6.1 is just a special case of 6.2!

6.2 $U = \exp\left[i \frac{\Theta}{2} (a_0^+ a_1 + a_1^+ a_0) \right]$

Note: $e^{i\lambda \hat{A}} \hat{B} e^{-i\lambda \hat{A}} = \hat{B} + i\lambda [\hat{A}, \hat{B}] + \frac{(i\lambda)^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$

So... $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^+ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} U$

$$= \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - i \frac{\Theta}{2} \left[\hat{A}, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] - \frac{\Theta^2}{8} \left[\hat{A}, \left[\hat{A}, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] \right] + \dots$$

where $\hat{A} = a_0^+ a_1 + a_1^+ a_0$

$$\left[a_0^+ a_1 + a_1^+ a_0, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] = a_0^+ \underbrace{\left[a_1, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right]}_{0} + \left[a_0^+, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] a_1 +$$

$$a_1^+ \underbrace{\left[a_0, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right]}_{0} + \left[a_1^+, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] a_0$$

$$= -\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

$$\Rightarrow \left[\hat{A}, \left[\hat{A}, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \right] \right] = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$\Rightarrow U \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} U^+ = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \left[1 - \frac{(\Theta/2)^2}{2} + \dots \right] + i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \left[\frac{\Theta}{2} - \frac{\left(\frac{\Theta}{2}\right)^3}{3!} + \dots \right]$$

$$\boxed{\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \cos(\Theta/2) + i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \sin(\Theta/2)}$$

for $\Theta = \pi/2$, $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_0 + i a_1 \\ a_1 + i a_0 \end{pmatrix}$ (problem 6.1)

from (6.8) $\begin{pmatrix} t' \\ r' \end{pmatrix} = \begin{pmatrix} \cos(\Theta/2) \\ i \sin(\Theta/2) \end{pmatrix}$

$$\begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} t' \\ r' \end{pmatrix}$$

$$\Rightarrow \boxed{\Theta = 2 \arctan\left(\frac{|r'|}{|t'|} \right)}$$

6.6 $|\alpha\rangle_0|\beta\rangle_1 = D_0(\alpha)D_1(\beta)|0\rangle_0|0\rangle_1$

$$D_n = \exp[\alpha a_n^+ - \alpha^* a_n]$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_0^+ \\ a_1^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_2 - i a_3 \\ -i a_2 + a_3 \\ a_2^+ + i a_3^+ \\ i a_2^+ + a_3 \end{pmatrix}$$

$$D_0(\alpha) = \exp(\alpha a_0^+ - \alpha^* a_0)$$

$$= \exp\left[\frac{\alpha}{\sqrt{2}}(a_2^+ + i a_3^+) - \frac{\alpha^*}{\sqrt{2}}(a_2 - i a_3)\right]$$

$$= \exp\left[\frac{\alpha}{\sqrt{2}}(a_2^+) - \frac{\alpha^*}{\sqrt{2}}(a_2)\right]\exp\left[\frac{i\alpha}{\sqrt{2}}(a_3^+) - \frac{(i\alpha)^*}{\sqrt{2}}(a_3)\right]$$

$$= D_2(\alpha/\sqrt{2})\ D_3(i\alpha/\sqrt{2})$$

similarly: $D_1(\beta) = D_2(i\alpha/\sqrt{2})\,D_3(\alpha/\sqrt{2})$

$$\Rightarrow |\alpha\rangle_0|\beta\rangle_1 \xrightarrow{BS} D_2(\alpha/\sqrt{2})\,D_2(i\beta/\sqrt{2})\,D_3(i\alpha/\sqrt{2})\,D_3(\beta/\sqrt{2}) \times |0\rangle_2|0_3\rangle$$

$$\boxed{|\alpha\rangle_0|\beta\rangle_1 \xrightarrow{BS} \left|\tfrac{1}{\sqrt{2}}(\alpha + i\beta)\right\rangle_2 \left|\tfrac{1}{\sqrt{2}}(i\alpha + \beta)\right\rangle_3}$$

(ignoring an <u>overall</u> phase factor which is not significant!)

