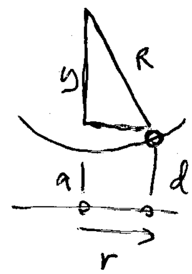
2. plane-sphere:



$$d(r) = a + R - y = a + R - \sqrt{R^2 - r^2} = a + R\left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right)$$

$$\cong a + \frac{r^2}{2R} + \mathcal{O}(r^4)$$

from 2.182: $P_{casimir} = -\dfrac{\pi^2 \hbar c}{240\, d^4}$

so $F_{p-s} \cong -\displaystyle\int dA \frac{\pi^2 \hbar c}{240\, d(r)^4} = -\frac{\pi^2 \hbar c}{240} \underbrace{\int_0^\infty 2\pi r\, dr \left(a + \frac{r^2}{2R}\right)^{-4}}_{\frac{2\pi R}{3a^3}}$

$$\boxed{F_{p-s} \cong -\frac{\pi^3 \hbar c}{360}\left(\frac{R}{a^3}\right)}$$

for 2 spheres, $d(r) \cong a + \dfrac{r^2}{2R} + \dfrac{r^2}{2R} = a + \dfrac{r^2}{R}$

this is equivelent to $R \to R/2$, so:

$$\boxed{F_{s-s} \cong -\frac{\pi^3 \hbar c}{720}\left(\frac{R}{a^3}\right)}$$

3. (3.12)

For a coherent state $|\beta\rangle$:

$$Q(\alpha) = \frac{|\langle\psi|\alpha\rangle|^2}{\pi} = \frac{e^{-|\alpha-\beta|^2}}{\pi} \quad \text{(from 3.61)}$$

$$C_A(\lambda) = \int d^2\alpha\, Q(\alpha)\, e^{\lambda\alpha^* - \lambda^*\alpha}$$ (you can evaluate this in Mathematica or equivalent.)

$$= e^{-|\lambda|^2} e^{\lambda\beta^* - \beta\lambda^*}$$

$$W(\alpha) = \int d^2\lambda\, C_A(\lambda)\, e^{|\lambda|^2/2} e^{\lambda^*\alpha - \lambda\alpha^*} \quad \text{(From 3.136)}$$

$$\boxed{W(\alpha) = \frac{2}{\pi} \exp\left[-2|\alpha-\beta|^2\right]}$$

For a number state $|n\rangle$:

First you need to find C_N. There are 2 ways to do this, either start with (3.103) and (3.134) or evaluate (3.128b) directly. Here is the second method:

$$C_N = \mathrm{Tr}\left[\hat{\rho}\, e^{\lambda\hat{a}^\dagger} e^{-\lambda^*\hat{a}}\right]$$

$$= \langle n|n\rangle\langle n| e^{\lambda\hat{a}^\dagger} e^{-\lambda^*\hat{a}}|n\rangle$$

$$= \left(\sum_m \langle n| \frac{(\lambda\hat{a}^\dagger)^m}{m!}\right)\left(\underbrace{\sum_m \frac{(-\lambda^*\hat{a})^m}{m!}|n\rangle}_{\sum_{m=0}^{n} \frac{(-\lambda^*)^m}{m!}\sqrt{\frac{n!}{(n-m)!}}\,|n-m\rangle}\right)$$

$$= \sum_{m=0}^{n} \frac{(-1)^m |\lambda|^{2m}}{(m!)^2} \frac{n!}{(n-m)!}$$ (due to the orthogonality of number states)

$$C_N = \sum_{m=0}^{n} \frac{(-1)^m (|\lambda|^2)^m}{m!} \binom{n}{m}$$

$$= L_n(|\lambda|^2)$$

$$\Rightarrow W(\alpha) = \int d^2\lambda \, C_N(\lambda) e^{-|\lambda|^2/2} e^{\lambda^* \alpha - \lambda \alpha^*}$$

