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Real tunneling geometries and the large-scale topology of the universe

G. W. Gibbons
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England

J. B. Hartle
Department of Physics, University of California, Santa Barbara, California 93106
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If the topology and geometry of spacetime are quantum-mechanically variable, then the particular classical large-scale topology and geometry observed in our universe must be statistical predictions of its initial condition. This paper examines the predictions of the "no boundary" initial condition for the present large-scale topology and geometry. Finite-action real tunneling solutions of Einstein's equation are important for such predictions. These consist of compact Riemannian (Euclidean) geometries joined to a Lorentzian cosmological geometry across a spacelike surface of vanishing extrinsic curvature. The classification of such solutions is discussed and general constraints on their topology derived. For example, it is shown that, if the Euclidean Ricci tensor is positive, then a real tunneling solution can nucleate only a single connected Lorentzian spacetime (the unique conception theorem). Explicit examples of real tunneling solutions driven by a cosmological constant are exhibited and their implications for cosmic baldness described. It is argued that the most probable large-scale spacetime predicted by the real tunneling solutions of the "no-boundary" initial condition has the topology $\mathbf{R}\times S^3$ with the de Sitter metric.

I. INTRODUCTION

A geometry is a manifold with a metric. In a sum-over-histories quantum mechanics of spacetime it is, therefore, as natural to sum over manifolds as it is over metrics. Admitting different manifolds to the sums over geometries defining quantum amplitudes means allowing different possibilities for the topology of physical spacetime and allowing also for the possibility of quantum transitions between the topology of space at one time and another.[1] This raises interesting questions.

As far as we know today, there are no certain reasons of principle or consistency which mandate the inclusion of topology as a variable in the quantum mechanics of spacetime. The idea has been opposed by some as inconsistent with the spirit of field theory.[2] However, the seminal considerations of Wheeler[1] and the suggestive work of Hawking on the evaporation of black holes[3] have provided considerable physical motivation for this mathematically natural idea. Further, allowing quantum amplitudes for different topologies may have important consequences even on familiar scales. The most striking of these is certainly the suggestion of Hawking,[4] Coleman,[5] Giddings and Strominger[6] and others that the inclusion of manifolds with wormholes in a sum-over-histories prescription for the universe's initial condition can lead to a determination, in part, of the coupling constants of the low-energy effective interactions of the elementary particles. In particular, the effective cosmological constant may be forced to a vanishing value.

If different topologies and metrics are allowed quantum mechanically, what determines the topology and metric of spacetime on the scales of the laboratory and the scales of cosmology? There is no evidence that any topologies other than the simplest possibilities (e.g., $\mathbf{R}\times S^3$) are realized on the scales and domains of the universe accessible to us. The metric properties are similarly simple (homogeneous and isotropic on average). The answer to these questions, such as that of why spacetime behaves classically at all in these regimes, is not to be found generally in the quantum theory of gravity. For, if metric and manifold are quantum-mechanically variable, the number of states which imply a particular behavior of the metric (e.g., classical) or a particular manifold (e.g., $\mathbf{R}\times S^3$) are but a negligible fraction of the total number of states available to the universe. Rather, the answer is to be sought in the theory of the initial condition of the universe which prescribes a particular quantum state which predicts probabilities for these possibilities.[7]

The "no-boundary" proposal is a promising theory of the initial condition.[8] Here, among other amplitudes, the wave function of a closed universe on a spacelike surface consisting of n disconnected parts is prescribed as a sum over histories of the form[9]

$$\Psi_0(h_1,\chi_1,\partial M_1,\ldots,h_n,\chi_n,\partial M_n) = \sum_M \nu(M)\int_{\mathcal{C}}\delta g\,\delta\phi\,\exp(-I[g,\phi,M])\ . \tag{1.1}$$

The arguments of the wave function are the spacelike three-metrics h_A, and matter fields χ_A, on the disconnected parts of the three-manifold ∂M_A, $A=1,\ldots,n$. The functional I is the Euclidean action for metric g and matter field configuration ϕ on a four-manifold M. The sum over manifolds, weighted by $\nu(M)$, is over a class which have the boundaries ∂M_A and no other boundaries. The functional integral is over four-metrics g and matter field configurations ϕ which induce h_A and χ_A on ∂M_A. To make a construction such as this definite, the class of manifolds, the measure for the functional integrals, and the contour $\mathcal{C}$ over which these integrals are taken must be specified. Various possibilities have been suggested for these.[10–12] The "no-boundary" proposal, such as any other theory of the initial condition, has the heavy burden to show that when the universe is large it predicts classical spacetime with a topology and metric consistent with observations on familiar and large scales.

