1. 4.5 We start with (4.155):

$$|\psi(t)\rangle = \sum_n C_n \left[\cos(\phi_n/2)|n,+\rangle e^{-i\Omega_n t/2} - \sin(\phi_n/2)|n,-\rangle e^{+i\Omega_n t/2} \right] e^{+in\omega t}$$

(Note: $E_\pm(n) = (n+\frac{1}{2})\hbar\omega \pm \hbar\Omega_n(\Delta)/2$; the book has a typo in 4.145!)

$$|n,+\rangle = \cos(\phi_n/2)|e\rangle|n\rangle + \sin(\phi_n/2)|g\rangle|n+1\rangle$$
$$|n,-\rangle = -i\sin(\phi_n/2)|e\rangle|n\rangle + \cos(\phi_n/2)|g\rangle|n+1\rangle$$

w/ $\phi_n = \tan^{-1}\left(\frac{\Omega_n(0)}{\Delta}\right)$

$$\Rightarrow |\psi(t)\rangle = \sum_n C_n \Big[\left(\cos^2(\phi_n/2) e^{-i\Omega_n t/2} + \sin^2(\phi_n/2) e^{+i\Omega_n t/2}\right)|e\rangle|n\rangle + \sin(\phi_n/2)\cos(\phi_n/2)\underbrace{\left(e^{-i\Omega_n t/2} - e^{+i\Omega_n t/2}\right)}_{-2i\sin(\Omega_n t/2)}|g\rangle|n+1\rangle\Big] e^{+in\omega t}$$

$$w(t) = P_e(t) - P_g(t) = 1 - 2P_g(t)$$

$$P_g(t) = \sum_n |(\langle n|\langle g|)|\psi(t)\rangle|^2 = \sum_n |(\langle n+1|\langle g|)|\psi(t)\rangle|^2$$

$$= \sum_n |c_n|^2 \underbrace{\sin^2(\phi_n/2)\cos^2(\phi_n/2)}\, 4\sin^2(\Omega_n t/2)$$

from (4.149): $\frac{1}{4}\frac{\Omega_n^2 - \Delta^2}{\Omega_n^2} = \frac{\lambda^2(n+1)}{\Omega_n^2}$

$$P_g(t) = \sum_n |C_n|^2 \frac{4\lambda^2(n+1)}{\Omega_n^2}\sin^2(\Omega_n t/2)$$

$$\Rightarrow w(t) = \sum_n |C_n|^2 \left[1 - \frac{8\lambda^2(n+1)}{\Omega_n^2}\sin^2(\Omega_n t/2)\right]$$

This agrees w/ 4.123 for $\Delta \to 0$!

$$\Rightarrow \boxed{w(t) = \sum_n \left(1 - \frac{4\lambda^2(n+1)}{\Omega_n^2}\right) + |c_n|^2 \frac{4\lambda^2(n+1)}{\Omega_n^2}\cos(\Omega_n t)}$$

(graph on next page)

$$W = \text{Exp}[-nb]\,\frac{nb^n}{n!}\left(1 - \frac{8\,\lambda^2\,(n+1)}{\Omega^2}\,\text{Sin}[\Omega\,t\,/\,2]^2\right)\;/.\;\Omega \to \sqrt{\Delta^2 + 4\,\lambda^2\,(n+1)}\;;$$

