69

The Conformal Rotation in Linearised Gravity

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'Cheshire Puss,' [*said Alice*]... *'would you tell me, please, which way I ought to go from here?' 'That depends a good deal on where you want to get to,' said the cat.*
Lewis Carroll, *Alice in Wonderland*

1 INTRODUCTION

Functional integrals have proved to be powerful tools for the investigation of quantum field theory. Functional integrals over Minkowski space field configurations of the form

$$\int \delta\varphi(x) \exp(\mathrm{i}S[\varphi(x)]) \tag{1.1}$$

express concretely the sum over histories formulation of quantum mechanics for field theory. Such integrals provide a direct route from classical action $S[\varphi]$ to quantum amplitude in a way which is easily accessible to formal manipulation. Functional integrals of the form

$$\int \delta\varphi(x) \exp(-I[\varphi(x)]) \tag{1.2}$$

where $I[\varphi]$ is a Euclidean action and $\varphi(x)$ a Euclidean field configuration, express ground state wavefunctions or generating functions in a way which can be made tractable for practical computation. The work of Professor Fradkin, whose sixtieth birthday we celebrate with this volume, provides striking evidence for the power, richness and subtlety of functional methods when applied to field theory.

Functional methods are particularly useful in the development of theories with invariances, such as gauge theories or parametrised theories, because they allow these invariances to be displayed explicitly. One expects these methods to be especially useful in the search for a quantum theory of gravity, which has invariances of both types. Indeed, Euclidean functional integrals for amplitudes have been proposed as the fundamental starting point of a quantum gravitational theory, an idea which has many novel consequences (see, for example, Hawking 1979, 1984). A natural action for such a theory is the Euclidean version of that for Einstein's general relativity,

$$l^2 I[g] = -2\int_{\partial M} d^3x h^{1/2} K - \int_M d^4x g^{1/2} R \tag{1.3}$$

where we use units in which $\hbar = c = 1$ and $l = (16\pi G)^{1/2}$ is the Planck length. This programme immediately encounters a difficulty. The Euclidean Einstein action is not positive definite and integrals over it of the form (1.2) will diverge (Gibbons *et al* 1978). As Gibbons *et al* showed, the Euclidean functional integrals can be made convergent by an additional formal manipulation as follows: write the metric g, which is the integration variable in a gravitational functional integral, as

$$g = \Omega^2 \tilde{g} \tag{1.4}$$

where $\tilde{g}$ is a representative metric in the conformal equivalence class of g, fixed, say, by the condition $R(\tilde{g}) = 0$. The integration over metrics g can be written as an integration over metrics $\tilde{g}$ which satisfy this condition and an integration over conformal factors Ω. If the contour of the Ω integration is distorted to complex values, the action can be made positive definite and the Euclidean functional integrals convergent. This is called a conformal rotation.

There is no direct analogue of the conformal rotation in most familiar gauge theories such as electrodynamics. The actions of these theories are typically positive when expressed in terms of the natural Euclidean variables. A conformal rotation is, however, needed to construct the Euclidean functional integrals of linearised gravity very much as it is in the full theory of general relativity (Gibbons and Perry 1978, Hartle 1984). In view of this lack of analogy between Einstein gravitational theories and familiar gauge theories, it would be helpful to have a more physically based motivation for the Euclidean gravitational integrals in their conformally rotated form. In this article we shall provide such motivation for linearised gravity by deriving the conformally rotated Euclidean functional integrals from the quantum mechanics of the theory expressed in terms of its physical degrees of freedom.

Gauge theories are formulated in terms of redundant variables. Configurations of the variables which differ by gauge transformations are physically equivalent. The true physical degrees of freedom of the theory are those which distinguish physically distinct configurations. Theories in which time is parametrised display similar properties although there are important differences (see, for example, Hartle and Kuchař 1984a,b).

The quantum mechanics of a theory with redundant variables is most simply discussed in terms of its physical degrees of freedom if they can be explicitly identified. The sums over histories for quantum amplitudes, for example, have a simple form when expressed in terms of the physical degrees of freedom. When so expressed they may not manifestly display all the invariances of the theory or its locality in the redundant variables. Quantum amplitudes, however, can also be expressed by functional integrals over the extended space of redundant variables so as to explicitly display invariance and locality. Such expressions are not only useful for constructing manifestly invariant perturbation theory. They are the starting point for the quantum mechanics of those theories with redundant variables for which, like general relativity, the physical degrees of freedom cannot be explicitly solved for.

The expressions for amplitudes in terms of functional integrals over the extended variables can be derived from those over the physical degrees of freedom by systematically adding integrals over the redundant variables (for example, Faddeev 1969, Faddeev and Popov 1967, 1973, Fradkin and Vilkovisky 1977, Henneaux 1985). It is through the exploration of this connection that one arrives at the correct form and measure for the functional integrals for gauge theories on the extended variables and makes the connection between Hamiltonian and Lagrangian quantum mechanics. The connection has mostly been discussed for the 'Lorentzian' functional integrals of the form (1.1) but it can also be derived for the Euclidean functional integrals using analogous techniques. It is a natural place to look for an understanding of the conformal rotation.

When the physical degrees of freedom can be explicitly identified, the process of connecting functional integrals in terms of the physical degrees of freedom with those in terms of the extended variables can be explicitly carried out. This will be the case for linearised gravity in contrast to the full general theory of relativity. We shall, therefore, explore the connection in the linearised theory with an eye to understanding the conformal rotation. The techniques for adding redundant integrations to Euclidean functional integrals will first be developed in the context of a simple model in §2 and then applied to linearised gravity in §3. There, for linearised gravity, we shall derive the conformally rotated Euclidean functional integral for a quantum amplitude from the functional integral for that amplitude expressed in terms of the physical degrees of freedom.

2 EUCLIDEAN FUNCTIONAL INTEGRALS FOR GAUGE AND PARAMETRISED THEORIES

In selecting an action to summarise the dynamics of a field theory one frequently has in mind two goals: to find an action which (1) is a local functional of a certain set of field variables and which (2) expresses manifestly the invariances of the theory in terms of these variables. In electrodynamics we seek an action which is local in the potentials $A_\mu(x)$ and which is Lorentz invariant and gauge invariant. In gravity we might seek an action which is a local function of the metric $g_{\alpha\beta}(x)$ and which is invariant under the group of diffeomorphisms. Meeting both goals (1) and (2) typically means that the action involves not only the physical degrees of freedom—those freely specifiable on an initial value surface—but redundant variables as well. In electrodynamics, the physical degrees of freedom are the two transverse components of the vector potential, $A_i^{\mathrm{T}}(\boldsymbol{x})$. The invariant action also involves $A_t(\boldsymbol{x})$ and the longitudinal component $A_i^{\mathrm{L}}(\boldsymbol{x})$. In the linearised theory of gravity, the physical degrees of freedom are the transverse–traceless parts of the metric perturbation h_{ij}^{TT} while the Einstein Lagrangian involves all the other components of the metric perturbation $h_{\alpha\beta}$ as well. In general relativity, the action is a functional of the metric $g_{\alpha\beta}$. There are two physical degrees of freedom at each point on an initial value surface although the constraints cannot be solved to exhibit them explicitly.

