Measures for Classicality

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This is a modestly edited section of the authors' paper Strong Decoherence [1] concerned with the general problem of a measure for classicality and a specific proposal for that measure. This section is largely self-contained and can be read separately from the rest. Neither the text or the references have been updated except for a few "to be published" references.

Quantum mechanics, along with the correct theory of the elementary particles (represented by the Hamiltonian H) and the correct initial condition in the universe (represented by the state vector $|\Psi\rangle$), presumably exhibits a great many essentially different strongly decohering realms, but only some of those are quasiclassical. For the quasiclassical realms to be viewed as an emergent feature of H, $|\Psi\rangle$, and quantum mechanics, a good technical definition of classicality is required. (One can then go on to investigate whether the theory exhibits many essentially inequivalent quasiclassical realms or whether the usual one is nearly unique.)

In earlier papers, [2–4] we have made a number of suggestions about the definition of classicality and it is appropriate to continue that discussion here. It is clear that from those earlier discussions that classicality must be related in some way to a kind of entropy for alternative coarse-grained histories. We must therefore begin with an abstract characterization of entropy and then investigate the application to histories. An entropy S is always associated with a coarse graining, since a perfectly fine-grained version of entropy in statistical mechanics would be conserved instead of tending to increase with time. Classically, if all fine-grained alternatives are designated by $\{r\}$, with probabilities p_r summing to one, that fine-grained version of entropy would be

$$S_{f-g} = -\sum_r p_r \log\, p_r \ , \tag{1}$$

where log means $\log_2$ and where, for convenience, we have put Boltzmann's constant k times $\log_e 2$ equal to unity. A true, coarse-grained entropy has the form

$$S \equiv -\sum_r \tilde{p}_r \log \tilde{p}_r \ , \tag{2}$$

where the probabilities $\tilde{p}_r$ are coarse-grained averages of the $\{p_r\}$. A coarse graining $p_r \to \tilde{p}_r$ must have certain properties (see [5] for more details):

$$1) \qquad \text{the } \{\tilde{p}_r\} \text{ are probabilities} \ , \tag{3a}$$

$$2) \qquad \tilde{\tilde{p}}_r = \tilde{p}_r \ , \tag{3b}$$

$$3) \qquad -\sum_r p_r \log\, \tilde{p}_r = -\sum_r \tilde{p}_r \log\, \tilde{p}_r \ . \tag{3c}$$

These properties are not surprising for an averaging procedure. The significance of the last one is easily seen if we make use of the well known fact that for any two sets of probabilities $\{p_r\}$ and $\{p'_r\}$ we have

$$-\sum_r p_r \log\, p_r \leq -\sum_r p_r \log\, p'_r \ . \tag{4}$$

Putting $p'_r = \tilde{p}_r$ for each r and using (1)-(4), we obtain

$$S_{f-g} = -\sum_r p_r \log\, p_r \leq -\sum_r p_r \log \tilde{p}_r = -\sum_r \tilde{p}_r \log \tilde{p}_r = S\ , \tag{5}$$

so that S_{f-g} provides a lower bound for the entropy S. If the initial condition and the coarse graining are related in such away that S is initially near its lower bound, then it will tend to increase for a period of time. That is the way the second law of thermodynamics comes to hold.

In order to know what nearness to the lower bound means, we should examine the upper bound on S. That upper bound is achieved when all fine-grained alternatives have equal coarse-grained probabilities $\tilde{p}_r$, corresponding in statistical mechanics to something like "equilibrium" or infinite temperature. Each $\tilde{p}_r$ is then equal to N^{-1}, where N (assumed finite) is the number of fine-grained alternatives, and the maximum entropy is thus

$$S_{\max} = \log\, N\ . \tag{6}$$

The simplest example of coarse graining utilizes a grouping of the fine-grained alternatives $\{r\}$ into exhaustive and mutually exclusive classes $\{\alpha\}$, where a class α contains N_α elements and has lumped probability

$$p_\alpha \equiv \sum_{r\in\alpha} p_r\ . \tag{7}$$

Of course we have

$$\sum_\alpha N_\alpha = N, \quad \sum_\alpha p_\alpha = 1\ . \tag{8}$$

The coarse-grained probabilities $\tilde{p}_r$ in this example are the class averages

$$\tilde{p}_r = p_\alpha / N_\alpha\ , \quad r \in \alpha\ , \tag{9}$$

and they clearly have the properties (3). The entropy comes out

$$S = -\sum_\alpha p_\alpha \log\, p_\alpha + \sum_\alpha p_\alpha \log N_\alpha\ , \tag{10}$$

where the second term contains the familiar logarithm of the number of fine-grained alternatives (or microstates) in a coarse-grained alternative (or macrostate), averaged over all the coarse-grained alternatives.

