Physics CS 140 **Set # 5** Winter 2014
(For T February 11, 5:00 PM)

Read Feynman "Quantum Mechanics and Path Integrals": Ch.3, Sects. 3-1 through 3-5 with special attention to 3-2 and 3-3.

Problem 1 (Feynman Problem 3-3). *Diffraction through a slit.*

By squaring the amplitude

$$\psi(x') = \int_{-b}^{b} \left(\frac{m}{2\pi i\hbar t'}\right)^{1/2} \exp\left\{\frac{im(x'-y)^2}{2\hbar t'}\right\} \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \exp\left\{\frac{im(X-y)^2}{2\hbar T}\right\} dy$$

and integrating over x, show that the probability of passage through the original sharp-edged slit is

$$P(going\ through) = \frac{m}{2\pi\hbar T} 2b$$

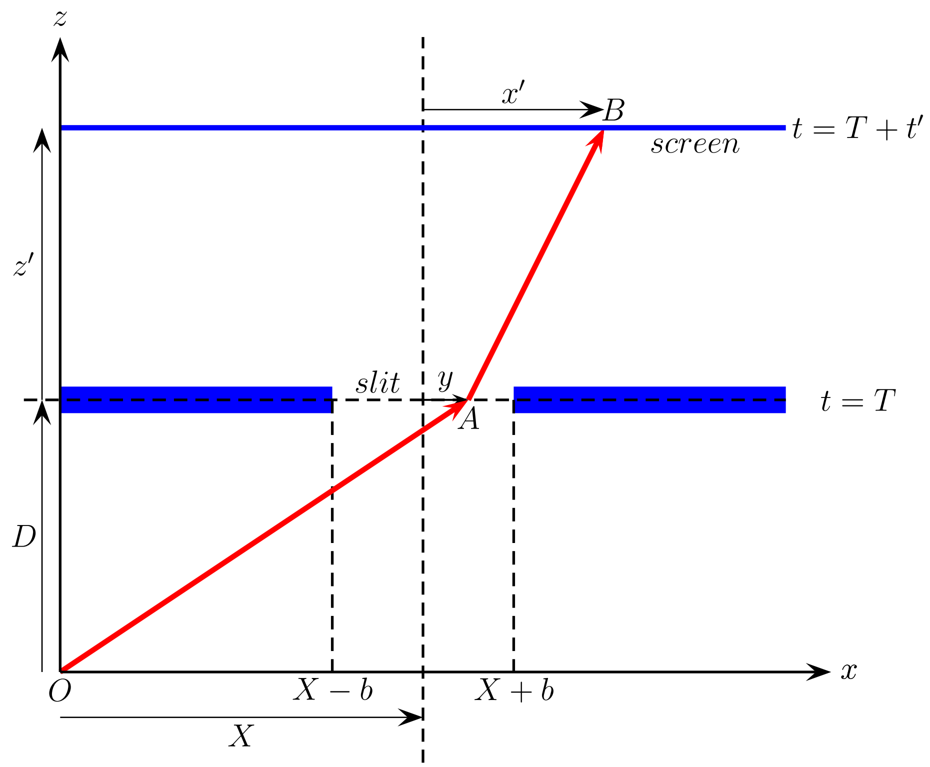
In the course of this problem the integral

$$\int_{-\infty}^{\infty} e^{iax} dx = 2\pi\delta(a)$$

will appear. this is the integral representation of the Dirac delta function of a.
Show that the probability per unit distance that the particle arrives at the point $X + y$ in the slit is

$$P(X+y)dy = \frac{m}{2\pi\hbar T} dy$$

Thus the quantum-mechanical results agree with the idea that the probability that a particle goes through a slit is equal to the probability that the particle arrives at the slit.



Problem 2 (Feynman Problem 3-12).

If the wave function for a harmonic oscillator is (at $t = 0$)

$$\psi(x,0) = \exp\left\{-\frac{m\omega}{2\hbar}(x-a)^2\right\}$$

then, using the propagator derived in class for the SHO , show that

$$\psi(x,t) = \exp\left\{-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}\left[x^2 - 2axe^{-i\omega T} + a^2\cos(\omega T)e^{-i\omega T}\right]\right\}$$

and find the probability distribution $|\psi|^2$.
How would you solve this problem using standard QM? Briefly describe.

