Phys 250 Quantum Optics, Homework #3, Solutions

1 Problem 2.7

We can express any state as a sum of number states:

$$|\psi\rangle = \sum_n C_n |n\rangle$$

So...

$$\begin{aligned}
|\psi'\rangle &= \hat{a}\,|\psi\rangle \\
&= N \sum_n C_n \sqrt{n}\,|n-1\rangle \\
\Rightarrow \langle\psi'|\psi'\rangle &= N^2 \sum_n C_n^2 n \quad \text{(due to orthogonality of } |n\rangle\text{)} \\
&= N^2 \bar{n} \\
\Rightarrow N &= \frac{1}{\sqrt{\bar{n}}}
\end{aligned}$$

What about $\bar{n}'$?

$$\begin{aligned}
\bar{n}' &= \langle\psi'|\,\hat{n}\,|\psi'\rangle \\
&= \frac{1}{\bar{n}} \sum_n C_n^2 n(n-1) \\
&= \frac{1}{\bar{n}} \left(\langle n^2\rangle - \bar{n}\right) \\
&= \frac{\langle n^2\rangle}{\bar{n}} - 1
\end{aligned}$$

This is not equal to $\bar{n} - 1$ unless $\langle n^2\rangle = \langle n\rangle^2$, or in other words $|\psi\rangle$ is a number state to begin with! This is because a photon has a higher probability of begin absorbed from a higher excited state (stimulated absorption).

2 Problem 2.8

It should be obvious that $\bar{n} = 5$.

Furthermore:

$$\begin{aligned}\bar{n}' &= \frac{\langle n^2 \rangle}{\bar{n}} - 1 \\ &= \frac{\frac{1}{2}(0^2) + \frac{1}{2}(10^2)}{5} - 1 \\ &= 9?\end{aligned}$$

This may seem strange, but the output from the (normalized) annihilation operator is weighted – the ground state ends up with a weight of 0, hence it does not contribute! ($N\hat{a}\left|\psi\right\rangle \to \left|9\right\rangle$, so the final state only has a term due to the $\left|10\right\rangle$ part of the wavefunction.)

Extended explanation: Having a process where you "know" a photon is absorbed implies some sort of "post-selection" is going on – this is actually relatively common in experimental quantum optics. If a photon was definitely absorbed, the state couldn't have been in the ground state to begin with; some how this is equivalent to measuring the state *before* the absorption process and finding it to be in the excited state, $\left|10\right\rangle$ (and ignoring the cases where it wasn't).

So... how does this work in theory? To conserve energy, an annihilation operator for one mode should be coupled to a creation operator for another state of the same energy (or several creation operators for different states whose energy adds up to the original state). A physically sensible operator should like more like this:

$$\hat{O} \sim 1 + N\left(\hat{a}^\dagger \hat{b} + \hat{a}^\dagger \hat{b}\right),$$

where $\hat{b}$ is the annihilation operator for the second mode, which was previously ignored. The $\hat{a}^\dagger \hat{b}$ term is needed for time-reversal symmetry, although it doesn't really matter here. Our initial state for the complete system must be something like this:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_a + \left|10\right\rangle_a\right) \otimes \left|0\right\rangle_b,$$

where we assume the second mode is in the ground state. We then apply the operator:

$$\hat{O}\left|\psi\right\rangle \sim \left(\left|0\right\rangle_a + \left|10\right\rangle_a\right) \otimes \left|0\right\rangle_b + N\sqrt{9}\left|9\right\rangle_a \otimes \left|1\right\rangle_b$$

If we wish to consider only states where we *know* a photon was absorbed from mode A, we can measure mode B and *post-select* for the states in which it is excited, which looks like:

$$\left\langle 1\right|_b \hat{O}\left|\psi\right\rangle \sim \left|9\right\rangle_a$$

Thus, we recover the effect of an annihilation operator by using a physical operator *plus* post-selection on a measurement. This implies, *by necessity*, that we are actually ignoring the part of the wavefunction where nothing happened.

3 Problem 2.10

$$
\begin{aligned}
\sum_n n(n-1)\dots(n-r+1)P_n &= \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} P_n \\
&= \frac{1}{1+\bar{n}} \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} x^n; \quad \text{from 2.145 with } x = \frac{\bar{n}}{1+\bar{n}} \\
&= \frac{x^r}{1+\bar{n}} \sum_{m=0}^{\infty} \frac{(m+r)!}{m!} x^m; \quad \text{where } m = n-r \\
&= \frac{x^r}{1+\bar{n}} r!\,(1-x)^{-(r+1)} \\
&= \left(\frac{r!}{1+\bar{n}}\right)\left(\frac{\bar{n}}{1+\bar{n}}\right)^r (1+\bar{n})^{r+1} \\
&= r!\bar{n}^r
\end{aligned}
$$

4 Problem 2.12

For the mixed state, start with (2.229):

$$
\begin{aligned}
P(\phi) &= \frac{1}{2\pi} \langle\phi|\,\hat{\rho}\,|\phi\rangle \\
&= \frac{1}{4\pi}\big(\underbrace{\langle\phi|0\rangle\langle 0|\phi\rangle}_{1} + \underbrace{\langle\phi|1\rangle\langle 1|\phi\rangle}_{1}\big) \\
&= \frac{1}{2\pi}
\end{aligned}
$$

For the superposition, start with (2.226):

$$
\begin{aligned}
P(\phi) &= \frac{1}{2\pi}\left|\sum_n e^{-in\phi} C_n\right|^2 \\
C_n &= \frac{1}{\sqrt{2}}\left\{1, e^{i\phi}, 0\dots\right\} \\
\Rightarrow P(\phi) &= \frac{1}{4\pi}\left|1 + e^{i(\theta-\phi)}\right|^2 \\
&= \frac{1 + cos(\theta-\phi)}{2\pi}
\end{aligned}
$$

The mixed state has a flat distribution – this is expected because it is just a statistical mixture of two number states, and number states have a flat phase distribution (no phase information). A superposition of two number states, however, does have phase information, which results in a lumped distribution.