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Plot[{Sum[W /. {λ → 1, Δ → 5, nb → 5}, {n, 0, 20}], Sum[W /. {λ → 1, Δ → 0, nb → 5}, {n, 0, 20}]}, {t, 0, 25}, PlotRange → {-1, 1}]
(*Plot shows Δ = {0, 5}*)
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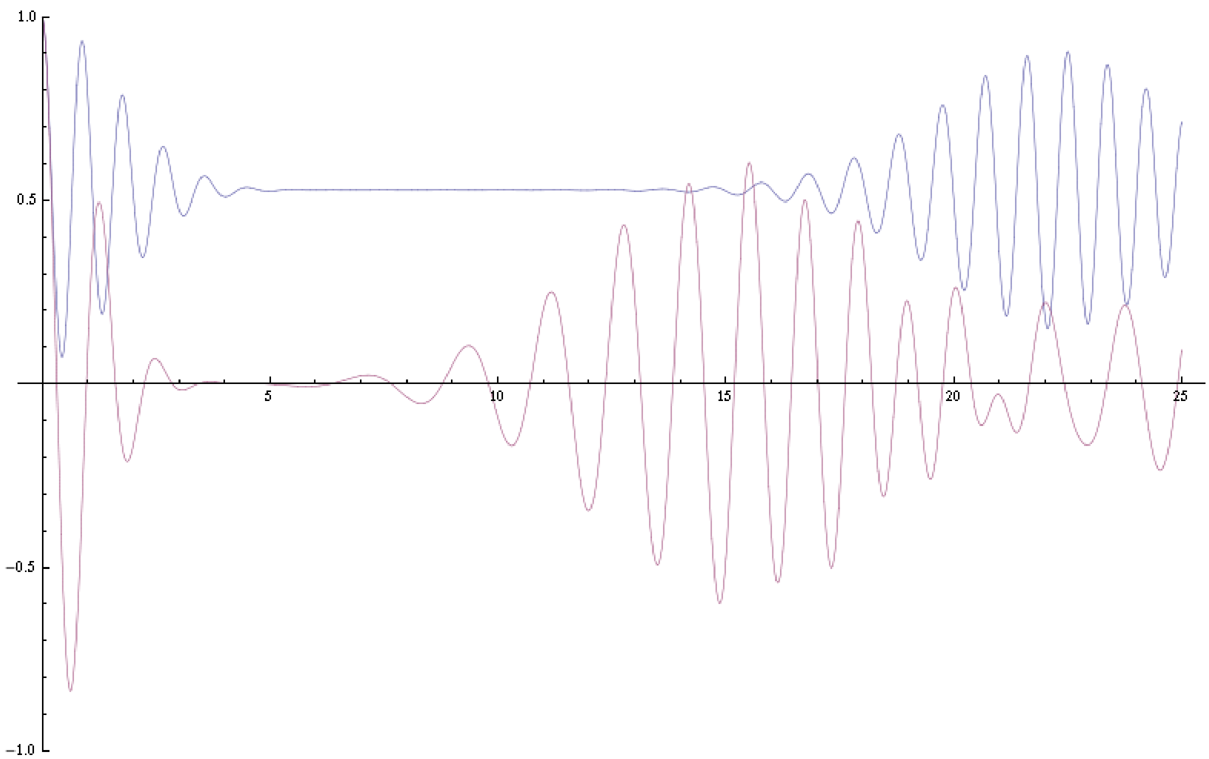

$$\Omega_n = \sqrt{\Delta^2 + 4\lambda^2(n+1)} \cong \Omega_{\bar{n}} + \frac{2\lambda^2(n-\bar{n})}{\Omega_{\bar{n}}} + \mathcal{O}(n-\bar{n})^2$$

$$\omega'(t) = \sum_n |c_n|^2 \frac{2\lambda^2(n+1)}{\Omega_n^2}\left(e^{i\Omega_n t} + e^{-i\Omega_n t}\right)$$

(just the oscillating part of $\omega(t)$)

$$\simeq e^{-\bar{n}} \sum_n \frac{\bar{n}^n}{n!} \underbrace{\frac{2\lambda^2(n+1)}{\Omega_n^2}}_{\cong \frac{2\lambda^2(\bar{n}+1)}{\Omega_{\bar{n}}^2} \text{ for } \bar{n}>>1} \left(e^{i\left(\Omega_{\bar{n}} + \frac{2\lambda^2\bar{n}}{\Omega_{\bar{n}}} - \frac{2\lambda^2 n}{\Omega_{\bar{n}}}\right)t} + e^{-\cdots}\right)$$