Problem 3

Th kernel for an infinite square well of width L was obtained in set #4:

$$K_L(x_f,t_f;x_i,t_i) = \sum_{n=-\infty}^{\infty}\left[K(2nL + x_f,t_f;x_i,t_i) - K(2nL - x_f,t_f;x_i,t_i)\right]$$

That is,

$$K(x_f,t_f;x_i,t_i) = \sum_{n=-\infty}^{\infty}\sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\left\{\exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(2nL+x_f-x_i)^2}{t_f-t_i}\right]\right.$$
$$\left. - \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(2nL-x_f-x_i)^2}{t_f-t_i}\right]\right\}$$

Next use the Fourier integral representation for the free particle kernel derived in class

$$K(x_2,t;x_1,0) = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}e^{(i/\hbar)(x_2-x_1)p}\exp\left[-\frac{i}{\hbar}\frac{p^2}{2m}t\right]$$

and rewrite $K(2nL + x_f,t_f;x_i,t_i)$ and $K(2nL - x_f,t_f;x_i,t_i)$ in terms of their Fourier integral representation. Thus get after some trig reductions

$$K_L(x_f,t_f;x_i,t_i) = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\frac{p^2}{2m}(t_f-t_i)\right]$$
$$\times\, 2ie^{(-i/\hbar)x_ip}\sin\left[(p/\hbar)\,x_f\right]\sum_{n=-\infty}^{\infty}\exp\left[\frac{2inLp}{\hbar}\right]$$

At this stage use Poisson's formula

$$\sum_{n=-\infty}^{\infty}e^{2\pi ixn} = \sum_{r=-\infty}^{\infty}\delta\left(x-r\right)$$

to finally obtain after some more trig and algebra manipulations the propagator

$$K_L(x_f, t_f; x_i, t_i) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-iE_n\left(t_f - t_i\right)/\hbar} \sin k_n x_i \sin k_n x_f$$

where $k_n = n\pi/L$ and $E_n = k_n^2\hbar^2/2m$ are the wave numbers and energy levels respectively for a particle of mass m inside an infinite square well of width L.
Now you have recovered the familiar results for the infinite square well in terms of energy eigenfunctions and eigenvalues.

Problem 4

In class we obtained the following expression for the kernel of a quadratic Lagrangian

$$K_L(x_2, t_2; x_1, t_1) = A(t_2, t_1)e^{(i/\hbar)S_{cl}(2|1)}$$

where the amplitude $A(t_2, t_1)$ is obtained by direct evaluation of the path integral for the fluctuations $y(t)$ from the classical path $\bar{x}(t)$

$$A(t_2, t_1) = \int_0^0 \delta\,[y(t)]e^{(i/\hbar)S[y(t)]}$$

that is,

$$A(t_2, t_1) = \lim_{\varepsilon\to 0}\frac{1}{A}\int_{-\infty}^{\infty}\frac{dy_1}{A}\cdot\cdots\cdot\int_{-\infty}^{\infty}\frac{dy_{N-1}}{A}\exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}S_{cl}(n+1, n)\right]$$

with $\quad \varepsilon = (t_2 - t_1)/N \quad$ and $\quad A = \sqrt{2\pi i\hbar\varepsilon/m}$

For a free particle of mass m

$$S_{cl}(n+1, n) = \frac{m}{2}\frac{\left(y_{n+1} - y_n\right)^2}{\varepsilon}$$

Use *mathematical induction* and show that the result of doing the first n-integrations in the expression above for $A(t_1, t_2)$ is

$$\sqrt{\frac{m}{2\pi i\hbar(n+1)\varepsilon}}\exp\left[\frac{i}{2\hbar}\frac{m}{(n+1)\varepsilon}y_{n+1}^2\right]$$

Note: *Assume the expression above is true then show that it is true for* $n+1$ *and finally check that it holds for* $n = 1$