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Boundary Terms in the Action for the Regge Calculus[1]

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Abstract

The boundary terms in the action for Regge's formulation of general relativity on a simplicial net are derived and compared with the boundary terms in continuum general relativity.

The Regge calculus [1] may play a useful role in the exploration of the sum over histories program for quantizing space-time. (For reviews of this program see [2-3].) The Regge calculus is a natural lattice formulation of general relativity. It can be used to supply manifestly coordinate-invariant approximations to the functional integrals which define the transition amplitudes for continuum quantum gravity and it is naturally suited to investigating global questions concerning these amplitudes. (See, e.g., [4].)

In continuum quantum gravity the typical functional integrals of interest are of the schematic form

$$\int \delta g \exp (iS[g]) \tag{1}$$

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where S is the action for general relativity and the sum is over all physically distinct four-geometries in a particular space-time region which match a given three-geometry on the boundary of that region. In the Regge calculus four-dimensional space-time is triangulated into a net of flat four-simplices. Curvature is concentrated at the two-simplices or bones of this net. To any functional integral of the form in equation (1) in the continuum theory there will therefore correspond an integral in the Regge calculus for a given simplicial decomposition of the space-time region of interest. The integral will be over the lengths of the edges in the interior of the net keeping fixed the lengths of the edges which define the boundary. The action will be a function of the edge lengths. Two things are therefore needed to define such an integral: a measure on the space of path edge lengths and an expression for the action in terms of these edge lengths. It is this expression for the action which concerns us in this note.

Regge [1] has already given an action in the case of a simplicial net where the boundaries are ignored. Variation of this action with respect to the edge lengths gives the analog of Einstein's equations for the simplicial net so that any other action can differ from Regge's only by boundary terms. In the quantum theory, however, these boundary terms are important. They are essential in order that the quantum mechanical amplitudes satisfy the correct composition law and in order that these amplitudes have the correct classical limit. In a number of examples [5] they give a contribution to the partition function which is important for the agreement with calculations based on straightforward thermodynamics. In this note we shall derive the boundary terms in the action for the Regge calculus.[2]

There are two requirements the action must satisfy in a path integral formulation of the quantum theory [2, 5]: (1) The action must be additive on space-time regions. If there are two contiguous regions with four-geometries g and g', then one must have

$$S[g+g'] = S[g] + S[g'] \tag{2}$$

where $g + g'$ is the four-geometry obtained by joining together the two regions. This condition is necessary in order that the usual law for the composition of probability amplitudes hold. (2) The classical equations of motion must be equivalent to requiring that the action be stationary under variations which fix the three-geometry on the boundary. This condition is necessary for the quantum mechanical theory to have the correct classical limit. In the continuum

[2] The boundary terms we shall derive are suited to boundary conditions which fix the edge lengths themselves on the boundary surface. This corresponds to fixing the three-geometry of the boundary in the continuum theory. A different choice of boundary conditions (e.g., fixing the conformal three-geometry and the trace of the second fundamental form) would require different boundary terms.

theory both of these requirements are satisfied if the action is taken to be [6, 5]

$$S[g] = \frac{1}{16\pi} \int_M R(-g)^{1/2}\, d^4x + \frac{1}{8\pi} \int_{\partial M} K(\pm h)^{1/2}\, d^3x + C \tag{3}$$

where K is the trace of the second fundamental form of the boundary, h_{ij} is the induced metric on the boundary, and C is a term which depends only on the boundary metric. The first integral in equation (3) is over the space-time volume of interest, the second is over its boundary, and the plus-or-minus sign is chosen according to whether the boundary is spacelike or timelike. (Our conventions are the same as those of Reference [7] with units where $G = 1$.)

We shall now consider these two requirements in the case of the Regge calculus and derive the form of the surface term in that case. For simplicity we shall consider only space-times with positive definite metric to avoid introducing the extra notation needed to formulate our results for space-times with indefinite metric (see [8], Appendices A and B). To begin we introduce some notation for which we follow References [8] and [9]. The collection of simplices which triangulate the space-time region we denote by Σ. The collection of four simplices we denote by Σ_4 and a particular four-simplex by σ. The collection of two-simplices (bones) of Σ we denote by Σ_2. These may be divided into bones which lie interior to Σ, which we denote by int Σ_2 and bones contained in the boundary which we denote by $\partial\Sigma_2$. A particular bone will be denoted by b. If a bone b is contained in a four-simplex σ we shall write $b \subset \sigma$ or $\sigma \supset b$.

