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Numerical Quantum Gravity

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ABSTRACT

Potential and actual applications of numerical simulations to issues in quantum gravity are discussed and compared with numerical simulations in the classical theory.

Numerical simulations are becoming an increasingly powerful and important tool in classical general relativity. They are increasingly powerful because of the accessibility of high speed computers and the efforts of many investigators[1] in developing the techniques with which to use them to solve Einstein's equation. They are increasingly important because for most astrophysical systems we lack the analytic methods and certainly the controlled experiments by which to answer the interesting questions.

Numerical simulation is also likely to be a powerful and important tool in the quantum theory of gravity. The reasons are similar to those which occur in the classical theory. The systems of interest, for example the universe, are complex, the interesting questions are only accessible qualitatively to analytic analysis. It is certainly difficult to imagine controlled experiments.

There are significant differences in the typical issues of the classical and quantum theories of gravity which lead to different numerical problems and therefore to different numerical techniques. The first major difference between classical and quantum gravity is that in the latter we do not have the theory. In classical gravity we have Einstein's well tested theory. We ask of this standard theory a variety of complex questions. Einstein's action, however, does not lead to a satisfactory quantum field theory. In quantum gravity we are therefore often engaged in the enterprise of presenting simple, standard questions to a variety of theories.

The second major difference between classical and quantum gravity is that in classical gravity we are typically interested in dealing with particular geometries while in quantum gravity we are interested in geometries "wholesale." The objects of interest in quantum gravity are quantum amplitudes. For example, one might be interested in scattering amplitudes in which gravitational processes play a part. In quantum cosmology one is interested in the wave function of the universe. These amplitudes are conveniently expressed as sums over geometries and it is in this form that they are most accessible to numerical approximation. For example, a possible wave function of the universe defined on the configuration space of three geometries is[2]

$$\Psi[{}^3\mathcal{G}] = \sum_{{}^4\mathcal{G}} \exp(-I[{}^4\mathcal{G}]), \qquad (1)$$

where I is the Euclidean action and the sum is over compact Euclidean 4-geometries ${}^4\mathcal{G}$ with a single boundary on which the induced 3-geometry is ${}^3\mathcal{G}$. In classical gravity we are interested in particular solutions of Einstein's differential equations. In quantum gravity we are interested in sums over classes of geometries.

This difference in aim dictates a difference in technique. In classical relativity we typically efficiently calculate using a differencing scheme adapted to the particular problem. In quantum gravity we need a general discretization method for dealing with geometries wholesale. Simplicial approximation and the methods of Regge calculus provide this.

A simplicial geometry is made up of flat simplices joined together. A two dimensional surface can be made out of flat triangles. A three dimensional manifold can be built out of tetrahedra; in four dimensions one uses 4-simplices and so on. The information about topology is contained in the rules by which the simplices are joined together. A metric is provided by an assignment of edge lengths to the simplices and a flat metric to their interiors.

A two dimensional surface made up of triangles is in general curved as, for example, the surface of the tetrahedron in Figure 1. The curvature is concentrated at the vertices because one cannot flatten the triangles meeting in a vertex without cutting one of the edges. If one does cut one edge and flatten, the angle by which they fail to meet is a measure of the curvature called the deficit angle. (See Figure 1). Concretely, the deficit angle is 2π minus the sum of the interior angles of the

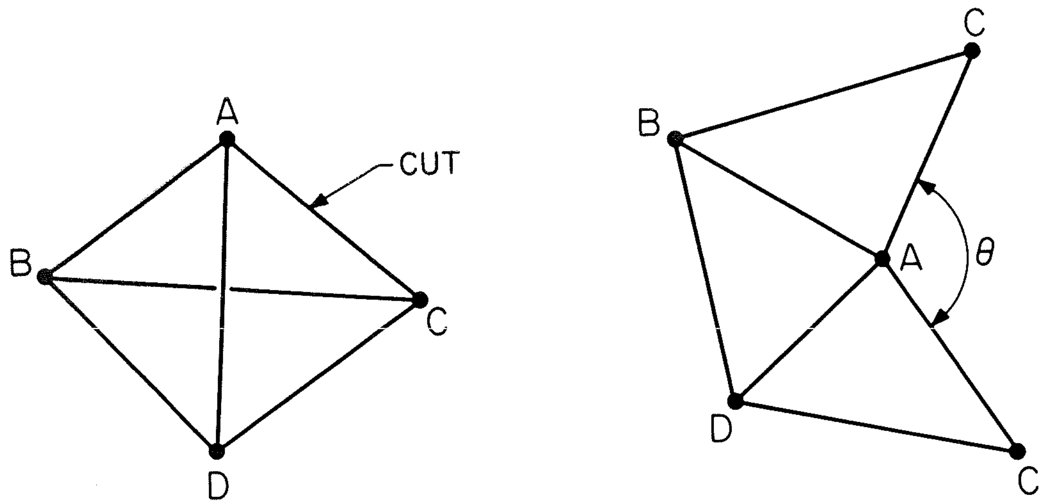

Figure 1: The surface of a tetrahedron is a two dimensional surface whose curvature is concentrated at its vertices. To flatten the three triangles meeting at verges A one could cut the tetrahedron along edge AC. The angle θ by which the edges AC fail to meet when flattened is a measure of the curvature at A called the deficit angle.