If one relaxes the goals of locality and invariance then there are many different forms of the action which express the physical content of a theory. In electrodynamics and linearised gravity, for example, one can express the action in terms of the physical degrees of freedom at the expense of Lorentz invariance.

How does one construct a quantum theory corresponding to a classical theory with redundant variables? If the physical degrees of freedom can be explicitly identified then one can proceed in two steps: (1) specify quantum amplitudes as sums over histories expressed in terms of the physical degrees of freedom; (2) if desired, add back into the resulting functional integral, additional integrals over the redundant degrees of freedom so as to not affect the value of the integral but to allow the integral to manifestly display the original invariance and locality. When the physical degrees of freedom cannot be explicitly identified, one can proceed formally and begin with the form of the results of this two-step process.

In the following, we would like to illustrate this procedure with a simple model (Hartle and Kuchař 1984b). The model is too simple to illustrate all the issues which arise but does display some typical ones in a transparent manner. In the succeeding section, we shall apply the techniques developed here to the case of linearised gravity.

The configuration space of the model consists of N variables $q^a(t)$, $a = 1, \cdots N$ which are the physical degrees of freedom and two variables $\varphi(t)$ and $\lambda(t)$ which represent the redundant variables. The Lagrangian is a sum of a Lagrangian for the physical degrees of freedom $l(q^a, \dot{q}^a)$ and a Lagrangian for the redundant variables $l^g(\varphi, \dot{\varphi}, \lambda)$. For l we take

$$l(q^a, \dot{q}^a) = \tfrac{1}{2} m\, \delta_{ab} \dot{q}^a \dot{q}^b - V(q) \tag{2.1}$$

and for l^g

$$l^g(\dot{\varphi}, \varphi, \lambda) = \tfrac{1}{2}\mu(\dot{\varphi} - \lambda)^2. \tag{2.2}$$

The result is a simple model of a gauge theory; l^g and the total Lagrangian are invariant under gauge transformations

$$\begin{aligned} \varphi(t) &\to \varphi(t) + \Lambda(t) \\ \lambda(t) &\to \lambda(t) + \dot{\Lambda}(t). \end{aligned} \tag{2.3}$$

Since the variable λ occurs in equation (2.2) without time differentiation, there is a constraint, which is that the momentum conjugate to φ vanishes

$$\pi = \partial l^g / \partial \dot{\varphi} = 0. \tag{2.4}$$

If we did not know it already, equation (2.4) would allow us to conclude that φ and λ are redundant variables and that the physical degrees of freedom are the q^a.

Of course, we are not typically given gauge theories in the simple form of (2.1) plus (2.2). Rather they are expressed in terms of other variables $Q^A = Q^A(q^a, \varphi, \lambda)$ in which some invariance is manifest. The above model, however, displays their characteristic structure. In electrodynamics for example, φ corresponds to $A_i^L(x)$ and λ corresponds to $A_t(x)$ while the q^a are analogous to $A_i^T(x)$. For the purposes of our model, let us imagine that invariance and locality have fixed the form (2.1) plus (2.2).

In the quantum theory corresponding to our simple model, states are labelled by the physical degrees of freedom, e.g. $|q^a, t\rangle$. Amplitudes may be constructed by sums over histories in terms of the physical degrees of freedom in both Hamiltonian and Lagrangian form. For example, the propagator may be expressed as

$$\langle q''^a t'' \,|\, q'^a t' \rangle = \int \delta^n p\, \delta^n q \exp\left(i \int_{t'}^{t''} dt (p_a \dot{q}^a - h(q, p)) \right) \tag{2.5}$$

where $h(q, p)$ is the Hamiltonian constructed from (2.1)

$$h(q^a, p_a) = \frac{1}{2m} \delta^{ab} p_a p_b + V(q). \tag{2.6}$$

The sum in (2.5) is over phase space paths which begin at q'^a at t' and end at q''^a at t''. The action in the exponent is the familiar canonical one while the measure is the usual invariant '$\mathrm{d}p\mathrm{d}q/(2\pi\hbar)$' measure on the space of phase space paths. One can think of the functional integral in (2.5) as being implemented in a variety of ways—time slicing for example. Corresponding to the different ways of 'putting coordinates' on the space of functions $q(t)$ and $p(t)$ there will be different explicit forms of the 'measure' for the functional integrals. We shall not consider these in any detail in this section although we shall supply explicit expressions in the case of linearised gravity†.

The integrals over the momenta in (2.5) can be carried out explicitly since the Hamiltonian is quadratic in them. This yields the Lagrangian form of the sum over histories for the propagator

$$\langle q''^a t'' \mid q'^a t' \rangle = \int \delta^n q \exp\left(\mathrm{i} \int_{t'}^{t''} \mathrm{d}t l(q^a, \dot{q}^a)\right). \tag{2.7}$$

The transition from (2.5) to (2.7) is important because in this way the form of the measure is derived from Hamiltonian quantum mechanics.

Some quantum amplitudes can be conveniently expressed in terms of Euclidean sums over histories. An example, on which we shall focus for concreteness, is the ground state wavefunction. If one expands the left-hand side of (2.5) or (2.7) in a complete set of energy eigenstates with energies E_n and wavefunctions $\Psi_n(q^a)$, one has, for example

$$\langle q^a, 0 \mid q'^a, t \rangle = \Sigma_n \Psi_n(q^a) \Psi_n^*(q'^a) \exp(\mathrm{i}E_n t). \tag{2.8}$$

If we fix q'^a to be at the minimum of $V(q)$, rotate $t \to -\mathrm{i}\tau$, and take the limit as $\tau \to -\infty$, the ground state will provide the dominant contribution to the right-hand side. Carrying out the same rotations on the right-hand sides of (2.5) and (2.7) we arrive at expressions for the ground state wavefunction $\Psi_0(q^a)$ up to a normalisation. From (2.7) one has

$$\Psi_0(q^a) = N \int \delta^n q \exp\left(-\int_{-\infty}^{0} \mathrm{d}\tau l_\mathrm{E}(q^a, \dot{q}^a)\right) \tag{2.9}$$

where N is a normalising constant and l_E is the Euclidean Lagrangian

$$l_\mathrm{E}(q^a, \dot{q}^a) = \tfrac{1}{2} m\, \delta_{ab} \dot{q}^a \dot{q}^b + V(q). \tag{2.10}$$

The exponent in (2.9) is minus the Euclidean action. From (2.5) we also have

$$\Psi_0(q^a) = N \int \delta^n p \delta^n q \exp\left(-\int_{-\infty}^{0} \mathrm{d}\tau (h(q, p) - \mathrm{i} p_a \dot{q}^a)\right). \tag{2.11}$$

†If the reader is in any doubt, these factors were considered in detail for this model in Hartle and Kuchař (1984b), although there is an unfortunate conflict in the use of the notation δq between that paper and this.