There is no easy way to show that this integral gives a simple answer without extensive knowledge of special functions! Instead we can calculate the first few:

for $|0\rangle$: $W(\alpha) = \frac{2}{\pi} e^{-2|\alpha|^2}$

$|1\rangle$: $W(\alpha) = -\frac{2}{\pi} e^{-2|\alpha|^2} (1 - 4|\alpha|^2)$

$|2\rangle$: $W(\alpha) = +\frac{2}{\pi} e^{-2|\alpha|^2} (1 - 8|\alpha|^2 + 8|\alpha|^4)$

$$= \frac{2}{\pi} e^{-2|\alpha|^2} \left(\frac{1}{2}\right) \left[(4|\alpha|^2)^2 - 4(4|\alpha|^2) + 2 \right]$$

$$\Rightarrow \boxed{W(\alpha) = \frac{2}{\pi} e^{-2|\alpha|^2} (-1)^n L(4|\alpha|^2)}$$

4. (3.13) a) $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta\rangle + |-\beta\rangle\right)$

$$\langle\psi|\psi\rangle = \frac{1}{2}\Big(\underbrace{\langle\beta|\beta\rangle + \langle-\beta|-\beta\rangle}_{2} + \underbrace{\langle\beta|-\beta\rangle + \langle-\beta|\beta\rangle}_{2\exp[-\beta^2]}\Big)$$

$$= 1 + e^{-2\beta^2} \cong 1 \text{ for } \beta >> 1$$

b) $$\langle n|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle n|\beta\rangle + \langle n|-\beta\rangle\right)$$

$$= \frac{1}{\sqrt{2}} e^{-|\beta|^2/2}\left(\frac{\beta^n + (-\beta)^n}{\sqrt{n!}}\right) \quad \text{from } 3.10$$

$$\Rightarrow P_n = |\langle n|\psi\rangle|^2 = \frac{e^{-|\beta|^2}}{2n!}\begin{cases} 4\beta^{2n} & n \text{ is even} \\ 0 & n \text{ is odd}\end{cases}$$

$$= \begin{cases} \frac{2e^{-|\beta|^2}}{n!}\beta^{2n} & n \text{ even} \\ 0 & n \text{ odd}\end{cases}$$

c) $$P(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 = \frac{1}{2\pi}\left|\sum_n e^{-in\phi}\langle n|\psi\rangle\right|^2$$

$$= \frac{e^{-|\beta|^2}}{4\pi}\left|\sum_n e^{-in\phi}\left(\frac{\beta^n + [-\beta]^n}{\sqrt{n!}}\right)\right|^2$$

You can approximate along the lines of (3.27)-(3.29), but in this case it is only defined on some wierd interval of ϕ.

d) $Q(\alpha) = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} = \frac{|\langle\psi|\alpha\rangle|^2}{\pi}$

$$\langle\psi|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\langle\beta| + \langle-\beta|\right)|\alpha\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\phi}e^{-\frac{1}{2}|\alpha-\beta|^2} + e^{-i\phi}e^{-\frac{1}{2}|\alpha+\beta|^2}\right) \quad \text{From (3.61)}$$

with $\phi = \mathrm{Im}(\beta^*\alpha)$

$$Q(\alpha) = \frac{1}{2\pi}\left[e^{-|\alpha-\beta|^2} + e^{-|\alpha+\beta|^2} + \underbrace{\left(e^{2i\phi} + e^{-2i\phi}\right)}_{2\cos(2\phi)} e^{\overbrace{-\frac{1}{2}\left(|\beta-\alpha|^2+|\beta+\alpha|^2\right)}^{-|\alpha|^2-|\beta|^2}}\right]$$

$$\underline{\underline{Q(\alpha) = \frac{1}{2\pi}\left[e^{-|\alpha-\beta|^2} + e^{-|\alpha+\beta|^2} + 2\cos\left(2\,\mathrm{Im}(\beta^*\alpha)\right)e^{-(|\alpha|^2+|\beta|^2)}\right]}}$$

$$C_A(\lambda) = \int d^2\alpha\, Q(\alpha)\, e^{\lambda\alpha^* - \lambda^*\alpha}$$

$$= \frac{e^{-|\lambda|^2}}{2}\left(e^{\lambda\beta^* - \beta\lambda^*} + e^{\beta\lambda^* - \lambda\beta^*}\right) + \frac{e^{-2|\beta|^2-|\lambda|^2}}{2}\left(e^{-\beta\lambda^* - \beta^*\lambda} + e^{\lambda\beta^* + \beta^*\lambda}\right)$$

$$= e^{-|\lambda|^2}\left[\cos\left(\mathrm{Im}[2\lambda\beta^*]\right) + e^{-2|\beta|^2}\cosh\left(2\mathrm{Re}[\lambda\beta^*]\right)\right]$$

$$W(\alpha) = \int d^2\lambda\, C_A(\lambda)\, e^{|\lambda|^2/2}\, e^{\lambda^*\alpha - \lambda\alpha^*}$$