triangles meeting at the vertex. It can thus be expressed as a function of their edge lengths.

In four dimensions the situation is similar with all dimensions increased by 2. The geometry is built from flat 4-simplices. Curvature is concentrated on the two dimensional *triangles* in which they intersect. There is a deficit angle associated with each triangle which is 2π minus the sum of the interior angles between the bounding tetrahedra of the 4-simplices which intersect the triangle.

As Regge showed,[3] Einstein's familiar gravitational action may be expressed as a function of the deficit angles and the volumes of the simplices. For example, the Euclidean Einstein action with cosmological constant for a connected closed manifold in n-dimensions is

$$g_n \ell^{n-2} I_n = -\int d^n x (g)^{\frac{1}{2}} (R - 2\Lambda). \tag{2}$$

Here, $\ell = (16\pi G)^{\frac{1}{2}}$ is the Planck length and g_n is dimensionless coupling. We use units where $\hbar = c = 1$ throughout. On a simplicial geometry (2) becomes exactly

$$g_n \ell^{n-2} I_n = -2 \sum_{\sigma \in \Sigma_{n-2}} V_{n-2}(\sigma)\theta_{n-2}(\sigma) + 2\Lambda \sum_{\tau \in \Sigma_n} V_n(\tau). \tag{3}$$

Here Σ_k is the collection of k-simplices and V_k is the volume of a k-simplex. The deficit angle θ_{n-2} is defined by

$$\theta_{n-2}(\sigma) = 2\pi - \sum_{\tau \supset \sigma} \theta_k(\sigma, \tau), \tag{4}$$

where the sum is over all the n-simplices τ which meet σ, and the $\theta_{n-2}(\sigma, \tau)$ are their interior angles at σ. Both V_k and $\theta_{n-2}(\sigma, \tau)$ are simply expressible in terms of the edge lengths through standard flat space formulae. By using these expressions in (3) the action becomes a function of the edge lengths.

As Hamber and Williams have shown[4] other gravitational actions, such as curvature squared Lagrangians, may be similarly expressed in an approximate form which becomes exact in the continuum limit. A similar formalism holds for Lorentzian geometries in which some of the squared edge lengths become "time-like" and, as a consequence, some of the angles become hyperbolic.[5] There is also an approximate $3+1$ version of the theory in which time is continuous but space is built up out of tetrahedra.[6,7]

A typical calculation of the gravitational action is shown in Figure 2. The manifold is CP^2 in the beautiful triangulation of Kühnel and Lassmann[8], CP^2_9. This has 9 vertices, 36 edges, 84 triangles, 90 tetrahedra and 36 4-simplices. Under the symmetry group of the triangulation the edges fall into two classes — 9 in one class (class I) and 27 in another (class II). Figure 2 shows the action[9] when all the class I edges have a value L_I and all the class edges a value L_{II}. At the value $L_I = L_{II} = (2.14)(\ell/H)$, where $H^2 = \ell\Lambda^2/3$, there is a saddle point — an extremum of the action and thus a solution of the discrete analog of Einstein's equation.

Extremizing the action with respect to the edge lengths gives a set of *algebraic* equations

$$\frac{\partial I}{\partial s_i} = 0, \qquad i = 0, \ldots n_1, \tag{5}$$

equal in number to the number of edge lengths. These are the discrete version of Einstein's equation.[10] In a sense they constitute a general differencing scheme for the Einstein equation. By and large, however, while some classical solutions have been obtained with these equations,[11,12] they have not proved to be the most efficient differencing scheme. The reason is that the Regge calculus is general. For any particular problem, *e.g.*, axisymmetric rotating collapse, better adapted schemes can typically be found. In addition, as we shall see, Regge calculus is particularly poorly adapted to dealing with asymptotically flat geometries.