(Note that the momenta are not rotated in passing from (2.5) to (2.11) and a divergent expression would result if they were.) Equation (2.11) is perhaps less familiar than (2.9) but it is still useful. Equation (2.9) can be derived from (2.11) by integrating out the momenta. Most importantly (2.11) shows that, if the Hamiltonian of the physical degrees of freedom has a lower bound, then the Euclidean functional integrals of the theory will converge. This will be the case for electrodynamics and for linearised gravity. It may also be of interest for general relativity where initial data which satisfy the constraints, and are thus restricted to the physical degrees of freedom, have positive energy (Schoen and Yau 1979b, Witten 1981).

By adding further integrations over the redundant variables, the functional integrals (2.5), (2.7), (2.9) and (2.11) can be expressed as integrals over the extended variables involving the full action. Consider for example the functional integral for the transition amplitude (2.7). For any function $\Phi(\varphi)$ such that $\Phi(\varphi)=0$ has a unique solution, the following identity is true

$$1=\int \delta\varphi\,\delta\lambda\,\det\left[\left|\frac{\partial\Phi}{\partial\varphi}\right|\right]\delta[\Phi(\varphi)]\,\exp\left(\mathrm{i}\int_{t'}^{t''}\mathrm{d}t l^{g}(\varphi,\dot\varphi,\lambda)\right). \qquad (2.12)$$

The identity can be verified by carrying out the integral over λ—it is a Gaussian—and then the integral over φ using the δ function. The term $\det[\,|\partial\Phi/\partial\varphi|\,]$ is the product of factors which depend on Φ and are necessary to make the integral unity. In a time slicing implementation of (2.12) there would be one factor of $|\partial\Phi/\partial\varphi|$ for each time slice. Together, these factors make up the familiar Faddeev–Popov determinant for the simple gauge transformation (2.3) and the 'gauge fixing condition' $\Phi(\varphi)=0$. To emphasise this they can be written $\det(\,|\partial\Phi^{\Lambda}/\partial\Lambda|\,)=\det|(\partial\Phi(\varphi+\Lambda)/\partial\Lambda)|$. Other numerical factors necessary to make the integral exactly unity have been absorbed into $\delta\varphi\,\delta\lambda$. If the identity (2.12) is inserted in the functional integral (2.7), the following expression for the transition amplitude results:

$$\langle q''^{a}t''\,|\,q'^{a}t'\rangle=\int \delta^{n+2}q\,\det\left[\left|\frac{\partial\Phi^{\Lambda}}{\partial\Lambda}\right|\right]\delta[\Phi(\varphi)]\,\exp(\mathrm{i}S[q^{\alpha}]) \qquad (2.13)$$

where we have written $q^{\alpha}=\{q^{a},\varphi,\lambda\}$ for the extended variables and S is the total action constructed from the sum of l and l^{g}. Equation (2.13) is the familiar form of the functional integral for the propagator in a gauge theory and the analysis above is the familiar derivation of it (see for example Faddeev 1969).

The repertoire of identities which can be used to create a path integral with the action $S[q^{\alpha}]$ is not limited to (2.12). For example, one might have used

$$1=\int \delta\varphi\,\delta\lambda\,\delta^{s}[\varphi]\delta[\lambda]\,\det\left[\left|\frac{\partial\lambda^{\Lambda}}{\partial\Lambda}\right|\right]\exp\left(\mathrm{i}\int_{t'}^{t''}\mathrm{d}t l^{g}(\varphi,\dot\varphi,\lambda)\right) \qquad (2.14)$$

where $\delta^s[\varphi]$ is a δ function enforcing the condition $\varphi = 0$ only on the final surface $t = t''$. This identity follows because the λ integration is fixed by its δ function and the φ integration is a Gaussian or is fixed by the δ function on the surface. Inserting this in (2.7) we recover a path integral of the form (2.13) but with a different set of gauge fixing δ functions which involve both φ and λ. The condition $\lambda = 0$ fixes the gauge freedom of (2.3) up to transformations of the form $\varphi \to \varphi + \Lambda$ where Λ is constant. Fixing φ on the surface fixes this last bit of gauge freedom.

The above model does not display the most general type of action involving redundant variables and the identities (2.12) and (2.14) are not the most general ways of adding integrations over such variables to functional integrals. For example, one might want to add gauge *invariant* redundant variables (we shall see an example in linearised gravity) and certainly there are many other forms of gauge fixing. Considerable insight into the various possibilities and the issues that they raise can be gained by studying the theory in its Hamiltonian form and by a study of the gauge and reparametrisation transformations on the space of extended variables. From the Hamiltonian theory, for example, one learns that the characteristic form (2.13) emerges naturally from (2.5) by introducing a δ function on the extended phase space to enforce the constraints depending on momenta, 'exponentiating' that δ function via $\delta(\pi) = (2\pi)^{-1}\int d\lambda \exp(i\lambda\pi)$ (thereby introducing a further integration over the multiplier) and integrating out the momenta. From the study of the theory on the extended space of variables one learns that the different possibilities for introducing redundant variables exemplified by (2.12) and (2.14) correspond to different ways of slicing the gauge orbits on the extended space so that only physically distinct configurations contribute to the sum over histories. We shall not review these general insights here and indeed there is no need to do so since they have been thoroughly discussed (Faddeev and Popov 1973, Fradkin and Vilkovisky 1977, Hartle and Kuchař 1984a,b, Henneaux 1985 and many other references). Rather we shall only note that it is possible to add integrations over redundant variables to the functional integrals in terms of the physical degrees of freedom with two identities

$$1 = \int_{-\infty}^{+\infty} dx\, \delta(x) \tag{2.15a}$$

and

$$1 = \frac{1}{\sqrt{i\pi}} \int_{-\infty}^{+\infty} dx\, e^{ix^2}. \tag{2.15b}$$

Where one goes with these identities depends on where one wants to get to.