5 Problem 2.13

The density operator of the thermal state in the number basis is given by (2.138):

$$\hat{\rho}_{Th} = \sum_n P_n \left|n\right\rangle \left\langle n\right| ,$$

but this is just a statistical mixture of number states. As with the last problem, this means that the distribution is flat: $P(\phi) = \frac{1}{2\pi}$. Of course it has to be, there should be no reason for a thermal state to prefer a specific phase!

6 Problem 3.1

$$\begin{aligned}
\hat{a}^\dagger \left|\alpha\right\rangle &= \alpha \left|\alpha\right\rangle \\
&= \sum_n C_n \sqrt{n+1} \left|n+1\right\rangle \\
&= \alpha \sum_n C_n \left|n\right\rangle \\
\Rightarrow C_n \sqrt{n+1} &= \alpha C_{n+1} \\
\Rightarrow C_{n+1} &= \frac{\sqrt{n+1}}{\alpha} C_n
\end{aligned}$$

Which means that C_n monotonically increases with n. This means the state is not normalizable and hence non-physical.

7 Problem 3.4

The easiest way to prove this is to calculate the partial derivative.

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\left|\alpha\right\rangle &= \frac{\partial}{\partial\alpha}e^{-\frac{\alpha\alpha^*}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \\
&= -\frac{\alpha^*}{2}\left|\alpha\right\rangle + e^{-\frac{\alpha\alpha^*}{2}}\sum_n \frac{n\alpha^{n-1}}{\sqrt{n!}}\left|n\right\rangle \quad \text{(Mathematical reminder: } \frac{\partial}{\partial\alpha}\alpha^* = 0\text{)} \\
&= -\frac{\alpha^*}{2}\left|\alpha\right\rangle + e^{-\frac{\alpha\alpha^*}{2}}\sum_n \frac{(n+1)\alpha^{n}}{\sqrt{(n+1)!}}\left|n+1\right\rangle \quad \text{(Because the n=0 term is 0!)} \\
&= -\frac{\alpha^*}{2}\left|\alpha\right\rangle + e^{-\frac{\alpha\alpha^*}{2}}\sum_n \frac{\sqrt{n+1}\alpha^{n}}{\sqrt{n!}}\left|n+1\right\rangle \\
&= -\frac{\alpha^*}{2}\left|\alpha\right\rangle + \hat{a}^\dagger\left|\alpha\right\rangle
\end{aligned}
$$

$$
\frac{\partial}{\partial\alpha}\left\langle\alpha\right| = -\frac{\alpha^*}{2}\left\langle\alpha\right|
$$

$$
\begin{aligned}
&\Rightarrow \frac{\partial}{\partial\alpha}\left|\alpha\right\rangle\left\langle\alpha\right| = \left(\hat{a}^\dagger - \alpha^*\right)\left|\alpha\right\rangle\left\langle\alpha\right| \\
&\Rightarrow \hat{a}^\dagger\left|\alpha\right\rangle\left\langle\alpha\right| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)\left|\alpha\right\rangle\left\langle\alpha\right|
\end{aligned}
$$

The proof of second identity is nearly identical.

8 Problem 3.5

First we calculate $\left\langle\hat{X}\right\rangle$ and $\left\langle\hat{X}^2\right\rangle$:

$$
\begin{aligned}
\hat{X}_1 &= \frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right) \\
\left\langle\hat{X}_1\right\rangle_\alpha &= \frac{1}{2}\left\langle\alpha\right|\hat{a}+\hat{a}^\dagger\left|\alpha\right\rangle \\
&= \frac{1}{2}\left(\alpha+\alpha^*\right)\left\langle\alpha|\alpha\right\rangle \\
&= \mathrm{Re}\left(\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{X}_1^2 \right\rangle_\alpha &= \frac{1}{4} \langle\alpha| \hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} |\alpha\rangle \\
&= \frac{1}{4} \langle\alpha| \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}^{\dagger 2} |\alpha\rangle \\
&= \frac{1}{4} \left(\alpha^2 + 2\alpha\alpha^* + \alpha^{*2} + 1\right) \\
&= \frac{1}{4} \left[(\alpha + \alpha^*)^2 + 1\right] \\
&= \frac{1}{4} + \mathrm{Re}\,(\alpha)^2
\end{aligned}
$$

$$
\Rightarrow \left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_\alpha = \left\langle \hat{X}_1^2 \right\rangle_\alpha - \left\langle \hat{X}_1 \right\rangle_\alpha^2 = \frac{1}{4}
$$

$$
\begin{aligned}
\hat{X}_2 &= \frac{1}{2i} \left(\hat{a} - \hat{a}^\dagger\right) \\
&\cdots \\
\left\langle \hat{X}_2 \right\rangle_\alpha &= \mathrm{Im}\,(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{X}_2^2 \right\rangle_\alpha &= -\frac{1}{4} \langle\alpha| \hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} |\alpha\rangle \\
&= -\frac{1}{4} \langle\alpha| \hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1 + \hat{a}^{\dagger 2} |\alpha\rangle \\
&\cdots \\
&= \frac{1}{4} + \mathrm{Im}\,(\alpha)^2
\end{aligned}
$$

$$
\Rightarrow \left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_\alpha = \left\langle \hat{X}_2^2 \right\rangle_\alpha - \left\langle \hat{X}_2 \right\rangle_\alpha^2 = \frac{1}{4}
$$