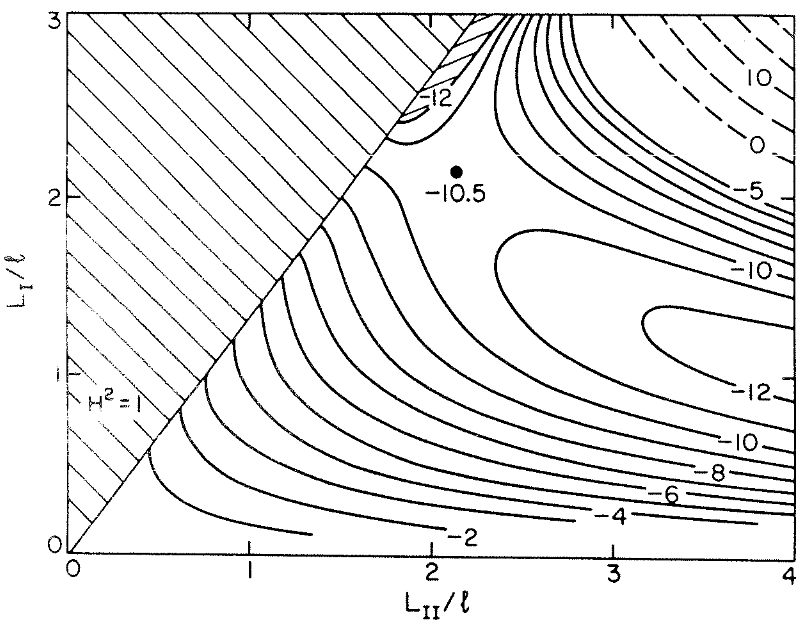

Figure 2: The action on a triangulation of CP^2. The nine vertex triangulation CP^2_9 has two classes of edges which transform among themselves under the action of its symmetry group. The figure shows a contour map of the action (divided by 100) when all the edges of the first class have the value L_I and all of the second class have the value L_{II}. The cosmological constant has the value specified by $H^2 = 1$. In the shaded region at upper left the simplicial inequalities are violated. There is a saddle point extremum and a solution of the Regge equations when $L_I = L_{II} = 2.14\ell/H^2$.

The Regge calculus *can* be useful in the classical theory when one is interested in dealing with geometries wholesale. A good example is the recent investigation of Piran and Williams into the evolution of cosmologies driven by the energy of a massive scalar field.[13] They used the $3+1$ Regge formalism to examine the space of initial conditions and show that a large class of them lead to an inflationary expansion.

If one attempts to solve the Regge equations on larger and larger simplicial nets in an effort to get a better and better approximation to a classical solution it becomes more and more difficult to do. Assume, for example, one uses the

Newton-Raphson method to solve the algebraic equations,

$$f_i(s_j) = \frac{\partial I}{\partial s_j} = 0. \tag{6}$$

From any given point $\vec{s}$ in the space of squared edge lengths one seeks a displacement $\vec{\Delta s}$ to a new set of edge lengths which satisfy the equation. Expanding the condition $\vec{f}(\vec{s} + \Delta\vec{s}) = 0$ to linear order in $\overrightarrow{\Delta s}$ to find

$$\overrightarrow{\Delta s} = -\left(\mathbf{I}^{(2)}\right)^{-1} \vec{f}(\vec{s}), \qquad (\mathbf{I}^{(2)})_{ij} = \frac{\partial^2 I}{\partial s_i \partial s_j}. \tag{7}$$

By iterating this equation one hopes to converge to a solution for given n_1 and then with larger and larger nets to get a better and better approximation to a continuum solution.

As the net is made larger, however, $\mathbf{I}^{(2)}$ becomes increasingly close to singular. The reason is the diffeomorphism invariance of the continuum theory. Regge calculus is formulated directly in terms of the edge lengths — physical gauge invariant quantities. For a general curved simplicial geometry we expect if we change the edge lengths we will change the geometry. There is however an important exception to this and that is flat space. Imagine scattering some vertices through a region of flat space, connecting them with edges to form a simplicial net, and assigning lengths to these edges using the flat metric. Now displace the vertices in some manner. One obtains a new assignment of edges but the same geometry and same action. There is a $4 \times$ (number of vertices) parameter family of such assignments.

In increasingly refined simplicial approximations to a curved geometry there will eventually be regions with large numbers of vertices in which the geometry is approximately flat. There will therefore be variations of the edge lengths which leave the action approximately unchanged. These directions, $4 \times$ (the number of vertices) in number, correspond in the continuum limit to the 4-fold per point family of coordinate transformations under which the continuum action is invariant. Thus, while there is not an exact notion of gauge invariance in the Regge calculus, there is an approximate invariance which in the continuum limit becomes the diffeomorphism invariance of the continuum theory.[14]

The approximate invariance of the action means that $\mathbf{I}^{(2)}$ will have small eigenvalues and become increasingly singular as the continuum limit is approached.

