Copenhagen Quantum Mechanics as an Approximation to Decoherent Histories Quantum Mechanics

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Reproduced here two are modestly edited excerpts from the authors's lectures [1, 2] concerning the connection between decoherent histories quantum theory and the Copenhagen formulation of the quantum mechanics of measurement situations that is typically found in contemporary textbooks. A rudimentary knowledge of decoherent histories quantum mechanics and its notation is assumed such as can be found in [1]. The equation numbering corresponding to the original source has been preserved. The references have not ben updated.

Copenhagen (textbook) quantum mechanics predicts the probabilities of the histories of measured subsystems. Measurement situations can be described in a closed system that contains both measured subsystem and measuring apparatus. In a typical measurement situation the values of a variable not normally decohering become correlated with alternatives of the apparatus that decohere because of *its* interactions with the rest of the closed system. The correlation means that the measured alternatives decohere because the alternatives of the apparatus with which they are correlated decohere.

The recovery of the Copenhagen rule for when probabilities may be assigned is immediate. Measured quantities are correlated with decohering histories. Decohering histories can be assigned probabilities. Thus in the two-slit experiment (Figure 1) , when the electron interacts with an apparatus that determines which slit it passed through, it is the decoherence of the alternative configurations of the apparatus that enables probabilities to be assigned for the electron.

As far as we know, the predictions of Copenhagen quantum mechanics are correct in its domain of applicability — measurements. Copenhagen quantum mechanics is not opposed to the decoherent histories quantum mechanics of closed systems. Rather Copenhagen quantum mechanics is an *approximation* to the more general framework . It is an approximation that is appropriate in the special cases of measurement situations and when the decoherence of alternative configurations of the apparatus may be idealized as exact and instantaneous. However, while measurement situations imply decoherence, they are only special cases of decohering histories. Probabilities may be assigned to alternative positions of the moon and to alternative values of density fluctuations near the big bang in a universe in which these alternatives decohere, whether or not they were participants in a measurement situation and certainly whether or not there was an observer registering their values.

In conventional discussions of measurement in quantum mechanics it is useful to consider ideal models of the measurement process (See, e.g. von Neumann [3] London and Bauer [4] , Wigner [5] or almost any current text on quantum mechanics). Such models idealize various approximate properties of realistic measurement situations as exact features of the model. For example, configurations of an apparatus corresponding to different results of an experiment are typically represented by *exactly* orthogonal states in these models. This kind of ideal model is useful in isolating the essential features of many laboratory measurement situations in an easily analyzable way. Ideal measurement models are useful in quantum

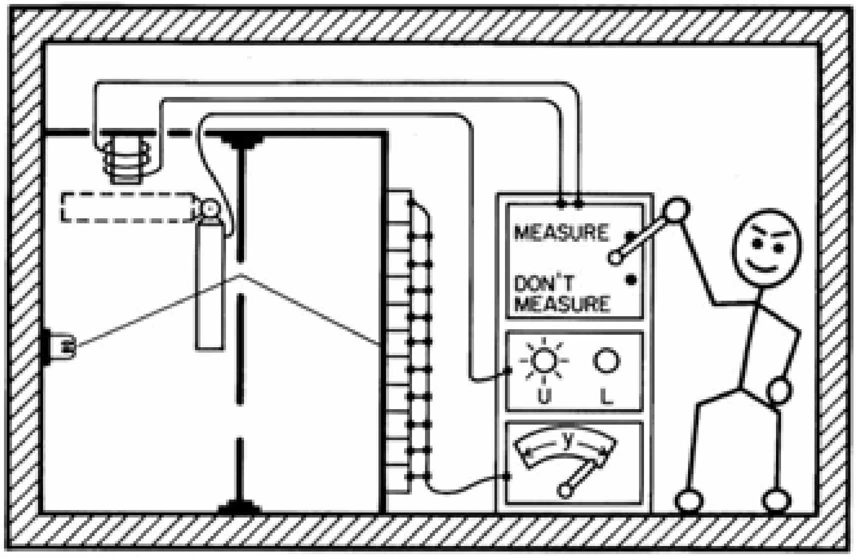

FIG. 1: A model closed quantum system containing an observer together with the necessary apparatus for carrying out a two-slit experiment. The observer can decide to measure whether the electron went through the upper slit by lowering a detector in front of it. Alternatives for the system include whether the observer measured which slit the electron passed through or did not, whether the electron passed through the upper or lower slit, the alternative positions of arrival of the electron at the screen, the alternative arrival positions registered by the apparatus, the registration of these in the brain of the observer, etc., etc., etc. Each exhaustive set of exclusive alternatives is represented by an exhaustive set of orthogonal projection operators on the Hilbert space of the closed system. Time sequences of such sets of alternatives describe sets of alternative coarse-grained histories of the closed system –deciding to measure, running the expeiment, recording the result, etc. Quantum theory assigns probabilities to the individual alternative histories in such a set when there is negligible quantum mechanical interference between them, that is, when the set of histories decoheres. A more refined model might consider a quantity of matter in a closed box. One could then consider alternatives such as whether the box contains a two-slit experiment, as is assumed here, or contains something else.