To proceed from Euclidean functional integrals in terms of the physical degrees of freedom to equivalent ones on an extended space of variables is a completely analogous process to that described above. The identity

(2.15a) is still of use, but because the exponents in the Euclidean integrals are real, (2.15b) is typically replaced by

$$1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx\, e^{-x^2}. \tag{2.15c}$$

As an example, consider adding integrations over φ and λ to the integral (2.9) for the ground state wavefunction of our model so that the resulting integral involves the Euclidean action for the theory. To obtain a Euclidean version of (2.2) one may rotate $t \to -i\tau$ and also $\lambda \to i\lambda$. Thus, a Euclidean gauge action is

$$I^g = \int d\tau \tfrac{1}{2}\mu(\dot{\varphi} - \lambda)^2 \tag{2.16}$$

and a Euclidean action for the whole theory is

$$I[q^\alpha] = \int d\tau l_E(q^\alpha, \dot{q}^\alpha) + I^g. \tag{2.17}$$

The form of the Euclidean action is determined by the goals of locality and invariance in the extended space of variables $\{q^a, \varphi, \lambda\}$ and in turn this dictates how the rotations are to be carried out. Thus, in the above example we rotate $\lambda \to i\lambda$ and not $\lambda \to \lambda$ or $\lambda \to -i\lambda$ so that gauge invariance in the form (2.13) is maintained. This can be the only motivation since the additional variables have no physical content. The process is familiar from electrodynamics where we rotate $A_t \to iA_\tau$ as we rotate $t \to -i\tau$ to obtain a gauge and O(4) invariant Euclidean action.

We can pass from a path integral of the form (2.9) to one involving the action (2.17) by making use of the identity

$$1 = \int \delta\varphi\, \delta\lambda \det\left[\left|\frac{\partial\Phi^A}{\partial\Lambda}\right|\right] \delta[\Phi(\varphi)] \exp(-I^g[\varphi, \lambda]) \tag{2.18}$$

analogous to (2.12). It can be verified by using (2.15c) to carry out the integrations over λ and (2.15a) to do those over φ. Inserted in (2.9) we find

$$\Psi_0[q^a] = \int \delta^{n+2}q \det\left[\left|\frac{\partial\Phi^A}{\partial\Lambda}\right|\right] \delta[\Phi(\varphi)] \exp(-I[q^\alpha]) \tag{2.19}$$

where I is the desired form of the action (2.17).

The above procedure works when the constant μ in (2.16) is positive. It fails when μ is negative. This can be seen either from the final answer or from the steps through which it was derived. In the final answer, the action I is neither positive definite nor bounded below if μ is negative. In the intermediate step, the integral (2.18) diverges.

Has the sum over histories formulation of quantum mechanics then somehow failed for the theory (2.17) with negative μ? Are Euclidean methods inapplicable in such a theory? The answer to both questions is certainly no. The theory in terms of the physical variables is well defined and Euclidean methods can be applied as long as the energy is positive on the physical degrees of freedom.

In the case of negative μ we *have* failed to cast the Euclidean functional integrals of the theory into a form constructed from the action (2.17). That action, in particular the sign of μ, was assumed fixed by the requirements of locality and invariance. There may, however, be many actions on the extended variables which meet these requirements *partially*, which are physically equivalent, and for which the corresponding Euclidean functional integrals are convergent. For example, if we change μ to $-\mu$ in (2.17) we obtain an action which is positive definite, which is gauge invariant, and which is physically equivalent since the gauge variables are redundant. It only fails to meet some requirement of locality expressed in terms of variables which mix q^a, φ and λ. This action could formally be regarded as arising from (2.17) by a further complex rotation of φ and λ. A Euclidean functional integral for the ground state wavefunction which involves this new action can be derived from (2.9) because the corresponding identity (2.18) is now convergent. Such an expression can be useful.

Starting from a quantum theory formulated in terms of physical degrees of freedom, there are many paths leading from its Euclidean functional integrals to those involving extended variables. How one proceeds depends not only on where one wants to get but also on whether there is a path leading there. The issue of whether the quantum theory is well defined, however, depends not on the properties of the theory expressed in terms of extended variables but rather on its properties expressed in terms of the true physical degrees of freedom.

3 LINEARISED GRAVITY

The transition between Euclidean functional integrals over physical degrees of freedom and those over extended variables can be explicitly worked out for the linearised version of Einstein's general relativity. This is because the physical degrees of freedom of linearised gravity can be explicitly identified and because its action is a quadratic functional. In this section we shall make this transition for the Euclidean integral defining the ground state wavefunctional for linearised gravity using the techniques reviewed in §2.

The action for linearised gravity is obtained from that of general relativity by expanding the metric in small perturbations $h_{\alpha\beta}$ about flat space. We shall assume throughout that these metric perturbations fall off spatially as $1/r^{3/2}$ or better at infinity. This will be a sufficient class of perturbations for our purposes. The action is then†

$$l^2 S_2[h_{\alpha\beta}] = \tfrac{1}{2}\int_M d^4x(h^{\alpha\beta}G_{\alpha\beta}) + \tfrac{1}{2}\int_{\partial M} d^3x h^{ij}(K_{ij} - \delta_{ij}K^k_k) \qquad (3.1)$$

where, in this section, $G_{\alpha\beta}$ is the *linearised* Einstein tensor and K_{ij} is the linearised extrinsic curvature of a constant t boundary of the region of interest. The action is invariant under gauge transformations of the form

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} + \nabla_{(\alpha}\xi_{\beta)} \tag{3.2}$$

and as a consequence the theory has four constraints. The four constraints and the four gauge degrees of freedom mean that eight of the ten $h_{\alpha\beta}$ are redundant variables while the remaining two are the physical degrees of freedom of linearised gravity. These can be found by writing the theory in $3+1$ form to exhibit its initial value formulation and then solving the constraints on an initial constant t slice (see Arnowitt and Deser 1959). The familiar result is that the physical degrees of freedom are the two transverse–traceless components of the perturbation in the metric of a constant t three-surface, h_{ij}^{TT}. That is, if the metric h_{ij} of this surface (the spatial components of $h_{\alpha\beta}$) is analysed into Fourier components labelled by a wavevector k^i, then the two trace-free components of h_{ij} projected into the subspace transverse to k^i are the physical degrees of freedom. In terms of them, the action is

$$l^2 s_2 = \tfrac{1}{4} \int d^4x [(\dot{h}_{ij}^{\mathrm{TT}})^2 - (\nabla_i h_{jk}^{\mathrm{TT}})^2] \tag{3.3}$$

where we have introduced the obvious convention that for any tensor $(a_{ij\ldots})^2 = a_{ij\ldots}a^{ij\cdots}$ and a similar one in four dimensions. The corresponding Hamiltonian is

$$l^2 h_2 = \int d^3x [(\pi_{ij}^{\mathrm{TT}})^2 + \tfrac{1}{4}(\nabla_i h_{jk}^{\mathrm{TT}})^2] \tag{3.4}$$

where π_{ij}^{TT} is the momentum conjugate to h_{ij}^{TT}. We note that the Hamiltonian is positive definite. Indeed, this is just the Hamiltonian for an assembly of independent harmonic oscillators. The quantum theory is therefore certainly well defined.