Classical spacetime is predicted by a theory of the initial condition when two requirements are satisfied. (i) There is negligible interference between alternative histories for spacetime geometry determined on scales far above the Planck length. That is, the alternative histories decohere.[14] (ii) The geometries of successive spacelike surfaces in these histories are highly correlated according to classical laws. There may be many initial conditions which lead to decoherence and classical correlations. However, classical correlations are most commonly signaled in quantum cosmology when the wave function of the universe is well approximated semiclassically.[15] Consider, for example, a spacelike surface with a single component, ∂M. Suppose that $\Psi_0[h,\chi,\partial M]$ is well approximated by the form

$$\Psi_0[h,\chi,\partial M] \approx \sum_M \nu(M)\Delta[h,\chi,\partial M,M]\cos(S[h,\chi,\partial M,M]) \tag{1.2}$$

with S being a classical action satisfying the Hamilton-Jacobi equation of classical Einstein theory on real Lorentzian spacetimes. For the validity of the approximation the action S must vary much more rapidly with its arguments h and χ than Δ. Such a wave function predicts that the universe has one among the manifolds in the sum (1.2) and one among the Lorentzian metrics on that manifold that could produce the classical action S. Manifolds and metrics with greater weight $\nu\Delta$ are more likely than those with lower values. If, for example, only a single manifold M contributed significantly to the sum (1.2) we would predict a definite topology for the universe in the large.

The situation is similar if the spacelike surface consists of two disconnected pieces. If $\Psi_0[h_1,\chi_1,\partial M_1;h_2,\chi_2,\partial M_2]$ were well approximated semiclassically analogously to (1.2), then the theory of initial condition could predict two disconnected universes with classical evolutions corresponding to the contributing action weighted by the corresponding prefactors. The relative likelihood of two universes or one would be established by comparing this amplitude to (1.2).

Approximate forms such as (1.2) arise from the steepest-descents approximation to the defining sum over histories.[13] In that approximation, the sum over histories (1.1) is dominated by the extrema of the action I. In general, these will occur in pairs of complex conjugate geometries and the extremum value of the action will have both real and imaginary parts. For example,

$$I[h,\chi,\partial M;\hat{g},\hat{\phi},M]=I_R[h,\chi,\partial M;\hat{g},\hat{\phi},M] \pm iS[h,\chi,M;\hat{g},\hat{\phi},M] \tag{1.3}$$

for a single connected ∂M. Here, we have written out explicitly the dependence of I on the boundary data $(h,\chi,\partial M)$ and on the extremizing solutions $(\hat{g},\hat{\phi},M)$. If, when the three-metric h is large, S varies much more rapidly as a function of h and χ than I_R, then S will satisfy the Hamilton-Jacobi equation to a good approximation and a form such as (1.2) will result. The real part of the action will contribute a plausibly dominant slowly varying exponential factor $\exp(-I_R[h,\chi,\partial M;\hat{g},\hat{\phi},M])$ to the weight Δ. Thus, in a way which will be discussed in more detail in Sec. V, the possible predictions of the "no-boundary" proposal for the geometry and topology of the classical spacetime of the late universe are reduced, in part, to a study of the classical extrema of the action I on the contributing manifolds M.[13] One is, thereby, led to the mathematical problem of exhibiting the complex solutions of Einstein's equation on various manifolds M with boundary ∂M, which near that boundary become close to real Lorentzian cosmological spacetimes having spatial sections which are large compared with the characteristic scales entering the underlying dynamics (e.g., the Planck length or the scale set by the cosmological constant). The solutions of interest for the semiclassical approximation need not be differentiable or even continuous. They need only have finite action.

Among complex solutions one expects the purely real tunneling metrics to be of special importance. A real tunneling solution describes transitions from a purely Riemannian[16] metric to a purely Lorentzian one. Such solutions describe tunneling in a variety of other situations in physics (see Fig. 1). The production of electron-positron pairs in a uniform electric field of sufficient

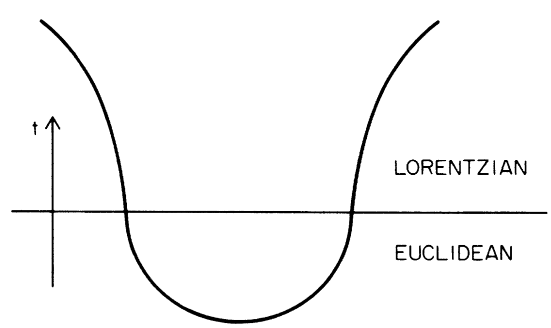

FIG. 1. The same figure describes schematically several different tunneling situations. (1) Pair creation from the vacuum in a uniform electric field. In this case the Euclidean solution for particle motion is a circle which joins smoothly onto the uniformly accelerated motion of the created pair in the uniform field. (2) The nucleation of a bubble of true vacuum in a false vacuum. The Euclidean instanton solution is a solution to the Euclidean classical field equations with a true vacuum inside and false outside. This joins to a Lorentzian solution in which the wall of the bubble expands. (3) In quantum cosmology, an example of a tunneling solution is the round Euclidean sphere joined on at an equator to the Lorentzian de Sitter space at its radius of maximum contraction.

