

PROBABILITIES FOR SINGLE EVENTS

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A slightly edited excerpt from the author's lectures *The Quantum Mechanics of Cosmology* in *Quantum Cosmology and Baby Universes: Proceedings of the 1989 Jerusalem Winter School for Theoretical Physics* (edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg), World Scientific, Singapore (1991).

II.1. Probability

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Even apart from quantum mechanics, there is no certainty in this world and therefore physics deals in probabilities. It deals most generally with the probabilities for alternative time histories of the universe. From these, conditional probabilities appropriate when information about our specific history is known may be constructed.

To understand what these probabilities mean, it is best to understand how they are used. We deal, first of all, with probabilities for *single* events of the *single* system that is the universe as a whole. When these probabilities become sufficiently close to zero or one there is a definite prediction on which we may act. How sufficiently close to 0 or 1 the probabilities must be depends on the circumstances in which they are applied. There is no certainty that the sun will come up tomorrow at the time printed in our daily newspapers. The sun may be destroyed by a neutron star now racing across the galaxy at near light speed. The earth's rotation rate could undergo a quantum fluctuation. An error could have been made in the computer that extrapolates the motion of the earth. The printer could have made a mistake in setting the type. Our eyes may deceive us in reading the time. We watch the sunrise at the appointed time because we compute, however imperfectly, that the probability of these things happening is sufficiently low.

Various strategies can be employed to identify situations where probabilities are near zero or one. Acquiring information and considering the conditional probabilities based on it is one such strategy. Current theories of the initial condition of the universe predict almost no probabilities near zero or one without further conditions. The "no boundary" wave function of the universe, for example, does not predict the present position of the sun on the sky. It will predict, however, that the conditional probability for the sun to be at the position predicted by classical celestial mechanics given a few previous positions is a number very near unity.

Another strategy to isolate probabilities near 0 or 1 is to consider ensembles of repeated observations of identical subsystems. There are no genuinely infinite ensembles in the world so we are necessarily concerned with the probabilities for deviation of a finite ensemble from the expected behavior of an infinite one. These are probabilities for a single feature (the deviation) of a single system (the whole ensemble). To give a quantum mechanical example, consider an ensemble of N spins each in a state $|\psi\rangle$. Suppose we measure whether the spin is up or down for each spin. The predicted relative frequency of finding n_{\uparrow} spin-ups is

$$f_N^{\uparrow} = \frac{n_{\uparrow}}{N} = |\langle \uparrow | \psi \rangle|^2, \quad (\text{II.1.1})$$

where $|\uparrow\rangle$ is state with the spin definitely up. Of course, there is no certainty that we will get this result but as N becomes large we expect the probability of significant deviations away from this value to be very small.

In the quantum mechanics of the whole ensemble this prediction would be phrased as follows: There is an observable f_N^\uparrow corresponding to the relative frequency of spin up. Its operator is easily defined on the basis in which all the spins are either up or down as

$$f_N^\uparrow = \sum_{s_1 \cdots s_N} |s_1 \rangle \cdots |s_N \rangle \left(\sum_i \frac{\delta_{s_i}^\uparrow}{N} \right) \langle s_n | \cdots \langle s_1 | . \quad (\text{II.1.2})$$

Here, $|s \rangle$, with $s = \uparrow$ or \downarrow , are the spin eigenstates in the measured direction. The eigenvalue in brackets is just the number of spin ups in the state $|s_1 \rangle \cdots |s_N \rangle$. The operator f_N^\uparrow thus has the discrete spectrum $1/N, 2/N, \cdots, 1$. We can now calculate the probability that f_N^\uparrow has one of these possible values in the state

$$|\Psi \rangle = |\psi \rangle \cdots |\psi \rangle \quad (N \text{ times}) , \quad (\text{II.1.3})$$

which describes N independent subsystems each in the state $|\psi \rangle$. The result is simply a binomial distribution. The probability of finding relative frequency f is

$$p(f) = \binom{N}{fN} p_\uparrow^{fN} p_\downarrow^{N(1-f)} \quad (\text{II.1.4})$$

where $p_\uparrow = |\langle \uparrow | \psi \rangle|^2$ and $p_\downarrow = 1 - p_\uparrow$. As N becomes large this approaches a continuum normal distribution that is sharply peaked about $f = p_\uparrow$. The width becomes arbitrarily small with large N as $N^{-1/2}$. Thus, the probability for finding f in some range about p_\uparrow can be made close to one by choosing N sufficiently large yielding a definite prediction for the relative frequency. In a given experiment how large does N have to be before the prediction is counted as definite? It must be large enough so the probability of error is sufficiently small to isolate a result of significance given the status of competing theories, competing groups, the consequences of a lowered reputation if wrong, the limitations of resources, etc.

The existence of large ensembles of repeated observations in identical circumstances and their ubiquity in laboratory science should not obscure the fact that in the last analysis physics must predict probabilities for the single system which is the ensemble as a whole. Whether it is the probability of a successful marriage, the probability of the present galaxy-galaxy correlation function, or the probability of the fluctuations in an ensemble of repeated observations, we must deal with the probabilities of single events in single systems. In geology, astronomy, history, and cosmology, most predictions of interest have this character.