cosmology for the same reasons. They are useful in indicating how the Copenhagen formulation of quantum theory can be derived as an approximation to the quantum mechanics of the universe described here.

Consider a closed system one part of which is a subsystem to be studied and the rest of which can be organized into various types of measuring apparatus. The latter includes any "observer" that may be present. Corresponding to this division, we assume a Hilbert space that is a tensor product, $\mathcal{H}_s \otimes \mathcal{H}_r$, of a Hilbert space for the subsystem and a Hilbert space for the rest. We assume an "initial condition" that is a product of a density matrix for the subsystem in $\mathcal{H}_s$ and another for the rest in $\mathcal{H}_r$

$$\rho = \rho_s \otimes \rho_r \ . \tag{II.10.1}$$

Various sets of yes-no alternatives for the subsystem are represented by exhaustive and exclusive sets of projection operators $\{S^k_\alpha(t)\}$, $\alpha = 1, 2, 3, \cdots$. Their Schrödinger picture representatives are of the form $S^k_\alpha = s^k_\alpha \otimes I_r$ where the s^k_α are a set of projection operators acting on $\mathcal{H}_s$. Of course, since the subsystem and the rest are interacting, the Heisenberg picture representatives $S^k_\alpha(t)$ will not in general have this product form. The various possible configurations of an apparatus which measures the set of alternatives $\{S^k_\alpha(t)\}$ are described by an exhaustive set of alternatives for the rest $\{R^{(k,\tau)}_\beta(t)\}$, $\beta = 1, 2, 3\cdots$. The operator $R^{(k,\tau)}_\beta(t)$ corresponds to the alternative that the apparatus has recorded the alternative β for the subsystem studied in the set k at time τ. We can ask about the value of this record at any time and so $R^{(k,\tau)}_\beta(t)$ itself depends on t. For example, we could ask whether the record of the result of the measurement persists. The S's and the R's at the same time are assumed to commute with one another.

Two of the three crucial assumptions defining the ideal measurement model are the following:

i. *Correlation*: The alternatives $\{S^k_\alpha(t)\}$ and $\{R^{(k,\tau)}_\beta(t)\}$ are exactly correlated, that is

$$Tr\left[R^{(n,t_n)}_{\beta'_n}(t_n)S^n_{\alpha'_n}(t_n)\cdots R^{(1,t_1)}_{\beta'_1}(t_1)S^1_{\alpha'_1}(t_1)\rho S^1_{\alpha_1}(t_1)R^{(1,t_1)}_{\beta_1}(t_1)\cdots S^n_{\alpha_n}(t_n)R^{(n,t_n)}_{\beta_n}(t_n)\right] \propto \delta_{\alpha'_n\beta'_n}\cdots\delta_{\alpha'_1\beta'_1}\delta_{\alpha_1\beta_1}\cdots\delta_{\alpha_n\beta_n} \ . \qquad \text{(II.10.2)}$$

This is an idealization of the measurement situation correlations discussed in the previous section. The existence of such correlations is not inherent in the properties of the operators $\{S^k_\alpha(t)\}$ and $\{R^k_\alpha(t)\}$. Their existence depends also on the Hamiltonian and on choosing an initial ρ that models an experimental preparation of apparatus and subsystem and which will lead to a measurement situation. We assume in the model that we have a ρ and H of this character.[1]

ii. *Persistent Records*: We assume that, as a consequence of ρ and H, distinguishable, persistent, non-interacting *records* form of the results of the measurements of the various times t. The $R^{(k,\tau)}_\beta(t)$ describe the alternative values of these records at time t. More precisely we assume that if $t_2 > t_1$ are any two times later than τ then these operators have the property

$$Tr\left[\cdots R^{(k,\tau)}_{\beta'_2}(t_2)\cdots R^{(k,\tau)}_{\beta'_1}(t_1)\cdots\rho\cdots R^{(k,\tau)}_{\beta_1}(t_1)\cdots R^{(k,\tau)}_{\beta_2}(t_2)\cdots\right] \propto \delta_{\beta'_2\beta'_1}\delta_{\beta_1\beta_1} \ , \qquad \text{(II.10.3)}$$

where the elipses $(\cdots)$ stand for any combination of R's and S's in the correct time order. That is, the record projections effectively commute with all other projections at time $t > \tau$. Eq. (10.3) is the statement that values of the records at later times are exactly correlated with those of earlier times. An assumption like (10.3) is not needed if only one measurement situation at one time is to be discussed, as is common in models of the measurement process. It is needed for discussions of sequences of measurements, as here, to ensure that subsequent interactions do not reëstablish the coherence of different measurement alternatives.