$$\cong \frac{2e^{-\bar{n}}\lambda^n(\bar{n}+1)}{\Omega_{\bar{n}}^2} \exp\left[it\left(\Omega_{\bar{n}} + \frac{2\lambda^2\bar{n}}{\Omega_{\bar{n}}}\right) + \underbrace{\bar{n}e^{\frac{-2i\lambda^2}{\Omega_{\bar{n}}}t}}_{\cong \bar{n} - \frac{2i\lambda^2}{\Omega_n}\bar{n}t - \frac{2\lambda^4}{\Omega_{\bar{n}}^2}\bar{n}t^2 + \cdots}\right] + \exp[---]$$

by comparing with 4.133 – 4.135, we see the quadratic term gives the decay time:

$$\boxed{t_c = \frac{\Omega_{\bar{n}}}{\sqrt{2\bar{n}}\,\lambda^2}}$$

for $\Delta = 0$, $\Omega_{\bar{n}} = 2\lambda\sqrt{\bar{n}+1} \Rightarrow t_c = \frac{\sqrt{2}}{\lambda}\sqrt{\frac{\bar{n}+1}{\bar{n}}}$,

which agrees w/(4.136)

from (4.137): $\underbrace{\left(\Omega(\bar{n}+1) - \Omega(\bar{n})\right)}_{\simeq \frac{2\lambda^2}{\Omega_{\bar{n}}} \text{ from expansion above}} t_R = 2\pi k$

$$\Rightarrow \boxed{t_R = \frac{\pi\Omega_{\bar{n}}}{\lambda^2}k} \quad \left(\text{for } \Delta = 0,\ t_R = \frac{2\pi\sqrt{\bar{n}+1}}{\lambda}k \cong \frac{2\pi\sqrt{\bar{n}}}{\lambda}k\right)$$

2. 4.6 $\omega(t) = \sum_n P_n \frac{1}{2}\left(e^{2i\lambda t\sqrt{n+1}} + e^{-2i\lambda t\sqrt{n+1}}\right)$ from (4.183)

For a thermal state: $P_n = \frac{1}{(\bar{n}+1)}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$

as before: $\sqrt{n+1} \cong \sqrt{\bar{n}+1} + \frac{n-\bar{n}}{2\sqrt{\bar{n}+1}} + \dots$

$$\cong A + Bn \quad \text{with} \quad \begin{cases} A = \sqrt{\bar{n}+1} - \frac{\bar{n}}{2\sqrt{\bar{n}+1}} \\ B = \frac{1}{2\sqrt{\bar{n}+1}} \end{cases}$$

$$\Rightarrow \omega(t) \cong \frac{1}{2(\bar{n}+1)} \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \left[e^{2i\lambda t(A+Bn)} + e^{-2i\lambda t(A-Bn)}\right]$$

let $e^{2i\lambda t(A+Bn)} \to e^{i(A'+B'n)}$

$$\Rightarrow \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n e^{i(A'+B'n)} = (1+\bar{n})\frac{e^{iA'}}{1-(e^{iB'}-1)\bar{n}} \quad \text{with} \quad \begin{cases} A' = 2\lambda t A \\ B' = 2\lambda t B \end{cases}$$

$$1-(e^{iB'}-1)\bar{n} \cong 1 - iB'\bar{n} + \frac{B'^2}{2}\bar{n} + \dots$$

$$\cong 1 - iB'\bar{n} - \frac{B'^2\bar{n}^2}{2} + \frac{B'^2(\bar{n}^2+\bar{n})}{2} + \dots$$

$$\cong e^{-iB'\bar{n}}\, e^{+B'^2(\bar{n}^2+\bar{n})/2}$$

$$\Rightarrow \frac{e^{iA'}}{1-(e^{iB'}-1)\bar{n}} \cong e^{i(A'+B'\bar{n})}\, e^{-B'^2(\bar{n}^2+\bar{n})/2}$$

$$\cong e^{2i\lambda t\sqrt{\bar{n}+1}}\, e^{-\frac{\lambda^2 t^2 \bar{n}}{2}}$$

$$\Rightarrow \omega(t) \cong \frac{1}{2(\bar{n}+1)}\left([\bar{n}+1]e^{-\frac{\lambda^2 t^2 \bar{n}}{2}}\right)\left(e^{2i\lambda t\sqrt{\bar{n}+1}} + e^{-2i\lambda t\sqrt{\bar{n}+1}}\right)$$