strength can be described semiclassically by a circular Euclidean particle motion joined to hyperbolic Lorentzian motion.[17] The decay of the false vacuum can be described semiclassically by a Euclidean instanton solution joined onto a Lorentzian solution representing an expanding bubble of true vacuum.[18,19] In quantum cosmology such real tunneling solutions describe the universe "tunneling from nothing,"[20] and are the dominant contributors to the semiclassical approximations to the "no-boundary" proposal.[9,13] The simplest example occurs when the action describes pure gravity with a positive cosmological constant Λ. Then a tunneling solution of the Einstein equation is a round Riemannian four-sphere joined across an equator to Lorentzian de Sitter space at its minimal radius. Except for the conformal modes, this solution is a local minimum of the real part of the action.[13]

Gravitational tunneling solutions necessarily correspond to discontinuous metrics because the signature changes from one region to anther.[21,22] However, as we shall see, a discontinuous metric can have finite action. The Riemannian part of the metric contributes a real term to the (Euclidean) action while the Lorentzian part contributes an imaginary part. The Lorentzian part of tunneling solutions are, therefore, the possible spacetimes of the late universe predicted in the "no-boundary" proposal with a weight dominated by exp[$-$(action of the Riemannian part)].

In this paper we shall begin an attack on classifying the complex solutions of the "no-boundary" proposal by examining the *real* tunneling solutions for simple manifolds and simple (or no) matter fields. We shall exhibit some general requirements and some particular examples. In this way, among an admittedly small class, we shall be able to see what large scale topologies and metrics for spacetime the "no-boundary" proposal favors.

II. SOME GENERAL PROPERTIES OF REAL TUNNELING METRICS

A. Definition and matching conditions

We are interested in manifolds M without boundary which are the union of two manifolds M_L and M_R with common boundary Σ:

$$M = M_L \cup M_R \ , \tag{2.1}$$

$$\partial M_R = \Sigma = \partial M_L \ . \tag{2.2}$$

On M_R we have a Riemannian metric (signature: $++++$) and on M_L Lorentzian metric (signature: $-+++$). The metrics induced on the common boundary Σ agree and are necessarily spacelike (signature: $+++$). In keeping with the idea of the "no-boundary" proposal, we shall assume that M_R is connected but not necessarily that M_L is connected. Thus, Σ will in general have more than one connected component $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, say. Each connected component of M_L is to be thought of as a disjoint "daughter" universe. The Riemannian "mother" universe M_R is thus connected and compact with boundary Σ.

We wish to restrict attention to classical histories (g,ϕ,M) with finite action. Thus, on the whole of M we impose the classical Einstein equation

$$R_{\alpha\beta} = \frac{l^2}{2}(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T) \tag{2.3}$$

for appropriate matter fields whose energy-momentum tensor is $T_{\alpha\beta}$, including any cosmological constant. Here, as throughout, we use units where $\hbar = c = 1$; $l = (16\pi G)^{1/2}$ is the Planck length.

Equation (2.3) must hold on Σ. That is, we do not want a distributional contribution to the Ricci tensor $R_{\alpha\beta}$ at Σ. This means that the second fundamental form K_{ij} of the metric on Σ must not have a discontinuity across Σ. The second fundamental form K_{ij} of Σ when thought of as the boundary of M_R is given by

$$K_{ij} = \nabla_{(i} n_{j)} \ , \tag{2.4}$$

where n^α is the unit normal to Σ in M_R and latin indices range over directions in Σ. When Σ is thought of as the boundary of M_L, the definition of K_{ij} that is continuously related to that of (2.4) is obtained by analytically continuing the metric to Lorentzian signature. Then,

$$K_{ij} = \pm i \nabla_{(i} n_{j)} \ , \tag{2.5}$$

where n^α is the unit timelike normal to Σ in M_L.

With these definitions, it is an easy consequence[23] of the Einstein equation that for finite action K_{ij} must be continuous across Σ. For *real* solutions this requires that, on Σ,

$$K_{ij} = 0 \ . \tag{2.6}$$

The three-metric on Σ together with the condition $K_{ij} = 0$ will satisfy the Hamiltonian constraint

$$^3R = l^2 T_{nn} \ , \tag{2.7}$$

where T_{nn} is the projection of $T_{\alpha\beta}$ onto the normal to the surface. Any solution of (2.7) will provide initial data for Einstein's equations but not all such initial data can arise as the boundary geometry of a *compact* Riemannian manifold M_R. The "no-boundary" proposal picks out the subset of possible initial data which are the boundary values of Riemannian solutions on M_R. By this restriction it becomes predictive. If we assume strong cosmic censorship, or what is essentially the same thing that M_L is globally hyperbolic, then the topology of the universe is determined by the topology of Σ as $\mathbf{R} \times \Sigma$.

Since we shall always assume that in the Lorentzian sector $T_{nn} > 0$, Σ must have positive Ricci scalar. The possible topologies of Σ are then extremely limited. The results of Schoen and Yau and others[24] show that connected components of Σ must be connected sums of elliptic spaces and three wormholes (i.e., $S^1 \times S^2$'s). The vast majority of three-manifold topologies are ruled out. Thus, just from positive energy, the possible topologies for the universe nucleated by a real tunneling solution are very restricted.