The questions of interest in this model are whether the set of histories of "measured" alternatives $\{[S_\alpha]\}$ decoheres, and, if so, what their probabilities are. The answers are supplied by analysing the decoherence functional

$$D\left([S_{\alpha'}], [S_\alpha]\right) = Tr\left[S^n_{\alpha'_n}(t_n)\cdots S^1_{\alpha'_1}(t_1)\rho S^1_{\alpha_1}(t_1)\cdots S^n_{\alpha_n}(t_n)\right] \ . \qquad \text{(II.10.4)}$$

[1] A more realistic model would treat a more general ρ but include as the first set of projections in the string defining a history a set one member of which is the alternative "ready to measure the alternatives in the set k". The relevant probabilities defining the correlations of a measurement situation would then be conditioned on this alternative for the apparatus.

Alongside each $S^k_{\alpha_k}(t_k)$ in the above expression insert a resolution of the identy into record variables

$$\sum\nolimits_{\beta_k} R^{(k,t_k)}_{\beta_k}(t_k) = 1 \ . \tag{II.10.5}$$

Because of the assumption (i) of *exact* correlation between the subsystem alternatives $S^k_{\alpha_k}$ and the records [eq. (10.2)], only the term with $\beta_k = \alpha_k$ is this sum survives.

A consequence of condition (10.3) and the properties of projections (2.4) is that all the inserted R's can be dragged to the left of the decoherence functional and evaluated at the last time. The decoherence functional is then

$$D\left([S_{\alpha'}], [S_\alpha]\right) = Tr\Big[R^{(n,t_n)}_{\alpha'_n}(t_n)\cdots R^{(1,t_1)}_{\alpha'_1}(t_n) S^n_{\alpha'_n}(t_n)\cdots S^1_{\alpha'_1}(t_1)\rho$$
$$\times S^1_{\alpha_1}(t_1)\cdots S^n_{\alpha_n}(t_n) R^{(1,t_1)}_{\alpha_1}(t_n)\cdots R^{(n,t_n)}_{\alpha_n}(t_n)\Big] \ . \tag{II.10.6}$$

Then, since the record variables, $R^{(k,\tau)}_\beta$, are exclusive by construction, we may use the cyclic property of the trace to show that the off-diagonal terms in the α's of (10.6) vanish identically. The records decohere. However, since the records are exactly correlated with measured properties of the system studied according to assumption (i), this decoherence accomplishes the decoherence of the measured alternatives of the system. Thus, as a consequence of the *existence* of alternatives $\{R^{(k,\tau)}_\beta(t)\}$ with the properties (i) and (ii) the decoherence functional (10.4) is exactly diagonal and we can write

$$D\left([S_{\alpha'}], [S_\alpha]\right) = \delta_{\alpha'_n\alpha_n}\cdots\delta_{\alpha'_1\alpha_1} Tr\left[S^n_{\alpha_n}(t_n)\cdots S^1_{\alpha_1}(t_1)\rho S^1_{\alpha_1}(t_1)\cdots S^n_{\alpha_n}(t_n)\right] \tag{II.10.7}$$

Put differently, we can say that the decoherence of the records in the larger universe has accomplished the exact decoherence of the measured quantity of the subsystem studied.[2]

The third assumption of the ideal measurement model is the following:

(iii) *Measured Quantities are Undisturbed.* We assume that the diagonal elements of the decoherence functional (10.5), which give the probabilities of the histories $[S_\alpha]$, are the same as if they were calculated with the operators $s^k_\alpha(t)\otimes I_r$ where $s^k_\alpha(t)$ are the alternatives for the subsystem evolved with its *own* Hamiltonian. Thus,

$$p([S_\alpha]) = tr\left[s^n_{\alpha_n}(t_n)\cdots s^1_{\alpha_1}(t_1)\rho_s s^1_{\alpha_1}(t_1)\cdots s^n_{\alpha_n}(t_n)\right] \ . \tag{II.10.8}$$

where ρ_s, the projection operators $\{s^k_\alpha(t)\}$, and the trace tr refer to the Hilbert space $\mathcal{H}_s$. This is the assumption that the measurement interaction instantaneously reduces the off-diagonal elements of the subsystem's decoherence functional to zero while leaving the diagonal elements unchanged. The values of measured quantities are thus left undisturbed.