The Regge action function is

$$\Lambda[\Sigma] = (8\pi)^{-1} \sum_{b \in \operatorname{int}\Sigma_2} A(b)\, \theta(b) \tag{4}$$

where $A(b)$ is the area of the bone b and $\theta(b)$ is the bone's defect angle defined by

$$\theta(b) = 2\pi - \sum_{\sigma \supset b} \theta(\sigma; b) \tag{5}$$

Here, $\theta(\sigma; b)$ is the angle between the two three-simplices of σ (π minus the angle between their inward normals) which intersect in b. The sum in equation (5) is over all σ which intersect in a given bone b. (See Figure 1 for a two-dimensional analog.) If the function Λ is varied with respect to an *interior* squared edge length $l^2{}_{ij}$ one will have first

$$8\pi\delta\Lambda = \sum_{b \in \operatorname{int}\Sigma_2} [\delta A(b)\, \theta(b) + A(b)\, \delta\theta(b)] \tag{6}$$

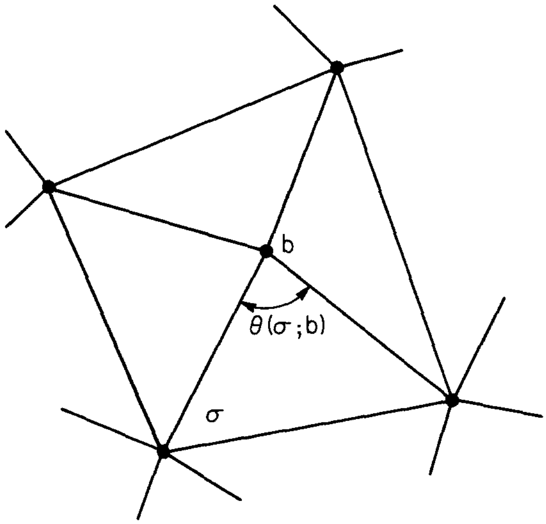

Fig. 1. A portion of a two-dimensional simplicial net. The net consists of flat two-simplices (triangles) and their edges and vertices. The bones of the net are the vertices. There the curvature is concentrated. A measure of the local curvature is given by the defect angle $\theta(b)$ defined by the difference between 2π and the sum of the interior angles $\theta(\sigma; b)$ of all the triangles σ which meet in b. A typical bone and angle $\theta(\sigma; b)$ are shown.

The last term can be rewritten using equation (5) as

$$\sum_{b \in \mathrm{int}\Sigma_2} A(b)\,\delta\theta(b) = -\sum_{b \in \mathrm{int}\Sigma_2} A(b) \sum_{\sigma \supset b} \delta\theta(\sigma; b)$$
$$= -\sum_{\sigma \in \Sigma_4} \sum_{b \subset \sigma,\, b \in \mathrm{int}\Sigma_2} A(b)\,\delta\theta(\sigma; b) \tag{7}$$

The last summation, arising from interchanging the two sums in the first line of equation (7), is over all interior bones contained in a given four-simplex σ.

Regge [1] proved the following identity (see [8] for a demonstration in the case of space-times with indefinite metric)

$$\sum_{b \subset \sigma} \delta\theta(\sigma; b)\, A(b) = 0 \tag{8}$$

There is one identity for each $\sigma \in \Sigma_4$ with the sum being over all bones lying in a given σ. If Σ has no boundary equation (8) would show that the last term in equation (6) vanished identically and the result of setting $\delta\Lambda = 0$ would be

$$\sum_{b \in \mathrm{int}\Sigma_2} \theta(b) \frac{\partial A}{\partial l^2{}_{ij}} = 0 \tag{9}$$

These are the equations of motion ("thatch equations") of the Regge calculus—the analogs of Einstein's equations. If, however, there is a boundary on which the edge lengths are held fixed, then the cancellation of the second term in equation (6) is incomplete and Λ will not be an extremum at a solution of the thatch equations, equation (9). In fact, assuming equation (9) for interior edge lengths $l^2{}_{ij}$ we have from equations (6), (7), (8) and the constraint that the edge lengths in the boundary not be varied

$$\begin{aligned} 8\pi\delta\Lambda &= \sum_{\sigma\in\Sigma_4}\ \sum_{b\subset\sigma,\, b\in\partial\Sigma_2} A(b)\,\delta\theta(\sigma;b) \\ &= \delta\left[\sum_{b\in\partial\Sigma_2} A(b) \sum_{\sigma\supset b} \theta(\sigma;b)\right] \end{aligned} \tag{10}$$

The angle $\Sigma_\sigma\theta(\sigma;b)$ occurring in equation (10) has a simple geometrical interpretation. It is just $\pi - \psi(b)$, where $\psi(b)$ is the angle between the normals of the two boundary three-simplices which intersect at b. Figure 2 gives a pictorial representation in two dimensions. Thus, for variations which keep fixed the boundary edge lengths and therefore the areas of the boundary two-simplices

$$8\pi\delta\Lambda = -\delta\left[\sum_{b\in\partial\Sigma_2} A(b)\,\psi(b)\right] \tag{11}$$

