
[11] I.P. Herman, J.C. MacGillivray, N. Skribanowitz and M.S. Feld, Proc. Vail Conf. on Laser Spectroscopy (Plenum, N.Y., 1973).
[12] H. Kuhn, J. Chem. Phys. 53 (1971) 101.
[13] K.H. Drexhage, Coherence and Quantum Optics, eds. L. Mandel and E. Wolf (Plenum, N.Y., 1973) p. 187.
[14] P.W. Milonni and P.L. Knight, Opt. Commun. 9 (1973) 119.
[15] T.H. Boyer, Phys. Rev. 182 (1969) 1374.
[16] T.H. Boyer, Phys. Rev. 186 (1969) 1304.
[17] T.H. Boyer, Phys. Rev. D1 (1970) 1526.
[18] T.H. Boyer, Phys. Rev. D2 (1970) 2257.
[19] T.H. Boyer, Phys. Rev. A5 (1972) 1799.
[20] T.H. Boyer, Phys. Rev. A7 (1973) 1832.
[21] T.H. Boyer, Phys. Rev. A9 (1974) 2078.
[22] V.F. Weisskopf and E. Wigner, Z. Physik 63 (1930) 54.
[23] T.A. Welton, Phys. Rev. 74 (1948) 1157.
[24] J.C. Slater, Nature 113 (1924) 307.
[25] J.R. Ackerhalt, P.L. Knight and J.H. Eberly, Phys. Rev. Lett. 30 (1973) 456.
[26] J.R. Ackerhalt, P.L. Knight and J.H. Eberly, Coherence and Quantum Optics, eds. L. Mandel and E. Wolf (Plenum, N.Y., 1973).
[27] I.R. Senitzky, Phys. Rev. Lett. 31 (1973) 955.
[28] P.W. Milonni, J.R. Ackerhalt and W.A. Smith, Phys. Rev. Lett. 31 (1973) 958.
[29] P.W. Milonni and W.A. Smith, Phys. Rev. A11 (1975) 814.
[30] C.A. Kocher and E.D. Commins, Phys. Rev. Lett. 18 (1967) 575.
[31] J.F. Clauser, Phys. Rev. A6 (1972) 49.
[32] I.R. Senitzky, Phys. Rev. 111 (1958) 3.
[33] I.R. Seniztky, Phys. Rev. 115 (1959) 227.
[34] I.R. Senitzky, Phys. Rev. 121 (1961) 171.
[35] I.R. Senitzky, Phys. Rev. Lett. 20 (1968) 1062.
[36] I.R. Senitzky, Phys. Rev. 183 (1969) 1069.
[37] I.R. Senitzky, Phys. Rev. A3 (1971) 421.
[38] I.R. Senitzky, Phys. Rev. A6 (1972) 1171.
[39] I.R. Senitzky, Phys. Rev. A6 (1972) 1175.
[40] R.H. Lehmberg, Phys. Rev. A2 (1970) 883.
[41] P.A.M. Dirac, Lectures on Quantum Field Theory (Belfer Graduate School of Science, Yeshiva University, N.Y., 1966).
[42] K. Gottfried, Quantum Mechanics (Benjamin, N.Y., 1966).
[43] M. Göppert-Mayer, Ann. Physik Lpz. 9 (1931) 273.
[44] P.I. Richards, Phys. Rev. 73 (1948) 254.
[45] E.A. Power and S. Zienau, Nuovo Cim. 6 (1957) 7.
[46] H.B.G. Casimir and D. Polder, Phys. Rev. 73 (1948) 360.
[47] E.A. Power and S. Zienau, Phil. Trans. Roy. Soc. A251 (1959) 427.
[48] W. Heitler, The Quantum Theory of Radiation (3rd ed., Oxford Univ. Press, London, 1960).
[49] L. Davidovich, Ph.D. thesis (Univ. of Rochester, 1975).
[50] R.G. Woolley, Mol. Phys. 22 (1971) 10.
[51] H.A. Bethe, Phys. Rev. 72 (1947) 339.
[52] S. Pasternak, Phys. Rev. 54 (1938) 1113.