As we shall see in detail later, the further requirement that Σ bound a compact four-manifold restricts the topology of Σ even more. For example, consider the de Sitter metric. If its round three-sphere sections are identified across a diameter, one still has a regular Lorentzian solution to the Einstein equation. Such a solution, however, would be ruled out in the "no-boundary" proposal because the identified S^3 which is Σ can no longer be the boundary of a portion of S^4 with its round metric. Indeed, according to a theorem of Chern and Simons,[25] no such identified S^3 with round metric can be conformally immersed, let alone isometrically embedded, in S^4 with its round metric.

B. Doubling and Wick rotations

Because Σ has vanishing second fundamental form, we may construct two new manifolds from M_R by joining two copies of it, M_R^- and M_R^+, across Σ. A similar construction works for M_L. The new manifolds are called the "doubles" of the original ones and we denote them by $2M_R$ and $2M_L$, respectively (Fig. 2). Metrics on the copies induce a metric on the double which is at least C^1 because $K_{ij}=0$ on Σ. Clearly, $2M_R$ will be a compact Riemannian manifold without boundary and each connected component of $2M_L$ is a Lorentizian spacetime without boundary. If the Einstein equation holds, one expects the metric to be real analytic (see Ref. 26 in the Riemannian case). The metrics on the doubled manifolds would then not only be C^1, but analytic as well. In the Riemannian case, this may be thought of as a kind of Schwarz reflection principle for the Neumann problem for solutions of Einstein's equation.

The doubled manifolds admit an isometry θ which interchanges the two halves $M_R^\pm$ ($M_L^\pm$, respectively) and leaves points on Σ fixed. That is,

$$\theta M^\pm = M^\mp \ , \tag{2.8}$$

$$\theta \Sigma = \Sigma \ . \tag{2.9}$$

Assuming that M_R (respectively, M_L) is oriented (respectively, time oriented), θ is orientation reversing or time reversing, respectively. Physically, $2M_L$ admits a moment of time symmetry on Σ and the "daughter" universes are born in a momentarily static state, that is, with no initial kinetic energy. Note that by construction θ is well defined on all of the real manifold $2M_R$ (or $2M_L$).

The existence of the reflection map θ means that geometries on M_L and M_R are related by a Wick rotation. Near Σ, any metric which allows a reflection map θ may be cast in the form

$$ds^2 = d\tau^2 + h_{ij}(x^i,\tau)dx^i dx^j \ , \tag{2.10}$$

where $h_{ij}(x^k,\tau)$ is an even function of time. Clearly by a Wick rotation, $\tau \to it$, we obtain another real section. Conversely, if a Wick rotation, $\tau \to it$ takes (2.10) to a new real metric, then $h_{ij}(x^k,t)$ must be an even function of t and the new real section also admits a reflection map. The existence of a reflection map has other important consequences as we shall see in the next subsection.

Before turning to these consequences we would like to view θ from the point of view of the complex geometry of the complexification of our solution of the Einstein equations.[27,28] This is a four-complex-dimensional manifold M_C (or eight real dimensional) with local complex coordinates z^α and complex metric $g_{\alpha\beta}dz^\alpha dz^\beta$. The two four-dimensional real manifolds $2M_L$ and $2M_R$ correspond to real slices in M_C on which the restricted metric is real and Lorentzian, or, respectively, real and Riemannian. These two real slices intersect transversely in the real three-dimensional manifold Σ. As described in Refs. 26 and 27 each real slice is the fixed point set of an antiholomorphic involution of M_C which is compatible with the complex metric $g_{\alpha\beta}$. As a consequence $2M_R$ and $2M_L$ are totally geodesic submanifolds of M_C and hence their intersection Σ is also totally geodesic. Denoting these antiholomorphic involutions by J_R and J_L we have, pointwise,

$$J_R(2M_R) = 2M_R \ , \tag{2.11}$$

$$J_L(2M_L) = 2M_L \ , \tag{2.12}$$

$$J_R(\Sigma) = J_L(\Sigma) = \Sigma \ . \tag{2.13}$$

Furthermore, since $2M_R$ and $2M_L$ are locally unique, it is clear that, although J_L does not fix the Riemannian real slice M_R pointwise, it does leave it invariant as a set. In other words, restricted to $2M_R$, J_R coincides with the orientation reversing isometry θ. Similarly restricted to $2M_L$, J_L coincides with time reversal symmetry. The interesting question of the global actions of J_L and J_R on the complexification M_C will not be addressed here. For our purposes, it suffices that J_L and J_R are defined in a neighborhood of the two real slices. Near their intersec-