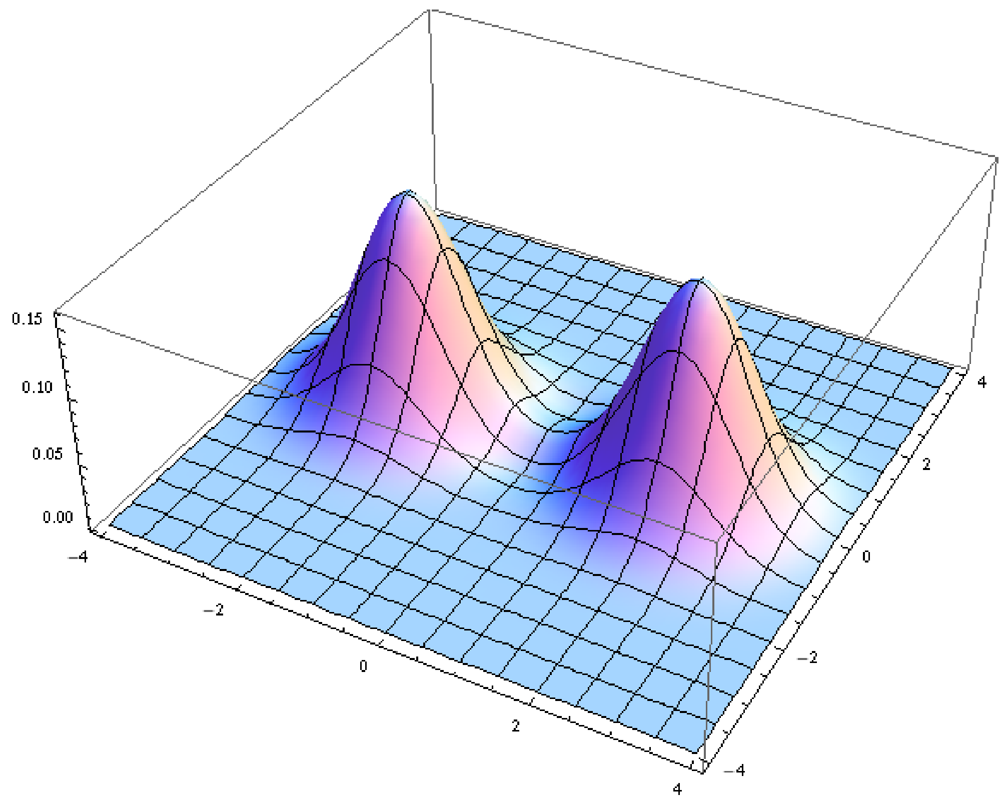
$$= \frac{1}{\pi}\left(e^{-2|\alpha-\beta|^2} + e^{-2|\alpha+\beta|^2} + 2e^{-|\alpha|}\cosh\left[2\beta\alpha^* - 2\alpha\beta^*\right]\right)$$

In[35]:= Plot3D[

$$\frac{1}{2\pi}\left(\mathrm{Exp}\left[-\mathrm{Abs}[\alpha-\beta]^2\right]+\mathrm{Exp}\left[-\mathrm{Abs}[\alpha+\beta]^2\right]+2\,\mathrm{Cos}[2\,\mathrm{Im}[\beta^*\alpha]]\,\mathrm{Exp}\left[-\mathrm{Abs}[\alpha]^2-\mathrm{Abs}[\beta]^2\right]\right)\ /.$$

{α → x + ⅈ y, β → 2}, {x, -4, 4}, {y, -4, 4}, PlotRange → All, PlotPoints → 50, ImageSize → Large]