$$\cong \cos\left(2\lambda t\sqrt{\bar{n}+1}\right) e^{-\frac{\lambda^2 \bar{n}^2 t}{2}}$$

$$\boxed{\Rightarrow t_c^{-1} = \lambda\sqrt{\frac{\bar{n}}{2}}}$$

(For a thermal state $\Delta n \cong \bar{n}$, so following 4.128 – 4.129, we find $t_c\left(\Omega_n(\bar{n}+\bar{n}) - \Omega_n(\bar{n}-\bar{n})\right) \cong \lambda\sqrt{n}\, t_c \cong 1$

or $t_c \propto \lambda\sqrt{n}$, in agreement w/ the above!)

3.4.7 a) $H = \hbar(\hat{n} + \cancel{1/2})\omega + \hbar\lambda\hat{n}(\sigma_+ + \sigma_-)$ (ignore)

Clearly this only mixes: $|e\rangle_n \leftrightarrow |g\rangle_n$ (no change in n!)

So... $H_n = \hbar \begin{bmatrix} n\omega & \lambda n \\ \lambda n & n\omega \end{bmatrix}$

and... $\boxed{|\pm\rangle_n = \frac{1}{\sqrt{2}}(|e\rangle \pm |g\rangle)|n\rangle \quad E_{\pm n} = n(\omega \pm \lambda)}$

b) $|\psi(t=0)\rangle = \sum_n C_n |g\rangle_n$

$= \sum_n C_n \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$

$\Rightarrow |\psi(t)\rangle = \sum_n C_n e^{-in\omega t} \frac{1}{\sqrt{2}}\left(e^{-in\lambda t}|+\rangle_n - e^{+in\lambda t}|-\rangle_n\right)$

$= \sum_n C_n e^{-in\omega t}\left[-i\sin(n\lambda t)|e\rangle + \cos(n\lambda t)|g\rangle\right]|n\rangle$

$\Rightarrow P_e = \sum_n P_n \sin^2(n\lambda t)$

$\Rightarrow \omega(t) = 2P_e - 1 = \sum_n P_n [2\sin^2(n\lambda t) - 1] = -\sum_n P_n \cos(2n\lambda t)$

For a coherent state:

$P_n = e^{-\bar{n}}\frac{\bar{n}^n}{n!} \Rightarrow \omega(t) = -\sum_n e^{-\bar{n}}\frac{\bar{n}^n}{n} \cos(2n\lambda t)$

$\Rightarrow \omega(t) = -\frac{1}{2}e^{-\bar{n}}\left[\exp(\bar{n}e^{-2i\lambda t}) + \exp(\bar{n}e^{+2i\lambda t})\right]$

(this same can be done in Mathematica if you don't know how...)

$\boxed{\omega(t) = -e^{-2\bar{n}\sin^2(\lambda t)}\cos[\bar{n}\sin(2\lambda t)]}$

Clearly, this function has period $T = \frac{\pi}{\lambda}$, and so the revivals are perfect! This is because all the terms are harmonic.

c) $P_n = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$

$$\omega(t) = -\sum_n P_n \cos(2\lambda n t)$$

$$= -\frac{1}{2(\bar{n}+1) - \frac{1+2\bar{n}}{1+\underbrace{\bar{n}-\bar{n}\cos(2\lambda t)}_{2\bar{n}\sin^2(\lambda t)}}}$$ (from mathematica)

$$\boxed{\omega(t) = -\frac{1+2\bar{n}\sin^2(\lambda t)}{1+4\bar{n}(\bar{n}+1)\sin^2(\lambda t)}}$$

4. 4.8 a) $\frac{H}{\hbar} = \eta\,(a^2\sigma_+ + a^{+2}\sigma_-) + \frac{1}{2}\omega_0\sigma_3 + \hat{n}\omega$