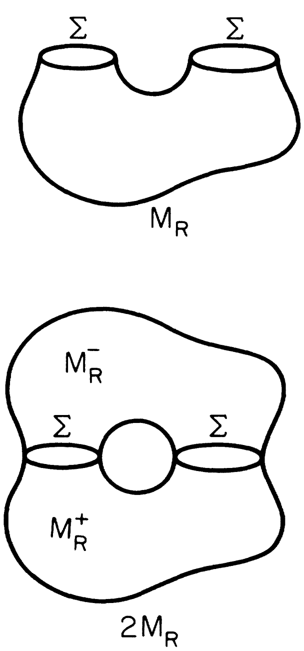

FIG. 2. From a compact geometry on a manifold M_R with boundaries Σ on which $K_{ij}=0$ we can construct a C^1 geometry on a closed manifold $2M_R$ by doubling.

tion Σ we may introduce a local complex coordinate τ such that J_R corresponds to $\tau \to \bar{\tau}$ and J_L to $\tau \to -\bar{\tau}$. Restricted to $2M_R$, J_L is just $\tau \to -\tau$, τ real, and so coincides with θ. Similarly restricted to $2M_L$, J_R is just $\tau \to -\tau$, τ imaginary, and so coincides with time reversal. See Refs. 27 and 28 for more details.

At this stage an example is probably in order. The simplest case is de Sitter space. Consider the complex quadric M_C in C^5 given by

$$(Z^1)^2+(Z^2)^2+(Z^3)^2+(Z^4)^2+(Z^5)^2=3/\Lambda \tag{2.14}$$

and the complex metric induced on this surface from the flat metric on C^5 in the natural way. The four-sphere $2M_R$ is the fixed point set of the antiholomorphic involution

$$J_R:\ (Z^1,Z^2,Z^3,Z^4,Z^5) \to (\bar{Z}^1,\bar{Z}^2,\bar{Z}^3,\bar{Z}^4,\bar{Z}^5)\ . \tag{2.15}$$

Lorentzian de Sitter space $2M_L$ is the fixed point set of the antiholomorphic involution

$$J_L:\ (Z^1,Z^2,Z^3,Z^4,Z^5) \to (\bar{Z}^1,\bar{Z}^2,\bar{Z}^3,\bar{Z}^4,-\bar{Z}^5)\ . \tag{2.16}$$

The intersection $\Sigma = 2M_R \cap 2M_L$ is the three-sphere

$$(Z^1)^2+(Z^2)^2+(Z^3)^2+(Z^4)^2=3/\Lambda \tag{2.17}$$

with (Z^1,Z^2,Z^3,Z^4) all real. M_R is the lower hemisphere of S^4, that is, it has Z^5 real and negative. M_L is the future half of de Sitter space, that is, it has $-iZ^5$ real and positive. In local coordinates we may set

$$(X^1,X^2,X^3,X^4)=\left[\frac{3}{\Lambda}\right]^{1/2}\cos\left[\left[\frac{\Lambda}{3}\right]^{1/2}\tau\right](\sin\xi\sin\theta\cos\psi,\ \sin\xi\sin\theta\sin\psi,\ \sin\xi\cos\theta,\ \cos\xi) \tag{2.18}$$

and

$$X^5=\left[\frac{3}{\Lambda}\right]^{1/2}\sin\left[\left[\frac{\Lambda}{3}\right]^{1/2}\tau\right]\ . \tag{2.19}$$

The metric thus becomes

$$ds^2=d\tau^2+\left[\frac{3}{\Lambda}\right]\cos^2\left[\left[\frac{\Lambda}{3}\right]^{1/2}\tau\right] \times[d\xi^2+\sin\xi^2(d\theta^2+\sin^2\theta\, d\psi^2)]\ . \tag{2.20}$$

Thus, M_R corresponds to

$$-\left[\frac{3}{\Lambda}\right]^{1/2}\frac{\pi}{2}\le\tau\le 0\ , \tag{2.21}$$

and M_L to

$$\tau=it,\quad t\ge 0\ . \tag{2.22}$$

Note that the metric form is an even function of the time variable τ. It thus makes sense to allow τ to be purely imaginary, and we get a new real, but Lorentzian metric, an example of (2.10). The Riemannian manifold M_R can be thought of as running along the real τ axis and the Lorentzian manifold M_R as running along the imaginary τ axis.

The example of de Sitter spacetime is rather a simple one. In Sec. IV we will give a rather more nontrivial example which also satisfies the Einstein equation.