This is the discrete realization of the fact that the continuum Einstein equation does not determine a solution uniquely but only up to a coordinate transformation.

I would now like to turn to the central problem of numerical quantum gravity — sums over geometries. Consider, by way of example, the sum over histories which gives the expectation value of some physical quantity A in the quantum state of the universe (1). Formally we would write this as

$$\langle A \rangle = \frac{\Sigma_{^4\mathcal{G}} A[^4\mathcal{G}] \exp(-I[^4\mathcal{G}])}{\Sigma_{^4\mathcal{G}} \exp(-I[^4\mathcal{G}])}, \tag{8}$$

where I is the Euclidean action and the sum is over compact $^4\mathcal{G}$ without boundary. To define such a sum there are three things to specify: the action, the $^4\mathcal{G}$'s which contribute, and the measure on the space of $^4\mathcal{G}$'s.

A geometry is a manifold with a metric. In general a sum over geometries will therefore involve a sum over manifolds as well as a sum over metrics. This sum over manifolds is usually called "summing over topology."

Giving meaning to formal sums such as (8) is the same problem as understanding how to compute them numerically. Sums over geometries can be given a concrete meaning by taking limits of sums of simplicial approximations to them. This is analogous to defining the Riemann integral of a function as the limit of sums of integrals of precise linear approximations to the function. For example, simplicial approximation could be used to give a concrete meaning to the formal sum for $\langle A \rangle$ as follows. (1) Fix a size of a simplicial net, say the number of vertices n_0. (2) Represent each manifold by a triangulation with n_0 vertices. (3) Approximate the sum over physically distinct metrics by a multiple integral over the squared edge lengths s_i. (4) Take the limit of these sums as n_0 goes to infinity. In short, express $\langle A \rangle$ as

$$\langle A \rangle = \lim_{n_0 \to \infty} \frac{\sum_{M(n_0)} \int_C d\Sigma_1 A(s_i, M) \exp[-I(s_i, M)]}{\sum_{M(n_0)} \int_C d\Sigma_1 \exp[-I(s_i, M)]}. \tag{9}$$

There remains the specification of the measure $d\Sigma_1$ and the contour C for the integral over edge lengths. Of course, today we understand little about the convergence of such a process but is is at lest definite enough to be discussed. We can hope to carry out the multiple integral over edge lengths by standard techniques, *e.g.,* the Monte-Carlo method, for each representative manifold, and sum the results up.

The sum over manifolds gives rise to some interesting questions.[15] The classification of 4-manifolds is an example of an undecidable question in mathematics. That is, roughly speaking, given two simplicial 4-manifolds A and B, each with n_0 vertices, there does not exist a computer program COMPARE(A, B, n_0) which will run, halt and print "yes" if the manifolds are the same and "no" if they are not. Nothing, however, prevents us from classifying 4-manifolds with any fixed, finite number of vertices. It is only that the techniques used to classify manifolds with one n_0 will not work for larger values. The job of numerically summing over manifolds will never become routine.[16]

Hamber and Williams have carried out pioneering Monte-Carlo sums over metrics on a 4-torus using a Regge calculus approximation to curvature squared gravitational actions.[17] Their nets were impressively large, having for example, several thousand edges. Space does not permit a review of all their interesting and suggestive results. As an example, however, one might note their calculation of "random geometry" — expectation values computed as in (8) with zero action! They found that expected dimensionless measures of the curvature squared were much larger than corresponding expected measures of the curvature indicating that random geometries are "rough" in the mean. Their calculations while preliminary in interpretation do show conclusively that numerical sums over metrics can be done.

An area in which numerical quantum gravity is likely to be of importance is quantum cosmology although few calculations have been attempted. Most proposals for a quantum state of the universe are explored in the context of minisuperspace models based on symmetries. To construct such models one restricts attention to geometries of particular symmetries, depending on fewer parameters, evaluates the action in terms of these parameters, and takes it to describe a mechanical system whose quantum mechanics one can construct and discuss. Minisuperspace models are easy to implement, generally easy to interpret, and usually suggestive. However, with their drastic truncations they are unlikely to be accurate approximations. Based on symmetries, they are not systematically improvable and necessarily suggest information about the quantum state only in a small region of superspace near the universe we know.

Regge calculus provides a potentially much more powerful way to explore the quantum state of the universe.[14] In a sense any given simplicial 4-manifold specifies a minisuperspace approximation in which the infinite number of degrees of freedom are replaced by a finite number of edge lengths. These models however, offer the prospect of systematic improvements by making the triangulation finer and finer. Further because they are not based on symmetries they can be used to explore the wave function in domains far from those close to the present universe. For problems such as these, numerical simulations may play as important a role in the quantum theory of gravity as they already do in the classical theory.

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