The ground state wavefunction for the theory (Kuchař 1970) is the wavefunction for the state with all the oscillators in their ground states. It can be constructed by the Euclidean functional integral analogous to (2.9) (Hartle 1984)

$$\psi_0[h_{ij}^{\mathrm{TT}}, T] = \int \delta h_{ij}^{\mathrm{TT}} \exp(-i_2[h_{ij}^{\mathrm{TT}}]) \tag{3.5}$$

where i_2 is the Euclidean action for linearised gravity and the sum is over all transverse–traceless tensor field configurations in the half space $x^0 < T$

†Throughout greek indices range over four dimensions while latin indices range over three. The signature is $(-, +, +, +)$ when we are discussing Lorentzian space–times and $(+, +, +, +)$ for Euclidean ones.

that match the argument of the wavefunction on the surface $x^0 = T$ and which fall off fast enough at Euclidean infinity so that the action is finite. We shall exhibit the measure in the Appendix. Explicitly, i_2 is

$$l^2 i_2 = \tfrac{1}{4}\int d^4x[(\dot{h}_{ij}^{\mathrm{TT}})^2 + (\nabla_i h_{jk}^{\mathrm{TT}})^2]. \tag{3.6}$$

It is positive definite and the integral (3.5) therefore converges. This could be seen in a different way from the positivity of the Hamiltonian and the analogue of (2.11).

Equation (3.5) is where we start. We would like to add redundant integrations to this expression until we arrive at an expression for ψ_0 which is manifestly gauge invariant and O(4) invariant. An O(4) and gauge invariant Euclidean action which is also local in the metric perturbations is the linearised version of (1.3),

$$l^2 I_2 = \tfrac{1}{4}\int_M d^4x[(\nabla_\alpha \bar{h}_{\beta\gamma})(\nabla^\alpha h^{\beta\gamma}) - 2(\nabla^\alpha \bar{h}_{\alpha\beta})^2] + \begin{pmatrix}\text{surface terms which involve}\\ \text{only the redundant variables}\end{pmatrix} \tag{3.7}$$

where

$$\bar{h}^\alpha{}_\beta = h^\alpha{}_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta h^\gamma{}_\gamma. \tag{3.8}$$

We cannot end up with a functional integral for ψ_0 involving this action. It is not positive definite. In particular on perturbations of the special form $h_{\alpha\beta} = -2\delta_{\alpha\beta}\chi$ we have

$$l^2 I_2 = -6\int d^4x(\nabla_\alpha \chi)^2. \tag{3.9}$$

However, (3.7) is not the only gauge invariant O(4) invariant action for linearised gravity.

To add back the redundant integrations we decompose $h_{\alpha\beta}$ into pieces corresponding to the physical degrees of freedom and pieces corresponding to the redundant integrations. As the result (3.9) suggests, it is convenient to begin by decomposing $h_{\alpha\beta}$ into conformal equivalence classes as

$$h_{\alpha\beta} = \varphi_{\alpha\beta} + 2\chi\delta_{\alpha\beta} \tag{3.10}$$

where the decomposition can be fixed by the O(4) invariant, gauge invariant condition

$$R(\varphi) = \nabla_\alpha \nabla_\beta \varphi^{\alpha\beta} - \nabla^2 \varphi^\beta_\beta = 0 \tag{3.11}$$

so that χ can be defined in terms of $h_{\alpha\beta}$ through

$$R(h) = -6\nabla^2\chi \tag{3.12}$$

and the boundary conditions that χ vanish on the surface $x^0 = T$ and at infinity.

The perturbation $\varphi_{\alpha\beta}$ may be further decomposed as

$$\varphi_{\alpha\beta} = t_{\alpha\beta} + l_{\alpha\beta} + \varphi^T_{\alpha\beta} + \varphi^L_{\alpha\beta} \tag{3.13}$$

where the components are defined as follows: let n^α be the unit vector orthogonal to the constant t surfaces. Consider the *families* of tensors $t_{\alpha\beta}$, $l_{\alpha\beta}$, $\varphi^T_{\alpha\beta}$ and $\varphi^L_{\alpha\beta}$ satisfying the following conditions

$$\nabla^\alpha t_{\alpha\beta} = 0 \qquad n^\alpha t_{\alpha\beta} = 0 \qquad t^\alpha{}_\alpha = 0 \tag{3.14a}$$

$$\nabla^\alpha l_{\alpha\beta} = 0 \qquad l^\alpha{}_\alpha = 0 \qquad \int_M d^4x\, t^{\alpha\beta} l_{\alpha\beta} = 0 \tag{3.14b}$$

$$\nabla^\alpha \varphi^T_{\alpha\beta} = 0 \qquad n^\alpha \varphi^T_{\alpha\beta} = 0 \qquad \int_M d^4x\, t^{\alpha\beta} \varphi^T_{\alpha\beta} = 0 \tag{3.14c}$$

$$\int_M d^4x\, t^{\alpha\beta}\varphi^L_{\alpha\beta} = \int_M d^4x\, l^{\alpha\beta}\varphi^L_{\alpha\beta} = \int_M d^4x\, \varphi^{T\alpha\beta}\varphi^L_{\alpha\beta} = 0. \tag{3.14d}$$

The orthogonality conditions are understood to hold for all tensors in the families. Then there is a unique decomposition of $\varphi_{\alpha\beta}$ into members of these families which we write as (3.13). A more explicit version of the decomposition is given in the Appendix. The condition (3.11) fixes $\varphi^T_{\alpha\beta} = 0$. The tensors $t_{\alpha\beta}$ correspond to the physical degrees of freedom. The rest are redundant.

Under gauge transformations only $t_{\alpha\beta}$, $l_{\alpha\beta}$ and χ are unchanged. Since the action (3.7) is gauge invariant it can be expressed as a Lorentz invariant combination of these quantities. In fact it has the form

$$l^2 I_2 = \tfrac{1}{4}\int_M d^4x[(\nabla_\alpha t_{\beta\gamma})^2 + (\nabla_\alpha l_{\beta\gamma})^2 - 24(\nabla_\alpha\chi)^2]$$

$$-\tfrac{1}{4}\int_{\partial M} d^3x\, n^\alpha \nabla_\alpha[2(n^\beta l_{\beta\gamma})^2 - \tfrac{3}{2}(n^\beta n^\gamma l_{\beta\gamma})^2]. \tag{3.15}$$

Using this decomposition of the metric we can proceed as in §2 to add in the redundant degrees of freedom by inserting in (3.5) identities composed of Gaussian integrals over the gauge invariant quantities and integrals over gauge fixing δ-functions for the gauge non-invariant ones. Although the final form is independent of the gauge fixing conditions it clarifies the argument to use a particular one. We shall choose

$$C_\alpha = \nabla^\beta \bar{\varphi}_{\alpha\beta} = 0 \tag{3.16}$$

which, when combined with (3.11), fixes the $\varphi^L_{\alpha\beta}$ components up to a transformation (3.2) satisfying

$$\nabla^2 \xi_\beta = 0. \tag{3.17}$$

By fixing a further condition on the $x^0 = T$ surface this remaining gauge freedom can be fixed. Additionally, conditions at the boundary and at infinity are needed on the remaining redundant components of $h_{\alpha\beta}$ to define the class of configurations over which we shall integrate. For simplicity we will take the approach of fixing all fields on the boundary by requiring $t_{\alpha\beta}$ to match the argument of the wavefunction at $x^0 = T$, by requiring the spatial part h_{ij} of the remaining components to vanish there†, and to satisfy the gauge condition (3.16). Finally all components of $h_{\alpha\beta}$ will be required to vanish at Euclidean infinity rapidly enough so that the action is finite. On such configurations the surface term in the action (3.15) vanishes.