Out[35]=

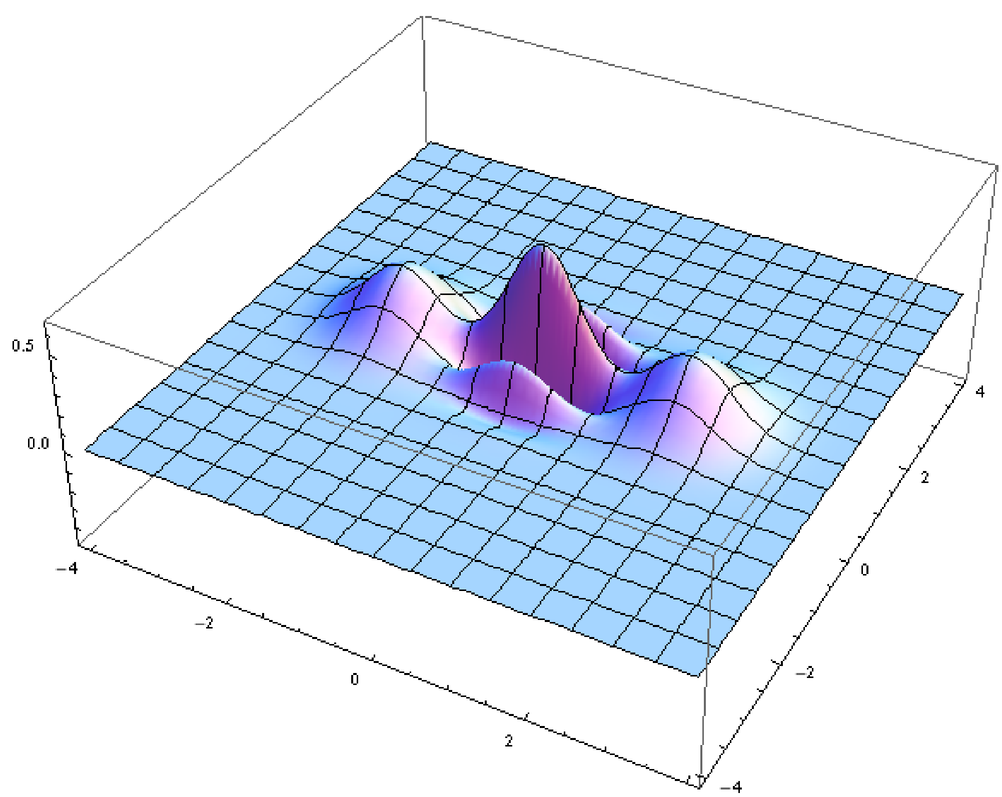


In[36]:= Plot3D[

$$\frac{1}{\pi}\left(\mathrm{Exp}\left[-2\,\mathrm{Abs}[\alpha-\beta]^2\right]+\mathrm{Exp}\left[-2\,\mathrm{Abs}[\alpha+\beta]^2\right]+2\,\mathrm{Exp}\left[-2\,\mathrm{Abs}[\alpha]^2\right]\mathrm{Cosh}[2\,\beta\,\alpha^*-2\,\alpha\,\beta^*]\right)\ /.$$

{α → x + ⅈ y, β → 2}, {x, -4, 4}, {y, -4, 4}, PlotRange → All, PlotPoints → 50, ImageSize → Large]

Out[36]=



5. (4.1) Start w/ eqn. (4.78)

$$C_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t} \quad ; \quad \lambda_\pm = \frac{1}{2}(\Delta \pm \Omega_R)$$

$$C_e(0) = 1, \text{ so } A_+ + A_- = 1$$

From (4.74)

$$\dot{C}_e = -\frac{i\nu}{2\hbar} e^{+i\Delta t} C_g = i\lambda_+ A_+ e^{i\lambda_+ t} + i\lambda_- A_- e^{i\lambda_- t}$$

$$\text{so...} \quad C_g(t) = -\frac{2\hbar}{\nu} e^{-i\Delta t}\left(\lambda_+ A_+ e^{i\lambda_+ t} + \lambda_- A_- e^{i\lambda_- t}\right)$$

$$C_g(0) = -\frac{\hbar}{\nu} \times \left[\Delta\underbrace{(A_+ + A_-)}_{1} + \Omega_R(A_+ - A_-)\right] = 0$$

$$\Rightarrow A_+ - A_- = -\frac{\Delta}{\Omega_R} \Rightarrow A_\pm = \frac{1}{2}\left(1 \mp \frac{\Delta}{\Omega_R}\right)$$

