# Phys 250 Quantum Optics, Homework #1, Solutions

For 1-3, just follow the text.

*4) Give an explicit expression for the conserved quantities corresponding to the invariance of the Lagrangrian under Lorentz boosts.*

Equations (3.41) - (3.46) in the reading give the general outline, but it's not as straightforward as the other problems. Start with the Lorentz boost:

$$\delta x^\mu = \epsilon^\mu_\nu x^\mu$$
$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$$

Making use of the anti-symmetry and (3.42):

$$\begin{aligned}
\delta x^\mu &= \frac{1}{2}\epsilon^{\rho\sigma}\left[\underbrace{\left(X^\mu_{\rho\sigma} + X^\mu_{\sigma\rho}\right)}_{0} - \left(X^\mu_{\rho\sigma} - X^\mu_{\sigma\rho}\right)\right] \\
&= -\frac{1}{2}\epsilon^{\rho\sigma}\left(X^\mu_{\rho\sigma} - X^\mu_{\sigma\rho}\right) \\
&= -\frac{1}{2}\epsilon_{\rho\sigma}\left(\delta_{\mu\rho}x^\sigma - \delta^{\mu\sigma}x^\rho\right) \\
&= -\frac{1}{2}\left(\epsilon_{\mu\sigma}x^\sigma - \epsilon_{\rho\mu}x^\rho\right)
\end{aligned}$$

The associated Noetherian conserved current is:

$$\begin{aligned}
J^{\mu\rho\sigma} &= -\Theta^\mu_\kappa X^{\kappa\rho\sigma} \\
&= -\frac{1}{2}\left(\Theta^{\mu\rho}x^\sigma - \Theta^{\mu\sigma}x^\rho\right)
\end{aligned}$$

Let us then define:

$$M^{\mu\nu} = \int 2J^{0\mu\nu} d^3x$$

Making use of the fact that $\Theta^{0i} = p^i$:

$$\begin{aligned}
M^{ij} &= -\int \left(\Theta^{0i}x^j - \Theta^{0j}x^i\right) d^3x \\
&= -\int \left(p^i x^j - p^j x^i\right) d^3x \\
&= -\int \left(\vec{p} \times \vec{x}\right)_k \varepsilon^{ijk} d^3x
\end{aligned}$$

In other words the off diagonal elements are the angular momentum of the field.

Additionally, noting that $\Theta^{00} = E$:

$$\begin{aligned}
M^{0j} &= -\int \left(\Theta^{00}x^j - \Theta^{0j}x^0\right) d^3x \\
&= -\int \left(Ex^j - p^j t\right) d^3x
\end{aligned}$$

Which is the field's center of mass minus its motion (momentum times time).