In terms of the decomposition (3.13), the action i_2 (equation (3.6)) on the physical degrees of freedom takes the form

$$l^2 i_2 = \tfrac{1}{4}\int d^4x (\nabla_\alpha t_{\beta\gamma})^2. \tag{3.18}$$

In the class over which we plan to integrate, the most general quadratic action in the redundant variables which is gauge invariant and O(4) invariant in the sense of being independent of n^α is

$$l^2 I_2^g = \tfrac{1}{4}\int d^4x [(\nabla_\alpha l_{\beta\gamma})^2 + a(\nabla_\alpha \chi)^2] \tag{3.19}$$

where a is an arbitrary positive constant. The coefficient of the $l_{\beta\gamma}$ terms is fixed by the requirement that the total action be independent of n^α. The coefficient of $(\nabla_\alpha\chi)^2$ is unrestricted by O(4) invariance since χ is an O(4) scalar. The constant a must be positive, however, for the action to be positive definite.

Integrals over the redundant variables involving the action (3.19) and the gauge fixing conditions (3.16) may be added to the Euclidean functional integral for the ground state wavefunction by forming the identities

$$1 = \int \delta l \delta\varphi^L\, \delta\chi\, \delta[C^\alpha]\, \det\left[\left|\frac{\delta C^\alpha}{\delta\xi^\beta}\right|\right] \exp(-I_2^g[l,\chi]) \tag{3.20a}$$

and

$$1 = \int \delta\varphi^T\, \delta[R(\varphi)]\, \det\left[\left|\frac{\delta R}{\delta\omega}\right|\right]. \tag{3.20b}$$

†Alternatively we could integrate over redundant variables which are not fixed on the boundary by inserting additional gauge fixing δ functions at the boundary surface (see for example Hartle 1984).

In equations (3.20) the functional integrals are over the configurations we have specified to the past of the surface $x^0 = T$. The determinant in (3.20a) is the Faddeev–Popov determinant of the operator constructed by varying the gauge fixing condition C^α(equation (3.16)) with respect to the gauge parameter ξ^α (equation (3.2)). The determinant in equation (3.20b) is of the operator constructed by varying the condition (3.11) which fixes the conformal equivalence class by an infinitesimal conformal transformation

$$h_{\alpha\beta} \to h_{\alpha\beta} + 2\delta_{\alpha\beta}\omega. \tag{3.21}$$

A specific measure is required in order for equations (3.20) to be true. This will be calculated explicitly in the Appendix.

Inserting the identities (3.20) into the Euclidean functional integral (3.5) we arrive at the following expression for the ground state wavefunction

$$\psi_0[h_{ij}^{TT}, T] = \int \delta\varphi\, \delta\chi\, \delta[C^\alpha(\varphi)]\, \delta[R(\varphi)]\, \det\left[\left|\frac{\delta C^\alpha}{\delta \xi^\beta}\right|\right] \times \det\left[\left|\frac{\delta R}{\delta\omega}\right|\right] \exp(-\hat{I}_2[\varphi, \chi]). \tag{3.22}$$

Here, $\hat{I}_2$ is the sum of i_2 and I_2^g

$$l^2\hat{I}_2[\varphi, \chi] = \tfrac{1}{4}\int d^4x[(\nabla_\alpha t_{\beta\gamma})^2 + (\nabla_\alpha l_{\beta\gamma})^2 + a(\nabla\chi)^2] \tag{3.23}$$

where a is any positive constant. The integral in equation (3.22) is over all *ten* components of $\varphi_{\alpha\beta}$ *and* over the 'conformal factor' χ in the class of configurations described above. The integration is thus of the form of an integration over all gauge inequivalent metrics in a conformal equivalence class specified by $R(\varphi) = 0$ together with an integration over conformal factor.

The action (3.23) is gauge invariant, O(4) invariant, and, for positive a, it is positive definite so that the integral in (3.22) converges. If this had been a Lorentzian functional integral we could have recovered an integral over the action S_2 (equation (3.1)) by choosing $a = -24$ and carrying out the integral over χ using the δ-function of R. In this Euclidean case the action cannot be made to coincide with the action I_2(equation (3.7)) because, as (3.15) shows, this would require a negative value of a and lead to a divergent functional integral. The action $\hat{I}_2$ is exactly that which would be formally obtained from I_2 by a rotation of the conformal factor $\chi \to i\chi$ and setting $a = 24$. The action $\hat{I}_2$ *can* be expressed in terms of the metric perturbations $h_{\alpha\beta}$ but only in a non-local manner. From (3.11)

$$\hat{I}_2[h] = I_2[h] - \frac{(a+24)}{144}\int d^4x R(h)\nabla^{-2}R(h). \tag{3.24}$$

This action is physically equivalent to I_2, gauge invariant and O(4) invariant. As long as $a > 0$ it is positive definite. Thus, at the expense of locality in the metric perturbations one can construct convergent functional integrals for linearised gravity which manifestly display the invariances of the theory. They are in fact the conformally rotated functional integrals of Gibbons *et al* (1978).

4 CONCLUSIONS

The Euclidean action for linearised gravity is not positive definite. This does not mean that there is not a satisfactory quantum theory of the linearised gravitational field. Neither does it mean that there is not a sum over histories formulation of this quantum theory or that Euclidean functional integrals cannot be used to construct appropriate amplitudes. There is a satisfactory quantum theory because the Hamiltonian expressed in terms of the physical degrees of freedom is positive. As a consequence there is also a sum over histories formulation of the theory in terms of the physical degrees of freedom and a corresponding Euclidean functional integral construction of the ground state wavefunction.

The non-positivity of the Euclidean action for linearised gravity *does* mean that we cannot express Euclidean functional integrals in a form in which the action is manifestly local in the metric perturbations $h_{\alpha\beta}$ and O(4) invariant. However, one can come close. One can express the Euclidean integrals of the theory in terms of an action which is O(4) invariant and which contains the same number of metric variables as the usual action. It is even local when expressed in terms of the variables $\varphi_{\alpha\beta}$ and χ used in §3. It is only that it is non-local when expressed in terms of the metric perturbations themselves. This action is the linearised version of the conformally rotated action of Gibbons *et al* (1978). (See also Gibbons and Perry (1978).)