Besides entropy, it is useful to introduce the concept of algorithmic information content (AIC) as defined some thirty years ago by Kolmogorov, Chaitin, and Solomonoff (all working independently).[1] For a string of bits s and a particular universal computer U, the AIC of the string, written $K_U(s)$, is the length of the shortest program that will cause U to print out the string and then halt. The string can be used as the description of some entity e, down to a given level of detail, in a given language, assuming a given amount of knowledge and understanding of the world, encoded in a given way into bits [7]. The AIC of the string can then be regarded as $K_U(e)$, the AIC of the entity so described.

We now discuss a way of approaching classicality that utilizes AIC as well as entropy. Some authors have tried to identify AIC in a straightforward way with complexity, and in

[1] For a discussion of the original papers see [6].

fact AIC is often called algorithmic complexity. However, AIC is greatest for a "random" string of bits with no regularity and that hardly corresponds to what is usually meant by complexity in ordinary parlance or in scientific discourse. To illustrate the connections among AIC, entropy or information, and an effective notion of complexity, take the ensemble $\widetilde{E}$ consisting of a set of fine-grained alternatives $\{r\}$ together with their coarse-grained probabilities $\widetilde{p}_r$. We can then consider both $K_U(\widetilde{E})$, the AIC of the ensemble, and $K_U(r|\widetilde{E})$, which is the AIC of a particular alternative r given the ensemble. For the latter we have the well known inequality (see, for example [8]):

$$\sum_r \widetilde{p}_r K_U(r|\widetilde{E}) \geq -\sum_r \widetilde{p}_r \log \widetilde{p}_r = S \ . \tag{11}$$

Moreover, it has been shown by R. Schack [9] that, for any U, a slight modification $U \to U'$ permits $K_{U'}(r|\widetilde{E})$ to be bounded on both sides as follows:

$$S + 1 \geq \sum_r \widetilde{p}_r K_{U'}(r|\widetilde{E}) \geq S, \tag{12}$$

so that we have

$$\sum_r \widetilde{p}_r K_{U'}(r|\widetilde{E}) \approx S. \tag{13}$$

(Previous upper bounds had $\mathcal{O}(1)$ in place of 1, but there was nothing to prevent $\mathcal{O}(1)$ from being millions or trillions of bits!)

Looking at the entropy S as a close approximation to $\sum_r \widetilde{p}_r K_{U'}(r|\widetilde{E})$, we see that it is natural to complete it by adding to it the quantity $K_{U'}(\widetilde{E})$ — the AIC of the *ensemble* with respect to the same universal computer U'. This last quantity can be connected with the idea of effective complexity — the length of the most concise description of the perceived regularities of an entity e. Any particular set of regularities can be expressed by describing e as a member of an ensemble $\widetilde{E}$ of possible entities sharing those regularities. Then $K_{U'}(\widetilde{E})$ may be identified with the effective complexity of e or of the ensemble $\widetilde{E}$ [7**?**]. Adding this effective complexity to S, we have:

$$\Sigma \equiv K_{U'}(\widetilde{E}) + S \ . \tag{14}$$

This sum of the the effective complexity and the entropy (or Shannon information) may be labeled either "augmented entropy" or "total information". If the coarse graining is the simple one obtained by partitioning the set of fine-grained alternatives $\{r\}$ into classes $\{\alpha\}$ with cardinal numbers N_α, then the total information becomes

$$\Sigma = K_{U'}(\widetilde{E}) - \sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha \log N_\alpha \ . \tag{15}$$

In (14), the first term becomes smaller as the set of perceived regularities becomes simpler, while the second term becomes smaller as the spread of possible entities sharing those regularities is reduced. Minimizing Σ corresponds to optimizing the choice of regularities and the resulting effective complexity thereby becomes less subjective. Thus, the total information or augmented entropy is useful in a wide variety of contexts [5, 7]. We apply it here to sets of alternative decohering coarse-grained histories in quantum mechanics.

The general idea of augmenting entropy with a term referring to algorithmic information content was proposed in a different context by Zurek [10]. However, as far as we know, the emphasis on the utility of the quantity Σ in (14) and (15) is new. We discussed the general idea of an entropy for histories in [2]. Earlier, Lloyd and Pagels [11] introduced a quantity they called thermodynamic depth, applicable to alternative coarse-grained classical histories α. They defined it as

$$D = \sum_\alpha p_\alpha \log(p_\alpha/q_\alpha) \ , \tag{16}$$

where q_α is an "equilibrium probability", which in our notation would be N_α/N for the simple coarse graining we have discussed. We clearly have

$$D = \log N + \sum_\alpha p_\alpha \log p_\alpha - \sum_\alpha p_\alpha \log N_\alpha \tag{17}$$

or

$$D = S_{\max} - S \tag{18}$$

for the set of alternative coarse-grained histories. We see that thermodynamic depth is intimately related to the notion of an entropy for histories.