C. General topological and geometrical restrictions on $2M_R$

We mentioned in Sec. II A that the topology of Σ was considerably restricted by the condition that it has positive three-Ricci scalar. In this section, we shall find further restrictions on the topology of the compact Riemannian double $2M_R$ arising from the existence of an orientation reversing isometry θ. The most immediate is that, since for manifolds of arbitrary dimension,

$$\chi(2M)=2\chi(M)-\chi(\partial M)\ , \tag{2.23}$$

and the Euler number of any compact three-manifold vanishes, the Euler number of $2M_R$ must be even. The Hirzebruch signature, $\tau(2M_R)$, must also vanish. To see this we express $\tau(2M_R)$ as

$$\tau(2M_R)=\int_{2M_R} S\sqrt{g}\, d^4x$$
$$=\int_{M_R^+} S\sqrt{g}\, d^4x+\int_{M_R^-} S\sqrt{g}\, d^4x \tag{2.24}$$
$$=\int_{M_R^+}(S+\theta * S)\sqrt{g}\, d^4x\ , \tag{2.25}$$

where

$$S=\frac{1}{48\pi^2}R_{\alpha\beta\lambda\rho}(*R)^{\alpha\beta\lambda\rho} \tag{2.26}$$

and $\theta * S$ is the pullback of S under θ. Since S is a pseudoscalar

$$\theta * S=-S \tag{2.27}$$

and the two contributions to τ in (2.4) cancel. In fact, a similar argument also works for the Euler number using the Gauss-Bonnet theorem on M_R with boundary term. Since the boundary $\partial M_R=\Sigma$ has vanishing second fundamental form, the boundary term vanishes and the integrand is now a scalar rather than a pseudoscalar. The two terms add to give an even Euler number.

One may also see these results using harmonic forms on $2M_R$. We have

$$\chi=2-2b_1+b_2^+ + b_2^-\ , \tag{2.28}$$

$$\tau=b_2^+ - b_2^-\ , \tag{2.29}$$

where b_1 is the dimension of the space of harmonic one-forms and b_2^+ and b_2^- are the dimension of the space of self-dual or anti-self-dual two-forms. If Ω is a self-dual harmonic two-form, then the pullback $\theta * \Omega$ is clearly an anti-self-dual two-form. Thus $b_2^+ = b_2^-$, whence $\tau = 0$ and $\chi = 2(1 - b_1 + b_2^+)$ is even.

The merit of thinking about two-forms is that is also becomes clear (as pointed out to us by Nigel Hitchin) that $2M_R$ cannot be a Kähler manifold except in the trivial case of a local product. If it were Kähler manifold, it would admit a covariantly-constant Kähler two-form Ω which, by convention, can be taken to be anti-self-dual. The pullback $\theta * \Omega$ would therefore be covariantly constant and self-dual. by taking a linear contribution of Ω and $\theta * \Omega$ one obtains a covariantly constant simple two-form, $\Omega_{\alpha\beta} = V_{[\alpha} U_{\beta]}$. This defines an integrable distribution in the tangent space of $2M_R$ which is what it means to say that the metric is locally a product. [Put another way, the holonomy reduces to SO(2)$\times$SO(2).]

We shall see later that the three conditions: (a) χ even, (b) $\tau = 0$, (c) $2M_R$ not a Kähler manifold unless a product, considerably restrict the possible manifolds on which there are real tunneling solutions of the Einstein equation.

D. Continuous isometries of $2M_R$

It may be that M_R and hence $2M_R$ has continuous isometries. If K^α is a Killing vector associated with a one-parameter subgroup of the isometry group, Isom($2M_R$), then the pullback $\theta * K^\alpha$ is also a Killing field. It follows that we can decompose K^α into two orthogonal pieces

$$K_\pm^\alpha = \tfrac{1}{2}(K^\alpha \pm \theta * K^\alpha) \qquad (2.30)$$

which are tangential (K_+^α) or normal (K_-^α) to Σ. Of course, either part may vanish.

The part K_-^α is normal to Σ and clearly by dragging Σ along the integral curves of K_-^α we obtain a family of surfaces orthogonal to K_-^α, that is, K_-^α is a hypersurface orthogonal Killing vector. K_+^α acts in these surfaces. If we analytically continue to the Lorentzian spacetime, M_L, then K_-^α is a static Killing vector. It follows that locally the metric may be cast in the form

$$ds^2 = V^2(x^i) du^2 + h_{ij}(x^i) dx^i dx^j , \qquad (2.31)$$

where $h_{ij}(x^i)$ depends only on the spatial coordinates x^i and the Wick rotation corresponds to replacing u by iu. This form of Wick rotation of a static spacetime is perhaps the most frequently encountered form but (from our earlier discussion) it is not the most general one.

Since Killing's equations imply that

$$\nabla_\alpha K_-^\alpha = 0 , \qquad (2.32)$$

we may integrate (2.30) over M_R^+ (or M_R^-) using the divergence theorem to obtain

$$\int_\Sigma K_-^\alpha d\sigma_\alpha = \int_\Sigma (K_-^\alpha n_\alpha) d\sigma = 0 , \qquad (2.33)$$

where n^α is the outward normal to Σ and $d\sigma$ the volume element. It follows from (2.33) that Σ must contain two open regions Σ_+ and Σ_- on one of which K_-^α is inward directed while on the other K_-^α is outward directed. If Σ is connected there must be a two-dimensional surface B (not necessarily connected) on which K_-^α vanishes, separating the two regions Σ_+ and Σ_-. This two-surface is the fixed point set of the one-parameter subgroup of isometries generated by K_-^α. It is sometimes called a "bolt of the first kind." It is not difficult to show that a bolt B is a totally geodesic submanifold of Σ and hence of $2M_R$ or $2M_L$.