<u>This state is not entangled.</u>

6.8 $|N\rangle_0 |N\rangle_1 = \frac{1}{N!}(a_0^+ a_1^+)^N |0\rangle_0 |0\rangle_1$

$$\xrightarrow{BS} \frac{1}{N!}\left[\frac{1}{2}(a_2^+ + i a_3^+)(i a_2^+ + a_3)\right]^N |0\rangle_2 |0_3\rangle$$

$$\rightarrow \frac{1}{2^N N!}\left[i a_2^{+2} - \cancel{a_3^+ a_2^+} + \cancel{a_3^+ a_2^+} + i a_3^{+2}\right]^N |0\rangle_2 |0\rangle_3$$

$$\rightarrow \frac{i^N}{2^N N!}\left[a_2^{+2} + a_3^{+2}\right]^N |0\rangle_2 |0\rangle_3$$

$$\rightarrow \left(\frac{i}{2}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \binom{N}{k} (a_2^{+2})^{N-k} (a_3^{+2})^k |0\rangle_2 |0\rangle_3$$

$$\rightarrow \sum_k \left(\frac{i}{2}\right)^N \frac{N!}{N!k!(N-k)!} \sqrt{(2N-2k)!(2k)!}\, |2N-2k\rangle_2 |2k\rangle_3$$

$$\rightarrow \sum_k i^N \left[\left(\frac{1}{2}\right)^{2N} \frac{(2k)!}{(k)!(2k-k)!} \frac{(2N-2k)!}{(N-k)!(N-k)!}\right]^{1/2} |2N-2k\rangle_2 |2k\rangle_3$$

$$\rightarrow \sum_k i^N \left[\binom{2k}{k}\binom{2N-2k}{N-k}\left(\frac{1}{2}\right)^{2N}\right]^{1/2} |2k\rangle_2 |2N-2k\rangle_3$$

The i^N term is an overall phase, and is not significant.

6.11 from 6.2: $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \cos\phi + i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \sin\phi$

$(\phi = \theta/2)$

$$\Rightarrow \begin{pmatrix} a_0^+ \\ a_1^+ \end{pmatrix} = \begin{pmatrix} a_2^+ \\ a_3^+ \end{pmatrix} \cos\phi + i \begin{pmatrix} a_3^+ \\ a_2^+ \end{pmatrix} \sin\phi .$$

$$|0\rangle|1\rangle \xrightarrow{BS1} |0\rangle|1\rangle \cos\phi_1 + i|1\rangle|0\rangle \sin\phi_1$$

$$\Rightarrow |0\rangle|1\rangle \xrightarrow{BS1}\xrightarrow{BS2} \cos\phi_1 \left(|0\rangle|1\rangle\cos\phi_2 + i|1\rangle|0\rangle \sin\phi_2\right) + \sin\phi_1 \left(i|1\rangle|0\rangle \cos\phi_2 - |0\rangle|1\rangle \sin\phi_2\right)$$

$$\to\to (\cos\phi_1\cos\phi_2 - \sin\phi_1 \sin\phi_2)|0\rangle_{D_1}|1\rangle_{D_2} + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)|1\rangle_{D_1}|0\rangle_{D_2}$$

We want all of the light to go to one detector (D_1) if no object is present.

Thus: $\cos\phi_1\cos\phi_2 - \sin\phi_1 \sin\phi_2 = 0 \quad (= \sqrt{P(D_2)})$

$$\Rightarrow \phi_1 = \pi/2 - \phi_2 \quad \text{or} \quad \theta_1 = \pi - \theta_2$$

(Note that this is satisfied for 50:50 BS.)

What if an object is present?

$$|0\rangle|1\rangle \xrightarrow{BS1} |0\rangle|1\rangle\cos\phi_1 + \cancel{i|1\rangle|0\rangle \sin\phi_1}\ |0\rangle|0\rangle \sin\phi_1$$

$$\xrightarrow{BS1}\xrightarrow{BS2} \cos\phi_1\left(|0\rangle|1\rangle\cos\phi_2 + i|1\rangle|0\rangle\sin\phi_2\right) + |0\rangle|0\rangle \sin\phi_1$$

$$\Rightarrow P(D_2) = [\cos\phi_1 \cos\phi_2]^2 = (\cos\phi_1 \sin\phi_1)^2$$

which is the chance of detecting the object.

It has a maximum of $1/4$ at $\phi = \frac{\pi}{4}(2n+1)$ or $\theta = \frac{\pi}{2}(2n+1)$

Thus a 50:50 BS is ideal.

(Note: I ignored the phase shifter in one arm, but it is easy to show that the ideal phase is 0, as in the 50:50 case.)