In applying augmented entropy to sets of coarse-grained histories in quantum mechanics, one must take into account that there are infinitely many different sets of fine-grained histories and that these sets do not generally have probabilities because they fail to decohere. The quantities N_α may therefore conceivably depart from their obvious definition as the numbers of fine-grained histories in the coarse-grained classes $\{\alpha\}$. In fact, there may be some latitude in the precise definition of the complexity and entropy terms in the total information (14). For example, one could consider instead of $\widetilde{E}$ an ensemble $\hat{E}$ consisting of the coarse-grained histories $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, their probabilities p_α, and the numbers N_α. A more general definition of the entropy S of histories may help to define these numbers. The generalized Jaynes construction for coarse-grained histories provides one framework for doing this [2]. In the most general situation, such a construction defines the entropy S as the maximum of $-Tr(\tilde{\rho} \log \tilde{\rho})$ over all density matrices $\tilde{\rho}$ that preserve the decoherence and probabilities of a given ensemble E of coarse-grained histories. Other Jaynes-like constructions may also be useful, for example ones that define entropy by proceeding step by step through the histories. We are investigating these various possibilities.

In any case, our augmented entropy in (15) for coarse-grained decohering histories in quantum mechanics is a negative measure of classicality: the smaller the quantity, the closer the set of alternative histories is to a quasiclassical realm. Reducing the first term in (15) favors making the description of the sequences of projections simple in terms of the field variables of the theory and the Hamiltonian H. It favors sets of projections at different times that are related to one another by time translations, as are many sequences of projections on quasiclassical alternatives at different times in the usual quasiclassical realm.

Reducing the second term favors more nearly deterministic situations in which the spread of probabilities is small. Approximate determinism is, of course, a property of a quasiclassical realm. Reducing the last term corresponds roughly to approaching "maximality", allowing the finest graining that still permits decoherence and nearly classical behavior. A quasiclassical realm must be maximal in order for it to be a feature exhibited by the initial condition and Hamiltonian and not a matter of choice by an observer.

Any proposed measure of closeness to a quasiclassical realm must be tested by searching for pathological cases of alternative decohering histories that make the quantity small without resembling quasiclassical realm of everyday experience. The worst pathology occurs for a set of histories in which the P's at every time are projections on $|\Psi\rangle$ and on states orthogonal to $|\Psi\rangle$. We see that in this pathological case the description of the histories and their probabilities is simple because the description of the initial state is simple, so that $K_{U'}(\widetilde{E})$ is small. The term $-\sum_\alpha p_\alpha \log p_\alpha$ is zero and the third term is also zero since the only α with $p_\alpha \neq 0$ corresponds to projecting onto the pure state $|\Psi\rangle$, so that N_α is one and $\log N_\alpha$ vanishes.

Evidently the smallness of Σ is not by itself a sufficient criterion for characterizing a quasiclassical realm. Further criteria can be introduced if we require that quasiclassical realms be strongly decohering with suitable restrictions on the sets $\{M\}_\alpha$ of operators from which the future histories are constructed. Requiring strong decoherence ensures a physical mechanism of decoherence and guarantees the permanence of the past. The sets $\{M\}_\alpha$ must be restricted so as to rule out pathologies such as discussed above. Presumably they must all belong to a huge set with certain straightforward properties. Those properties might be connected with locality, since quantum field theory is perfectly local. (Even superstring theory is local — although the string is an extended object, interaction among strings is always local in spacetime.) It would be in this way that strong decoherence enters a definition of classicality.

A quasiclassical realm would then be characterized in quantum mechanics as a realm that minimizes the augmented entropy given by (15) subject to further suitable conditions. Quasiclassical realms so defined would be an emergent feature of H, $|\Psi\rangle$, and quantum mechanics — a feature of the universe independent of human choice. In principle, given H and $|\Psi\rangle$, we could compute the quasiclassical realm that these theories exhibit. We could then investigate the important question of whether the usual quasiclassical realm is essentially unique or whether the quantum mechanics of the universe exhibits essentially inequivalent quasiclassical realms. Either conclusion would be of central importance for understanding quantum mechanics.

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