The physical significance of B is that on analytic continuation of $2M_R$ to the Lorentzian sector $2M_L$, B becomes the Boyer axis of a Killing horizon, i.e., the intersection of a past and future horizon, which may also be a cosmological event horizon. A good example is provided by de Sitter spacetime. Setting, in (2.14),

$$Z^5 = \left[\frac{3}{\Lambda} - r^2\right]^{1/2} \sin\left[u\left[\frac{\Lambda}{3}\right]^{1/2}\right] , \qquad (2.34)$$

$$Z^4 = \left[\frac{3}{\Lambda} - r^2\right]^{1/2} \cos\left[u\left[\frac{\Lambda}{3}\right]^{1/2}\right] , \qquad (2.35)$$

$$Z^i = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) , \qquad (2.36)$$

the metric becomes

$$ds^2 = \left[1 - \frac{\Lambda r^2}{3}\right] du^2 + \left[1 - \frac{\Lambda r^2}{3}\right]^{-1} dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) . \qquad (2.37)$$

The surface Σ corresponds to $Z^5 = 0$. The portion of Σ corresponding to Σ_+ is

$$u = 0, \quad 0 \le r \le (3/\Lambda)^{1/2} . \qquad (2.38)$$

The bolt corresponds to $r = \sqrt{\Lambda/3}$, i.e., to the two-sphere given by the intersection of $Z^5 = 0$ and $Z^4 = 0$. Note that u has the character of an angular coordinate. Another example which we shall not discuss in detail is provided by the product metric on $S^2 \times S^2$ (the Nariai metric).

III. THE UNIQUE CONCEPTION

Having described some general topological and geometrical properties of closed Riemannian manifolds admitting a reflection map θ, we now wish to see what the consequences of the Einstein equation are. We shall assume that on $2M_R$ the Ricci tensor is non-negative, i.e.,

$$R_{\alpha\beta} V^\alpha V^\beta \ge C V_\alpha V^\alpha, \quad C > 0 , \qquad (3.1)$$

for all vectors V^α. Condition (3.1) certainly holds for vacuum Einstein metrics with positive cosmological term (i.e., $\Lambda > 0$). It also holds for some (but not all) physically reasonable matter energy-momentum tensors when analytically contained to the Riemannian regime. This has some striking consequences for the geometry and topology of $2M_R$. It implies, for example, that the fundamental group $\pi_1(2M_R)$ is finite.[29] It also implies the following *unique conception theorem:*

Σ is connected .

This result follows from the finiteness of $\pi_1(2M_R)$. If Σ were not connected we could construct a compact manifold by taking copies M_R and joining them back to back to construct a manifold containing a closed loop (Fig. 2) that cannot be shrunk to zero no matter how many times it is traversed.

An alternative proof is provided by Fraenkel's result[30] that any two compact minimal hypersurfaces in a compact Riemannian manifold with positive Ricci tensor must intersect. Since Σ is totally geodesic ($K_{ij}=0$) it is automatically minimal ($h^{ij}K_{ij}=0$) and the theorem follows. An interesting illustration of Fraenkel's theorem is provided by the equatorial three-spheres on S^4. They intersect on S^2's. Using the coordinates introduced in (2.37) the family of equatorial three-spheres $u=$const and $u=\pi\sqrt{3/\Lambda}+$const intersect in the bolt $r=\sqrt{3/\Lambda}$.

From a physical point of view, the fact that Σ is connected has the important consequence that a tunneling solution of the sort we are considering (assuming $R_{\alpha\beta}>0$) must nucleate a unique Lorentzian spacetime. The situation illustrated in Fig. 3 cannot take place. It is important to realize, however, that the unique conception theorem is crucially dependent upon our assumption that the Ricci tensor is positive. If we consider matter for which this assumption is violated (such as the four-form used by Giddings and Strominger[6]) then one can find real tunneling solutions in which Σ has more than one connected component. In the four-form case there is a conserved global charge whose sum over the disconnected components of Σ must equal zero. In other words, the creation of universe-antiuniverse pairs is enforced by charge conservation. In the absence of a conserved global charge and, in particular, for pure gravity with a positive cosmological term, however, our theorem shows that only single universe production is allowed.[31]

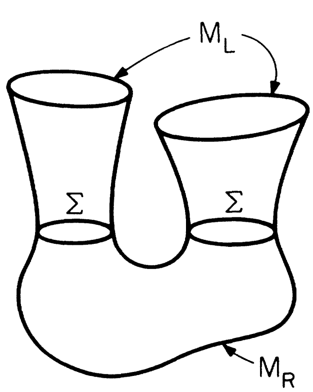

FIG. 3. The unique conception theorem shows that, when the Euclidean matter action is positive, a real tunneling solution cannot nucleate the disconnected Lorentzian spacetime pictured here.