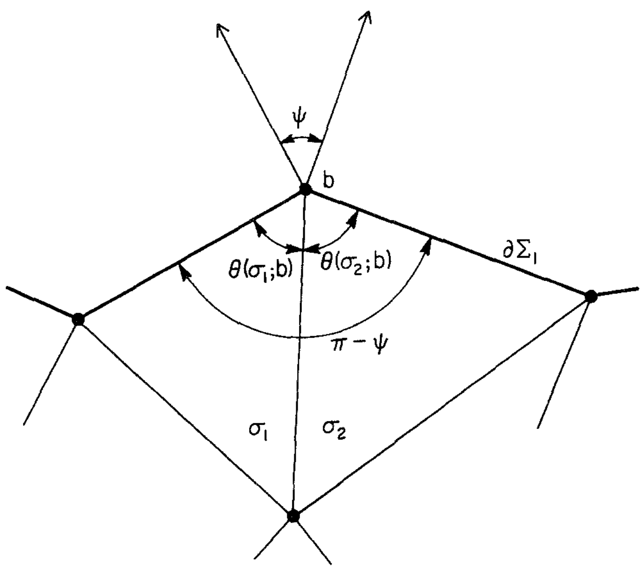

Fig. 2. A portion of the boundary of a two-dimensional simplicial net. The heavy lines are the edges of the net which make up the boundary, $\partial\Sigma_1$. At a typical boundary bone b the angle ψ is the angle between the normal to the boundary edges which meet at b. The angle ψ is equally the difference between π and the sum of the interior angles of the triangles of the net which meet in b.

Equation (11) shows that the function Λ fails to be an extremum at a solution of the thatch equations [equation (9)] only by the variation of a boundary term. We can therefore immediately write down an action which *is* an extremum. It is

$$S[\Sigma] = (8\pi)^{-1}\left[\sum_{b\in \mathrm{int}\,\Sigma_2} A(b)\,\theta(b) + \sum_{b\in\partial\Sigma_2} A(b)\,\psi(b) + C(\partial\Sigma)\right] \tag{12}$$

The first sum is over the bones interior to the net; the second is over the bones in the boundary. The function C depends only on the edge lengths in the boundary but is otherwise arbitrary.

Of the two requirements on the action mentioned earlier as necessary for sum over histories quantization, the second is met by equation (12): Variation of S will yield equation (9) when the edge lengths of the three simplices on the boundary are held fixed. The first requirement that the action be additive can also be met by equation (12).

Let Σ' and Σ'' be two contiguous simplicial nets with $\Sigma = \Sigma' \cup \Sigma''$ the combined net. We denote by Σ_2', Σ_2'', Σ_2 the bones of Σ', Σ'', Σ, respectively and divide Σ_2 into four types as portrayed in Figure 3:

Type I: bones in int Σ' or int Σ''
Type II: bones in $\partial\Sigma'$ but not Σ'' or vice versa
Type III: bones in $\partial\Sigma' \cap \partial\Sigma'' \cap \mathrm{int}\,\Sigma$
Type IV: bones in $\partial\Sigma' \cap \partial\Sigma'' \cap \partial\Sigma$

It helps also to reexpress the action for a general net by making use of (5) and the analogous formula for $\psi(b)$ when $b \in \partial\Sigma$,

$$\psi(b) = \pi - \sum_{\sigma\supset b} \theta(\sigma; b) \tag{13}$$

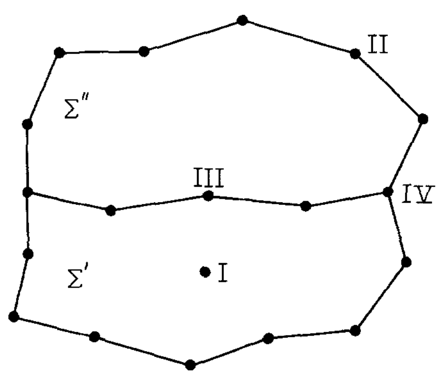

Fig. 3. A schematic two-dimensional representation of the classification of the bones of a net Σ which is the union of two nets Σ' and Σ''.