The goal of physical theory is, therefore, most generally to predict the probabilities of histories of single events of a single system. Such probabilities are, of course, not measurable quantities. The success of a theory is to be judged by whether its definite predictions (probabilities sufficiently close to 0 or 1) are confirmed by observation or not.

Probabilities need be assigned to histories by physical theory only up to the accuracy they are used. Two theories that predict probabilities for the sun not rising tomorrow at its classically calculated time that are both well beneath the standard on which we act are equivalent for all practical purposes as far as this prediction is concerned. For example, a model of the Earth's rotation that includes the gravitational effects of Sirius gives different probabilities from one which does not, but ones which are equivalent for all practical purposes to those of the model in which this effect is neglected.

The probabilities assigned by physical theory must conform to the standard rules of probability theory: The probability for both of two exclusive events is the sum of the probabilities

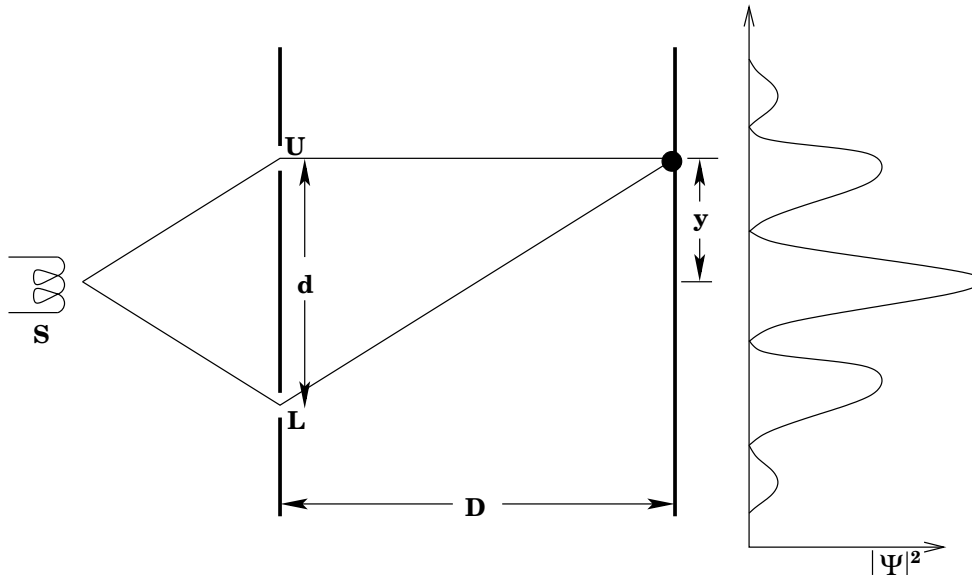


FIG. 1: The two-slit experiment. An electron gun at left emits an electron which is detected at a point y on a screen after passing through another screen with two slits. Because of quantum interference, it is not possible to assign probabilities to the alternative histories in which the electron arrives at y having gone through the upper or lower slit. The probability to arrive at y should be the sum of the probabilities of the two histories. But in quantum mechanics probabilities are squares of amplitudes and $|\psi_L(y) + \psi_U(y)|^2 \neq |\psi_L(y)|^2 + |\psi_U(y)|^2$. In a different physical situation where the electron interacts with apparatus that measures which slit it passed through, then quantum interference is destroyed and consistent probabilities can be assigned to these histories.

for each. The probabilities of an exhaustive set of alternatives must sum to unity. The probability of the empty alternative is zero. Because probabilities are meaningful only up to the standard by which they are used, it is useful to consider *approximate probabilities* which need satisfy the rules of probability theory only up to the same standard. A theory which assigns approximate probabilities in this sense could always be augmented by a prescription for renormalizing the probabilities so that the rules are exactly obeyed without changing their values in any relevant sense.

II.1.2. Probabilities in Quantum Mechanics

The characteristic feature of a quantum mechanical theory is that not every history that can be described can be assigned a probability. Nowhere is this more clearly illustrated than in the two slit experiment. In the usual “Copenhagen” discussion if we have not measured which of the two slits the electron passed through on its way to being detected at the screen, then we are not permitted to assign probabilities to these alternative histories. It would be inconsistent to do so since the correct probability sum rule would not be satisfied. Because of interference, the probability to arrive at y is not the sum of the probabilities to arrive at y going through the upper or lower slit:

$$p(y) \neq p_U(y) + p_L(y) \quad (\text{II.1.5})$$

because

$$|\psi_L(y) + \psi_U(y)|^2 \neq |\psi_L(y)|^2 + |\psi_U(y)|^2 \quad . \quad (\text{II.1.6})$$

If we *have* measured which slit the electron went through, then the interference is destroyed, the sum rule obeyed, and we *can* meaningfully assign probabilities to these alternative histories.

We cannot have such a rule in quantum cosmology because there is not a fundamental notion of “measurement”. There is no fundamental division into observer and observed and no fundamental reason for the existence of classically behaving measuring apparatus. In particular, in the early universe none of these concepts seem relevant. We need an observer-independent, measurement-independent rule for which histories can be assigned probabilities and which cannot. That rule is the condition for the decoherence of histories.