[53] W.E. Lamb Jr. and R.C. Retherford, Phys. Rev. 72 (1947) 241;
W.E. Lamb, Phys. Rev. 85 (1952) 259.
[54] W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley, Reading, 1962).
[55] E.J. Moniz and D.H. Sharp, Phys. Rev. D10 (1974) 1133.
[56] R.P. Feynman, F.L. Vernon Jr. and R.W. Hellwarth, J. Appl. Phys. 28 (1957) 49.
[57] V.M. Fain, Radiofizika 6 (1963) 207.
[58] E. Schrödinger, Nuovo Cim. 9 (1958) 162.
[59] G.S. Agarwal, Quantum Statistical Theories of Spontaneous Emission (Springer-Verlag, Berlin, 1974).
[60] R.K. Bullough, Coherence and Quantum Optics, eds. L. Mandel and E. Wolf (Plenum, New York, 1973) p. 121.
[61] G.S. Agarwal, Phys. Rev. A10 (1974) 717.
[62] W.E. Lamb Jr., Phys. Rev. 134 (1964) A1429.
[63] M. Jammer, The Conceptual Development of Quantum Mechanics (McGraw-Hill, New York, 1966) p. 114.

[64] M. Jammer, The Philosophy of Quantum Mechanics (Wiley, New York, 1974) p. 26.
[65] Letters on Wave Mechanics, ed. K. Przibram (Philosophical Library, New York, 1967) p. 10.
[66] Ref. [64] p. 53.
[67] M.O. Scully and M. Sargent, Physics Today 25, No. 3 (1972) 38.
[68] W. Heisenberg, Z. Physik 43 (1927) 172.
[69] P.L. Knight and L. Allen, Phys. Lett. 38A (1972) 99.
[70] E. Fermi, Rev. Mod. Phys. 4 (1932) 87.
[71] E.T. Jaynes and F.W. Cummings, Proc. IEEE 51 (1963) 89.
[72] L.A. Khalfin, Sov. Phys. JETP 6 (1958) 1053.
[73] C.R. Stroud Jr., Ph.D. thesis (Washington University, 1969) unpublished.
[74] L. Allen and J.H. Eberly, Optical Resonance and Two-Level Atoms (Wiley, N.Y., 1975).
[75] F. Bloch, Phys. Rev. 70 (1946) 460.
[76] Collected Papers of L.D. Landau, ed. D. ter Haar (Gordon and Breach, N.Y., 1965) pp. 8–18.
[77] R.J. Glauber, Phys. Rev. 130 (1963) 2529.
[78] J.H. Shirley, Phys. Rev. 181 (1969) 600.
[79] R.P. Feynman, Phys. Rev. 56 (1939) 340.
[80] W. Heisenberg, The Physical Principles of the Quantum Theory (The University of Chicago Press, Chicago, 1930).
[81] H.A. Lorentz, The Theory of Electrons (Dover, N.Y., 1952).
[82] H.A. Kramers, Nature 114 (1924) 310.
[83] W. Heisenberg, Z. Physik 33 (1925) 879.
[84] W. Kuhn, Z. Physik 33 (1925) 408.
[85] M. Born and P. Jordan, Z. Physik 34 (1925) 858.
[86] R. Ladenburg and H. Kopfermann, Nature 122 (1928) 438.
[87] J. Andrade e Silva and G. Lochak, Quanta (McGraw-Hill, New York, 1969) p. 148.
[88] R.K. Nesbet, Phys. Rev. A4 (1971) 259.
[89] Ref. [63] p. 283.
[90] M. Born, Z. Physik 37 (1926) 863.
[91] Ref. [64] pp. 24–33.
[92] E. Schrödinger, Ann. Physik 79 (1926) 734.
[93] Ref. [64] p. 120.
[94] The Born-Einstein Letters (Walker and Company, N.Y., 1971) p. 210.
[95] L.E. Ballantine, Rev. Mod. Phys. 43 (1970) 358.
[96] A. Einstein, J. Franklin Institute 221 (1936) 349.
[97] G. Baym, Lectures on Quantum Mechanics (Benjamin, New York, 1969) p. 22.
[98] P.A.M. Dirac, The Principles of Quantum Mechanics (4th ed., Oxford University Press, London, 1959).