This Hamiltonian mixes $|e\rangle|n\rangle \leftrightarrow |g\rangle|n+2\rangle$

$\Rightarrow |\psi_1\rangle_n \leftrightarrow |\psi_2\rangle_n$

$$\Rightarrow H_n = \hbar \begin{bmatrix} n\omega + \omega_0/2 & \eta\sqrt{(n+1)(n+2)} \\ \eta\sqrt{(n+1)(n+2)} & (n+2)\omega - \omega_0/2 \end{bmatrix}$$

The eigenstates are:

$$\frac{E_{\pm n}}{\hbar} = (n+1)\omega \pm \frac{\Omega_n(\Delta)}{2}$$

where $\Omega_n \equiv \sqrt{4(n+1)(n+2)\eta^2 + \Delta^2}$

$\Delta \equiv \omega_0 - 2\omega$

$$|\pm\rangle_n = \frac{1}{N_\pm}\left[\frac{\Delta \pm \Omega_n}{\Omega_n(0)}|\psi_1\rangle_n + |\psi_2\rangle_n\right]$$

This is identical to the Jaynes Cummings Hamiltonian, so...
(using 4.147 - 4.149)

$$|+\rangle_n = \cos\left(\Phi_n/2\right)|\psi_1\rangle_n + \sin\left(\Phi_n/2\right)|\psi_2\rangle_n$$

$$|-\rangle_n = -\sin\left(\Phi_n/2\right)|\psi_1\rangle_n + \cos\left(\Phi_n/2\right)|\psi_2\rangle_n$$

$$\Phi_n = \tan^{-1}\left(\frac{\Omega_n(0)}{\Delta}\right)$$

b) This is nearly identical to the first problem!

$$|\psi(t=0)\rangle = \sum_n C_n\left(\sin(\Phi_n/2)|+\rangle_n + \cos(\Phi_n/2)|-\rangle_n\right)$$

$$\Rightarrow \langle e|\psi(t)\rangle = \sum_n C_n |n\rangle e^{in\omega t} \sin(\Phi_n/2)\cos(\Phi_n/2) \times \underbrace{\left(e^{-i\Omega_n t/2} - e^{+i\Omega_n t/2}\right)}_{-2i\sin(\Omega_n t/2)}$$

$$\Rightarrow P_e(t) = \sum_n P_n \underbrace{\sin^2\left(\frac{\Phi_n}{2}\right)\cos^2\left(\frac{\Phi_n}{2}\right)}_{\frac{1}{4}\frac{\Omega_n^2-\Delta^2}{\Omega_n^2} \;=\; \frac{\eta^2(n+1)(n+2)}{\Omega_n^2}} 4\sin^2(\Omega_n t/2)$$

$$\omega(t) = 2P_e(t)-1 = \sum_n P_n\left(\frac{8\eta^2(n+1)(n+2)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)-1\right)$$

for a number state, $P_n = \delta_n$

for a coherent state, $P_n = e^{-\bar{n}}\frac{\bar{n}^n}{n!}$

for $|n\rangle$, the inversion is periodic in Ω_n.

from 4.137: $\underbrace{[\Omega(\bar{n}+1)-\Omega(\bar{n})]}_{\cong \frac{2\eta^2(3+2\bar{n})}{\Omega_{\bar{n}}}} t_R = 2\pi k$ (from expansion about $\bar{n}$)

for $\bar{n} \gg 1$, $\Omega_{\bar{n}} \to 2\bar{n}\eta \Rightarrow t_r \cong \left(\frac{\pi}{\eta}\right)\bar{n}k$

So the revivals are nearly periodic! See plot on next page.

c) $P_n = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$ ($\omega(t)$ above)

This obviously won't simplify. A plot is on the next page.