$$\Rightarrow C_e(t) = \frac{1}{2} e^{\frac{i\Delta t}{2}} \left[\left(1 - \frac{\Delta}{\Omega_R}\right) e^{\frac{+i\Omega_R t}{2}} + \left(1 + \frac{\Delta}{\Omega_R}\right) e^{\frac{-i\Omega_R t}{2}}\right]$$

$$= \frac{1}{2} e^{\frac{i\Delta t}{2}} \left[2\cos(\Omega_R t/2) - \overset{\left(\frac{\Delta}{\Omega_R}\right)}{2} i \sin(\Omega_R t/2)\right]$$

$$\boxed{C_e(t) = e^{\frac{i\Delta t}{2}} \left[\cos\left(\frac{\Omega_R t}{2}\right) - i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right]}$$

$$\Rightarrow C_g(t) = -\frac{\hbar}{2V} e^{-\frac{i\Delta t}{2}} \left[\underbrace{(\Delta+\Omega_R)\left(1-\frac{\Delta}{\Omega_R}\right)}_{= \frac{\Delta^2}{\Omega_R} - \Omega_R = \frac{1}{\Omega_R}(\Delta^2 - \Omega_R^2) = -\frac{V^2}{\Omega_R \hbar^2}} e^{+\frac{i\Omega_R t}{2}} + \underbrace{(\Delta-\Omega_R)\left(1+\frac{\Delta}{\Omega_R}\right)}_{= +\frac{V^2}{\Omega_R \hbar^2}} e^{-\frac{i\Omega_R t}{2}} \right]$$

$$\boxed{C_g(t) = \frac{iV}{\Omega_R \hbar} e^{-\frac{i\Delta t}{2}} \sin\left(\frac{\Omega_R t}{2}\right)}$$

$$w(t) = P_e(t) - P_g(t)$$

$$P_e(t) = |c_e|^2 = \cos^2\left(\frac{\Omega_R t}{2}\right) + \left(\frac{\Delta}{\Omega_R}\right)^2 \sin^2(\dots)$$

$$= \cos^2(\dots) + \left[1 - \left(\frac{V}{\Omega_R \hbar}\right)^2\right] \sin^2(\dots)$$

$$= 1 - \left(\frac{V}{\Omega_R \hbar}\right)^2 \sin^2\left(\frac{\Omega_R t}{2}\right)$$

$$P_g(t) = \left(\frac{V}{\Omega_R \hbar}\right)^2 \sin^2\left(\frac{\Omega_R t}{2}\right)$$

$$\Rightarrow \underline{w(t) = 1 - 2\left(\frac{V}{\Omega_R \hbar}\right)^2 \sin^2\left(\frac{\Omega_R t}{2}\right) = 1 - \left[\cos\left(\Omega_R t\right) - 1\right]\left(\frac{V}{\Omega_R \hbar}\right)^2}$$

6. (4.2)

For $\Delta = 0$, $\Omega_R = \frac{V}{\hbar}$

So... $C_e = \cos\left(\frac{Vt}{2\hbar}\right)$

$C_g = i\sin\left(\frac{Vt}{2\hbar}\right)$

$$|\psi\rangle = C_g(t)\, e^{-iE_g t/\hbar}|g\rangle + C_e(t)\, e^{-iE_e t/\hbar}|e\rangle \qquad (4.70)$$

$$= C_g(t)|g\rangle + C_e(t)\, e^{-i\omega_0 t}|e\rangle$$ (overall phase is not significant!)

$$\langle \hat{d}\rangle = \langle\psi|\hat{d}|\psi\rangle = d\left(\langle\psi|g\rangle\langle e|\psi\rangle + \langle\psi|e\rangle\langle g|\psi\rangle\right)$$

$$= 2d\, \mathrm{Re}\left(\langle\psi|g\rangle\langle e|\psi\rangle\right)$$

$$= 2d\, \mathrm{Re}\left(i e^{-i\omega_0 t}\right) \sin\left(\frac{Vt}{2\hbar}\right)\cos\left(\frac{Vt}{2\hbar}\right)$$

$$\boxed{\langle \hat{d}\rangle = d \sin(\omega_0 t)\sin\left(\frac{Vt}{\hbar}\right)}$$

$W(t)$ for $\Delta = 0$: $W(t) = 1 - \cos\left(\frac{Vt}{\hbar}\right)$

The inversion is out of phase with the dipole moment and doesn't have a fast rotating part.