[99] R.P. Feynman, R.B. Leighton and M. Sands, Lectures on Physics (Addison-Wesley, New York, 1965) vol. III, p. 18–9.
[100] R.P. Feynman, Proc. Second Berkeley Symposium on Mathematical Statistics and Probability (University of California Press, Berkeley, 1951) p. 533.
[101] P.A.M. Dirac, Fields and Quanta 3 (1972) 154.
[102] J.A. Wheeler, Ann. N.Y. Acad. Sci. 48 (1946) 219.
[103] C.N. Yang, Phys. Rev. 77 (1950) 242.
[104] W.W. Chow, M.O. Scully and J.O. Stoner Jr., Phys. Rev. A11 (1975) 1380.
[105] R.M. Herman, H. Grotch, R. Kornblith and J.H. Eberly, Phys. Rev. A11 (1975) 1389.
[106] Ref. [99] p. 3–7.
[107] R.R. Chance, A Prock and R. Silbey, J. Chem. Phys. 62 (1975) 771.
[108] P.W. Milonni and P.L. Knight, Phys. Rev. A10 (1974) 1096.
[109] R.D. Colegrove, P.A. Franken, R.R. Lewis and R.H. Sands, Phys. Rev. Lett. 3 (1959) 420.
[110] P.A. Franken, Phys. Rev. 121 (1961) 508.
[111] G. Breit, Rev. Mod. Phys. 5 (1933) 91.
[112] A.P. Thorne, Spectrophysics (Chapman and Hall and Science Paperbacks, London, 1974) p. 92.
[113] E.T. Jaynes, Phys. Rev. A2 (1970) 260.
[114] H.M. Gibbs, Coherence and Quantum Optics, eds. L. Mandel and E. Wolf (Plenum, New York, 1973) p. 83.
[115] J.M. Wessner, D.K. Anderson and R,T. Robiscoe, Phys. Rev. Lett. 29 (1972) 1126.
[116] R.K. Nesbet, Phys. Rev. Lett. 27 (1971) 553.
[117] H.B.G. Casimir, J. Chim. Phys. 46 (1949) 407.
[118] H.B.G. Casimir, Proc. Kon. Ned. Akad. Wetenschap. 51 (1948) 793.
[119] W. Nernst, Verh. Dtsch. Phys. Ges. 18 (1916) 83.

[120] T.W. Marshall, Proc. Roy. Soc. A276 (1963) 475.
[121] T.W. Marshall, Proc. Camb. Philos. Soc. 61 (1965) 537.
[122] T.H. Boyer, Phys. Rev. D11 (1975) 790.
[123] M. Planck, Theory of Heat Radiation (Dover, New York, 1959).
[124] A. Einstein and L. Hopf, Ann. Physik 33 (1910) 1105.
[125] M. von Laue, Ann. Physik 47 (1915) 853.
[126] F. London, Z. Physik 63 (1930) 245.
[127] R. Schiller and H. Tesser, Phys. Rev. A3 (1971) 2035.
[128] E.J.W. Verwey and J.Th.G. Overbeek, J. Phys. and Colloid. Chem. 51 (1947) 631.
[129] J.A. Wheeler, Phys. Rev. 59 (1941) 928(A).
[130] T.H. Boyer, Phys. Rev. D11 (1975) 809.
[131] M. Surdin, Ann. Inst. Poincaré 15 (1971) 203.
[132] E. Santos, Lett. Nuovo Cim. 4 (1972) 497.
[133] L. de le Pena-Auerbach and A.M. Cetto, Phys. Lett. 47A (1974) 183.
[134] E. Santos, Nuovo Cim. 22B (1974) 201.

Note added in proof

Since this article was submitted for publication some further work on the subject came to the author's attention. Clauser [Phys. Rev. D9 (1974) 853] has performed an experiment testing the semiclassical prediction that the radiation from a single atomic spontaneous decay can excite more than one detector atom. His results refute this prediction and support the quantum-electrodynamical prediction. Nash and Gordon [Phys. Rev. A12 (1975) 2472] have examined in detail the problem of radiative equilibrium in the neoclassical theory and have found the predictions of the theory incompatible with experimental results.