As its name suggests, the conformally rotated action for linearised gravity can be obtained from the Euclidean action by a formal rotation of the conformal factor χ. In a similar way, a functional integral using the conformally rotated action may be obtained from the corresponding integral expressed in terms of the Euclidean action by a formal rotation of the contour of integration of the conformal factor. This is not a very satisfactory procedure, however, because the integral involving the Euclidean action does not exist. Neither can one start from the Lorentzian functional integral and perform simultaneous rotations of the conformal factor and time to obtain a Euclidean functional integral over the conformally rotated action. There appears to be no simple distortion of both contours such that the functional integral remains convergent at every intermediate step. Thus the Euclidean functional integral for linearised gravity over the conformally rotated action is not best seen as arising from some convergent functional integral involving the usual action through a distortion of contours†. Rather, it is best viewed as arising from the standard process of quantising a theory with gauge and reparametrisation invariance: (1) expressing the theory in terms of its physical degrees of freedom; (2) then formulating the quantum sum over histories in terms of these degrees of freedom; and (3) finally adding back in integrations over redundant variables to manifestly express the invariance of the theory. How one adds back in these integrations *is* limited in the Euclidean sums over histories by the convergence of the final expression but is mostly determined by what final expression one wishes to get.

That the quantum mechanics of the linearised gravitational field is well defined and the role of the conformal factor easy to understand is no surprise. The theory is mathematically equivalent to two harmonic oscillators for each mode of excitation. It is of considerable interest to see whether this understanding can be extended to linear perturbations off a curved background, to general relativity itself and to general relativity interacting with matter fields. The positive energy theorems of classical general relativity (Schoen and Yau 1979b, Witten 1981) and the closely related positive action theorems (Schoen and Yau 1979a) give hope that this will be possible.

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APPENDIX: THE MEASURE

To show that the functional integrals in (3.20) and (3.22) have a definite and concrete meaning, we shall evaluate the factors making up the measures $\delta\varphi$, $\delta\chi$, etc in a particular set of 'coordinates on the function spaces' with the specific gauge choice (3.16). The coordinates we shall use are the coefficients of a Fourier expansion of the integration variables. To make this concrete we shall take our spacetime to be a finite box of volume L^4 interior to the planes $x^0 = T$, $x^0 = T - L$; $x^i = \pm L/2$. This makes the Fourier momenta k^α

†It could be so seen starting from a non-local action.

discrete. The expansion is then,

$$t_{\alpha\beta}(x) = t^{\mathrm{cl}}_{\alpha\beta}(x) + \sum_{\nu=1}^{2} \sum_{k}{}' t^{(\nu)}(k) t^{(\nu)}_{\alpha\beta}(k, x)$$

$$l_{\alpha\beta}(x) = \sum_{\nu=1}^{3} \sum_{k}{}' l^{(\nu)}(k) l^{(\nu)}_{\alpha\beta}(k, x)$$

$$\varphi^{L}_{\alpha\beta}(x) = \sum_{\nu=1}^{4} \sum_{k}{}' \varphi^{L(\nu)}(k) \varphi^{L(\nu)}_{\alpha\beta}(k, x) \tag{A.1}$$

$$\varphi^{T}_{\alpha\beta}(x) = \sum_{k}{}' \varphi^{T}(k) \varphi^{T}_{\alpha\beta}(k, x)$$

$$\chi(x) = \sum_{k}{}' \chi(k) s(k, x).$$

Here, $t^{\mathrm{cl}}_{\alpha\beta}$ is the classical solution of the linearised Einstein equations which matches the argument of the wavefunction on the surface $x^0 = T$ and vanishes on the other boundary surfaces. (We choose our finite volume box large enough so that the compact support of the initial data at $x_0 = T$ is interior to it.) The class of field configurations that is summed over for the additional piece of $t_{\alpha\beta}$ and the other tensor fields is specified by the boundary conditions and gauge choice described in §3. The boundary conditions are that the spatial components of the fields vanish at $x^0 = T$ and $x^0 = T - L$ and are periodic in the spatial directions. The gauge choice is equation (3.16). The modes on the right-hand side of (A.1) will be constructed to satisfy these conditions. The notation Σ_k' in (A.1) means the sum over all $k^0 > 0$ and $\boldsymbol{k}$ such that $k^i \neq 0$. Modes with $k^i = 0$ will have infinite action in the infinite volume limit and thus will not contribute to the functional integrals; we omit them for convenience in defining the tensor modes. To explicitly construct these modes we first define, for given k^α satisfying the above restrictions,

$$s(k, x) = \begin{cases} \dfrac{2}{L^2} \sin k_0(x^0 - T) \sin \boldsymbol{k} \cdot \boldsymbol{x} & k^3 > 0 \\ \dfrac{2}{L^2} \sin k_0(x^0 - T) \cos \boldsymbol{k} \cdot \boldsymbol{x} & k^3 < 0 \end{cases}$$

$$p_\alpha(k, x) = \frac{1}{k k_0} s^\gamma \nabla_\gamma \dot{s}(k, x) n_\alpha + \frac{k_0}{k} s(k, x) s_\alpha \tag{A.2}$$

$$p_{\alpha\beta}(k, x) = \frac{1}{k^2} \left[(s^\gamma \nabla_\gamma)^2 s n_\alpha n_\beta - 2 s^\gamma \nabla_\gamma \dot{s} n_{(\alpha} s_{\beta)} + \ddot{s} s_\alpha s_\beta \right]$$

where $s_\alpha(k)$ is the unit normalised projection of k^α onto the space orthogonal to n^α and $k = (k_\alpha k^\alpha)^{1/2}$. Using $\varepsilon^{(\nu)}_\alpha(k)$, two orthonormal vectors transverse to both k^α and n^α, we then construct the unit tensors

$$t^{(1)}_{\alpha\beta}(k) = \sqrt{2}\varepsilon^{(1)}_{(\alpha}\varepsilon^{(2)}_{\beta)} \qquad t^{(2)}_{\alpha\beta}(k) = \frac{1}{\sqrt{2}}(\varepsilon^{(1)}_\alpha\varepsilon^{(1)}_\beta - \varepsilon^{(2)}_\alpha\varepsilon^{(2)}_\beta)$$
$$\varphi^T_{\alpha\beta}(k) = \frac{1}{\sqrt{2}}(\varepsilon^{(1)}_\alpha\varepsilon^{(1)}_\beta + \varepsilon^{(2)}_\alpha\varepsilon^{(2)}_\beta). \tag{A.3}$$