In idealized models of this kind, the fundamental formula is exact and the rule for assigning probabilities can be restated: *Probabilities can be assigned to histories that have been* **measured** *and the probability is* (10.8). This is the rule of the Copenhagen interpretations for assigning probabilities. Eq. (10.8) may be unfamiliar to those used to working

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[2] A slightly different idealization leading to the same result would be to assume that the correlations expressed in (10.2) are with projections $\{R^{(k,\tau)}_\alpha(t)\}$ that *always* exactly decohere because of the properties of the initial ρ no matter where located in a string of projections in the decoherence functional. Such variables are typically described as "macroscopic". There is some economy in such a sweeping idealization but the model of persistent records suggests a mechanism by which such decoherence might be accomplished.

with a state vector that evolves unitarily in between measurements and by reduction of the state vector at a measurement. In fact, it is a compact and efficient expression of these two forms of evolution as has been stressed by Groenewold (1952), Wigner (1963), Aharonov, Bergmann, and Lebovitz (1964), Unruh (1986), and Gell-Mann (1987) among others. I shall demonstrate this equivalence explicitly below but for the moment let us discuss the significance of the ideal measurement model.

The ideal measurement model shows how the Copenhagen rule for assigning probabilities fits into the more general post-Everett framework of quantum cosmology. The rule holds in the model because certain approximate features of some measurement situations have been idealized as exact. Specifically, these idealizations include the *exact* factorization of the initial density matrix ρ [eq. (10.2)], the *exact* correlation between measured system and registering apparatus [eq. (10.2)], and the *exact* persistence and independence of measurement records [eq. (10.3)]. In practice none of these idealizations will be *exactly* true. There are many typical experimental situations involving measurements at a single time, however, where they are true to an excellent approximation[3]. The further idealization that measured quantities are undisturbed almost never holds for measurements of microscopic quantities but is typical for measurements of macroscopic ones. For experimental situations where the idealizations of measurement model are approximately true, the Copenhagen rule supplies an *approximation* for the probabilities of the fundamental formula. The fundamental formula, however, applies more generally and precisely, for example, to situations in the early universe where nothing like the idealizations of this measurement model may be appropriate.

Let's return to the equivalence of eq.(10.8) with the usual picture of a unitarily evolving state vector reduced on measurement. To see the equivalence let us calculate the probability for a sequence of just two measurements at times t_1 and t_2 according to the usual story in the Heisenberg picture, given and an initial pure $\rho_s = |\psi >< \psi|$ at time t_0. The state $|\psi>$ is constant from t_0 to t_1. The probability that the outcome of the first measurement is α_1 is

$$p(\alpha_1) = <\psi|s^1_{\alpha_1}(t_1)|\psi> \ . \tag{II.10.9}$$

The normalized state after the measurement is reduced to

$$|\psi_{\alpha_1}> = \frac{s^1_{\alpha_1}(t_1)|\psi>}{\sqrt{<\psi|s^1_{\alpha_1}(t_1)|\psi>}} \ . \tag{II.10.10}$$

The probability of obtaining the result α_2 on the next measurement given the result α_1 on the first is

$$p(\alpha_2|\alpha_1) = <\psi_{\alpha_1}|s^1_{\alpha_1}(t_2)|\psi_{\alpha_1}\rangle = \frac{<\psi|s^1_{\alpha_1}(t_1)s^2_{\alpha_2}(t_2)s^1_{\alpha_1}(t_1)|\psi>}{<\psi|s^1_{\alpha_1}(t_1)|\psi>} \ . \tag{II.10.11}$$

The *joint* probability for α_2 followed by α_1 is

$$p(\alpha_2,\alpha_1) = p(\alpha_2|\alpha_1)p(\alpha_1) \ . \tag{II.10.12}$$

so that using (10.9) and (10.11) we have

$$p(\alpha_2,\alpha_1) = <\psi|s^1_{\alpha_1}(t_1)s^2_{\alpha_2}(t_2)s^1_{\alpha_1}(t_1)|\psi> \ . \tag{II.10.13}$$

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[3] See Section II.11 of [2] for some estimates of the degree of approximation.

This is just the formula (10.8) for the Copenhagen probabilities for the special case of a history with two times and a pure initial density matrix ρ_s.

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