In[67]:= $W2 = \mathrm{Exp}[-nb]\,\frac{nb^n}{n!}\left(\frac{8\,\eta^2\,(n+1)\,(n+2)}{\Omega^2}\,\mathrm{Sin}[\Omega\,t\,/\,2]^2-1\right) /.\ \Omega \to \sqrt{\Delta^2+4\,\eta^2\,(n+1)\,(n+2)}\,;$

```
Plot[Sum[W2 /. {η → 1, Δ → 0, nb → 5}, {n, 0, 20}], {t, 0, 25}, PlotRange → {-1, 1}]
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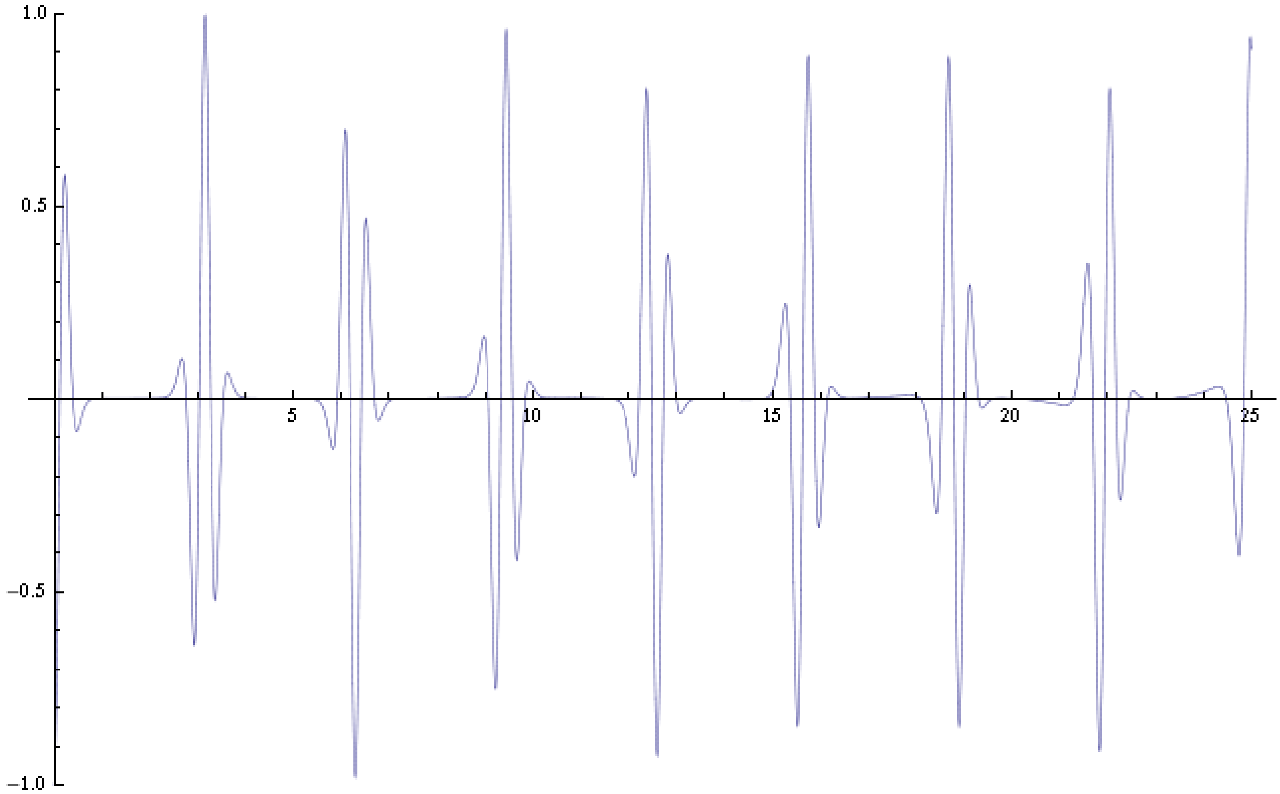
Out[68]=



In[71]:= $W3 = \frac{1}{nb+1}\left(\frac{nb}{nb+1}\right)^n\left(\frac{8\,\eta^2\,(n+1)\,(n+2)}{\Omega^2}\,\mathrm{Sin}[\Omega\,t\,/\,2]^2-1\right) /.\ \Omega \to \sqrt{\Delta^2+4\,\eta^2\,(n+1)\,(n+2)}\,;$

```
Plot[Sum[W3 /. {η → 1, Δ → 0, nb → 5}, {n, 0, 30}], {t, 0, 25}, PlotRange → {-1, 1}]
```

Out[72]=