Then the tensor modes are

$$\begin{aligned}
t^{(\nu)}_{\alpha\beta}(k,x) &= s(k,x)t^{(\nu)}_{\alpha\beta}(k) \qquad \nu = 1,2\\
\varphi^T_{\alpha\beta}(k,x) &= s(k,x)\varphi^T_{\alpha\beta}(k)\\
\varphi^{L(\nu)}_{\alpha\beta}(k,x) &= \frac{\sqrt{2}}{k}\varepsilon^{(\nu)}_{(\alpha}\nabla_{\beta)}s(k,x) \qquad \nu = 1,2\\
\varphi^{L(3)}_{\alpha\beta}(k,x) &= \frac{\sqrt{2}}{k}\nabla_{(\alpha}p_{\beta)}(k,x)\\
\varphi^{L(4)}_{\alpha\beta}(k,x) &= \frac{1}{k^2}\nabla_\alpha\nabla_\beta s(k,x)\\
l^{(\nu)}_{\alpha\beta}(k,x) &= \sqrt{2}\,\varepsilon^{(\nu)}_{(\alpha}p_{\beta)}(k,x) \qquad \nu = 1,2\\
l^{(3)}_{\alpha\beta}(k,x) &= \frac{1}{\sqrt{3}}\varphi^T_{\alpha\beta}(k,x) - \sqrt{\frac{2}{3}}\,p_{\alpha\beta}(k,x).
\end{aligned} \tag{A.4}$$

These tensor modes are real and normalised to one on the finite volume.

The real functions $t^{(\nu)}(k)$, $l^{(\nu)}(k)$, $\varphi^T(k)$, $\varphi^{L(\nu)}(k)$ and $\chi(k)$ become coordinates in the space of functions over which we integrate and the actions may be expressed in terms of them. For example, the action I^g (equation (3.19)) is

$$I^g = \sum_k{}' I^g(k)$$

where

$$I^g(k) = \frac{k^2}{4}\left[\sum_{\nu=1}^{3}(l^{(\nu)}(k))^2 + a(\chi(k))^2\right] \tag{A.5a}$$

and the action i_2 (equation (3.6)) is

$$i_2 = i_2^{\text{cl}}[h^{\text{TT}}] + \sum_k{}' i_2(k)$$

where

$$i_2(k) = \sum_{\nu=1}^{2} k^2[t^{(\nu)}(k)]^2 \tag{A.5b}$$

and $I_2^{\rm cl}[h^{\rm TT}]$ is the classical action evaluated for the appropriate classical solution. The functional δ-functions and Faddeev–Popov determinants are, in the gauge (3.16),

$$\begin{aligned} \delta[\nabla^\beta \varphi^L_{\alpha\beta}]\ \det\left[\left\|\frac{\delta C^\alpha}{\delta \xi_\beta}\right\|\right] &= \prod_k{}' D\left[\prod_{\nu=1}^{4} \delta(k\varphi^{L(\nu)}(k))\right][k^2]^4 \\ \delta[R(\varphi^T)]\ \det\left[\left\|\frac{\delta R}{\delta \omega}\right\|\right] &= \prod_k{}' D'\,\delta(k^2\varphi^T(k))[k^2] \end{aligned} \qquad \text{(A.6)}$$

where D and D' are numerical constants determined by the normalisation of the fields and implementation of the δ-functions and determinants. The products of factors $(k^2)^4$ and k^2 in equations (A.6) are Faddeev–Popov determinants when represented as modes. Defining the measures (3.20a,b) as

$$\delta l\,\delta\varphi^L\,\delta\chi = \prod_k{}' \frac{N}{D}\left[\prod_{\nu=1}^{3} \mathrm{d}l^{(\nu)}(k) \prod_{\mu=1}^{4} \mathrm{d}\varphi^{L(\mu)}(k)\right]\mathrm{d}\chi(k) \qquad \text{(A.7a)}$$

$$\delta\varphi^T = \prod_k{}' \frac{1}{D'}\,\mathrm{d}\varphi^T(k) \qquad \text{(A.7b)}$$

and using (A.5a) and (A.6), equations (3.20) become (suppressing the label k on the mode amplitudes)

$$1 = \int \prod_k{}' N\left[\prod_{\nu=1}^{3} \mathrm{d}l^{(\nu)}\right]\mathrm{d}\chi\left[\prod_{\mu=1}^{4} \mathrm{d}\varphi^{L(\mu)}\delta(k\varphi^{L(\mu)})\right][k^2]^4 \exp[-I^{\rm g}(k)] \qquad \text{(A.8a)}$$

$$1 = \int \prod_k{}' \mathrm{d}\varphi^T\delta(k^2\varphi^T)[k^2] \qquad \text{(A.8b)}$$

where N is a constant needed to set (A.8a) to unity including a factor of l^{-4} and the constant $a^{1/2}$ as well as other numerical constants.

Using the Fourier modes (A.1), the path integral over the physical degrees of freedom can also be made concrete. Defining the measure over the fluctuations around the classical solution to be

$$\delta t = \prod_k{}' \frac{\pi}{4l^2}\left[\prod_{\nu=1}^{2} \mathrm{d}t^{(\nu)}(k)\right] \qquad \text{(A.9)}$$

and using (A.5b), the wavefunctional is

$$\psi[h^{\rm TT}_{ij}, T] = \mathcal{N}\exp\{-I_2^{\rm cl}[h^{\rm TT}_{ij}]\}\int \prod_k{}' \frac{\pi}{4l^2}\left[\prod_{\nu=1}^{2} \mathrm{d}t^{(\nu)}(k)\right]\exp[-I_2(k)] \qquad \text{(A.10)}$$

where $\mathcal{N}$ is a normalisation parameter that can be computed explicitly by first evaluating the path integral over the fluctuations in the finite volume

L^4 and then fixing $\mathcal{N}$ by requiring that the resulting wavefunctional be the normalised product of ground state harmonic oscillator wavefunctions of the amplitudes of the Fourier transform of $h_{ij}^{\mathrm{TT}}(x)$ in the appropriate measure as $L \to \infty$. One finds that $\mathcal{N}$ is the properly normalised ground state wavefunctional for zero initial data on the boundary surface at $x_0 = T - L$.

Finally, for completeness we give the form of the parameterised wavefunctional (3.22) in the Fourier coordinates:

$$\psi_0[h_{ij}^{\mathrm{TT}}, T] = \mathcal{N}\left\{ \int \prod_k{}' \frac{\pi N}{4l^2} \mathrm{d}^{10}\varphi \, \mathrm{d}\chi \left[\prod_{\nu=1}^{4} \delta(k\varphi^{L(\nu)}) \right] \delta(k^2\varphi^T)[k^2]^5 \right\} \times \exp\{-\hat{I}[h_{ij}^{\mathrm{TT}}, t^{(\nu)}, l^{(\nu)}, \chi]\} \tag{A.11}$$

where

$$\hat{I} = i_2^{\mathrm{cl}}[h_{ij}^{\mathrm{TT}}] + \sum_k{}' [I^g(k) + i_2(k)]$$

and the measure

$$\prod_k{}' \frac{\pi N}{4l^2 DD'} \mathrm{d}^{10}\varphi \, \mathrm{d}\chi$$

is the product of (A.7a), (A.7b) and (A.9).

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