Substituting (5) and (13) into (12) yields

$$8\pi S[\Sigma] = -\sum_{\sigma\in\Sigma_4,\, b\in\Sigma_2,\, b\subset\sigma} A(b)\,\theta(\sigma;b) + \sum_{b\in\operatorname{int}\Sigma_2} 2\pi A(b) + \sum_{b\in\partial\Sigma_2} \pi A(b) + C[\partial\Sigma] \tag{14}$$

With these preliminaries, consider the difference $D = S[\Sigma'] + S[\Sigma''] - S[\Sigma]$ with $S[\Sigma']$, $S[\Sigma'']$, and $S[\Sigma]$ all expressed in the form of equation (14). Since $\Sigma_4 = \Sigma_4' \cup \Sigma_4''$ the terms $A(b)\,\theta(\sigma;b)$ cancel entirely; and clearly the terms $2\pi A(b)$ for b of type I and the terms $A(b)$ for b of type II cancel as well. For b of type III, the contributions $\pi A(b)$ from $S[\Sigma']$ and $S[\Sigma'']$ together cancel the $2\pi A(b)$ from $S[\Sigma]$. Finally bones b of type IV contribute to D

$$\pi A(b) + \pi A(b) - \pi A(b) = \pi A(b) \tag{15}$$

so that we get in all

$$S[\Sigma'] + S[\Sigma''] - S[\Sigma] = \pi \sum_{b\in\Delta_2} A(b) + C[\partial\Sigma'] + C[\partial\Sigma''] - C[\partial\Sigma] \tag{16}$$

where $\Delta_2 = \partial\Sigma_2' \cap \partial\Sigma_2'' \cap \partial\Sigma_2$. Thus the requirement of additivity reduces to a condition on the purely surface terms:

$$C[\partial\Sigma] = C[\partial\Sigma'] + C[\partial\Sigma''] + \pi \sum_{b\in\Delta_2} A(b) \tag{17}$$

When Δ_2 is empty, as might happen for example in a spatially closed space-time, the choice $C \equiv 0$ fulfills (17) but in general the additivity requirement implies nonzero values for the purely surface terms. Since the continuum theory is a limit of the simplicial theory, the same remark should apply there as well.

If all boundaries are embeddable in a flat four-dimensional space, *and* if Δ_2 is empty, one choice for $C(\partial\Sigma)$ which will satisfy equation (17) is

$$C(\partial\Sigma) = c \sum_{b\in\partial\Sigma} A(b)\,\psi_0(b) \tag{18}$$

where ψ_0 is the angle defined by equation (13) when the boundary is embedded in flat space and c is an arbitrary constant.

Having established the form of a suitable action for the Regge calculus it is of interest to investigate its correspondence with that of the continuum theory. Any simplicial net may be regarded as a continuum geometry with the curvature concentrated on bones. Previous calculations [8] have already shown that for such a geometry

$$\frac{1}{2}\int_\Sigma d^4x\,(g)^{1/2} R = \sum_{b\in\Sigma_2} A(b)\,\theta(b) \tag{19}$$

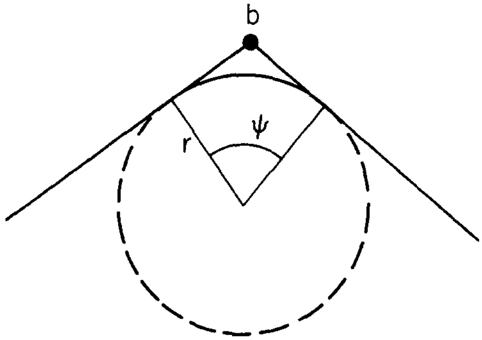

Fig. 4. To calculate the integral of $K(h)^{1/2}$ in equation (20) we may smooth out the surface with the extrinsic curvature concentrated on the bones into one in which the three simplices are interpolated by cylindrical surfaces of radius r and for which the extrinsic curvature therefore varies smoothly. A section transverse to the bone is shown here. The integral of $K(h)^{1/2}$ is then easily calculated and the limit $r \to 0$ taken to recover the corresponding expression for the simplicial net.

We shall now show that

$$\int_{\partial\Sigma} d^3x\,(h)^{1/2} K = \sum_{b \in \partial\Sigma_2} A(b)\,\psi(b) \tag{20}$$

To see this imagine smoothing out the curvature in a continuous way, computing the left-hand side of equation (20) and taking the limit as the curvature becomes concentrated on the bone. A simple way to do this is to approximate the boundary hypersurface at the bone by a portion of a hypercylinder curved in the direction normal to the bone and flat in directions parallel to it. (See Figure 4.) The trace of the extrinsic curvature of such a cylinder is $K = 1/r$, where r is its radius of curvature. The length in the normal direction of the cylinder which interpolates between the two flat three-simplices is $r\psi$. Thus equation (20) is satisfied, the integration in the normal direction giving a factor of ψ and that in the parallel directions giving the area of the bone.

Equations (19) and (20) show that the Regge calculus action in equation (12) is exactly the continuum action of equation (3) in the case when all the curvature is concentrated on the bones of a simplicial net and that further the choice in equation (18) for the purely surface terms in the action is (for an appropriate c) just that suggested [5, 6] for the corresponding continuum case.

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