IV. EXPLICIT TUNNELING SOLUTIONS

A. Known Einstein manifolds with positive Λ

There are very few known nonflat examples of solutions to

$$R_{\alpha\beta}=\Lambda g_{\alpha\beta} \tag{4.1}$$

with $\Lambda>0$. Those known to use are (1) S^4, with the round metric, (2) CP^2, with the Fubini-Study metric, (3) $S^2\times S^2$, with the product of two round metrics, (4) $CP^2\#\overline{CP^2}$, with the Page metric,[32] (5) $CP^2\#n\overline{CP^2}$, $3\le n\le 8$ with a Kähler metric (where $n\overline{CP^2}=CP^2\#\cdots\#\overline{CP^2}$, n times),[33] and (6) $K3$, with a Hyper-Kähler metric.[34] Of these, S^4 and $S^2\times S^2$ have been considered earlier. Of the rest, the Kähler examples will not exhibit a reflection map. The remaining example is the Page metric and this does admit a reflection map. We shall discuss it in detail later. Before doing so, we want to introduce a general class of Bianchi type-IX examples into which the Page metric falls.

B. Bianchi type-IX Einstein metrics

If we assume that Σ with its metric is homogeneous, it follows from Einstein's equation that $2M_R$ will admit an isometry group G with generically three-dimensional orbits. Such a metric is said to be of cohomogeneity-one and Σ will be a homogeneous space with respect to G. There are two cases to consider.

(1) The "Kantowski-Sachs" case in which G has no simply transitive three-dimensional subgroup. This leads us to the product metric on $S^2\times S^2$ so we discuss it no further.

(2) The case when G has a three-dimensional subgroup with a transitive action on Σ. This means that G must be one of the Bianchi groups, and the metric on Σ is a left invariant one for G. We shall now show that these lead to Bianchi type-IX Einstein metrics. The three-Ricci scalar ${}^3R$ of Σ satisfies

$${}^3R=2\Lambda \tag{4.2}$$

(by 2.7) and hence is positive unless $\Lambda=0$. This is excluded since no nonflat compact Ricci flat four-metric admits continuous isometries.[35] Of the Bianchi groups, only type IX [i.e., $G=$SO(3) or SU(2)] has a left-invariant metric with positive Ricci scalar[35] so the only cohomogeneity-one metrics to be considered are of Bianchi IX type. If $\sigma_1,\sigma_2,\sigma_3$ are left-invariant one forms on SU(2) such that

$$d\sigma_1=-\sigma_2\wedge\sigma_3 \tag{4.3}$$

and cyclic permutations, the metric may be cast in the form[36]

$$ds^2=a^2b^2c^2d\eta^2+a^2\sigma_1^2+b^2\sigma_2^2+c^2\sigma_3^2\ , \tag{4.4}$$

where a, b, and c are functions of the variable η which is related to τ by

$$d\tau=abc\,d\eta\ . \tag{4.5}$$

Evidently the metric (4.4) is of the form (2.10) provided a,

b, and c are *even* functions of τ and hence η.

If we let $a=\exp(\alpha)$, $b=\exp(\beta)$, $c=\exp(\gamma)$ and denote differentiation with respect to η by an overdot, the Einstein equation takes the form[37]

$$2\ddot{\alpha}=a^4-(b^2-c^2)^2-2\Lambda a^2b^2c^2 \tag{4.6}$$

together with the equations resulting from cyclic permutations of a,b,c and with the constraint:

$$4(\dot{\alpha}\dot{\beta}+\dot{\beta}\dot{\gamma}+\dot{\gamma}\dot{\alpha})=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4-4\Lambda a^2b^2c^2 \,. \tag{4.7}$$

From (4.6) we can obtain by subtraction the equations

$$\ddot{\alpha}-\ddot{\beta}=(a^2-b^2)(a^2+b^2-c^2) \tag{4.8}$$

and

$$\ddot{\alpha}-\ddot{\gamma}=(a^2-c^2)(a^2+c^2-b^2) \,. \tag{4.9}$$

Metrics of the form (4.4) admitting a reflection map θ have the property that on Σ, i.e., at $\eta=\tau=0$, the initial values of a,b,c and their derivatives satisfy

$$\dot{\alpha}=\dot{\beta}=\dot{\gamma}=0 \,, \tag{4.10}$$

$$a^4+b^4+c^4=2a^2b^2+2b^2c^2+2c^2a^2-4\Lambda a^2b^2c^2 \,. \tag{4.11}$$

Clearly there is a two-parameter family of initial data satisfying (4.10) and (4.11). If, as functions of η, no two of a, b, and c are equal, we call the solution triaxial. Otherwise it is called biaxial. (In analogous problems with quadratic forms in optics the terms biaxial and uniaxial are often used. However, this should cause no confusion here.)

Proposition: The only solutions satisfying (4.10) and (4.11) which give a compact Einstein metric are biaxial.

Proof: To have M_R compact a, b, and c cannot all remain nonzero, we must "close off the space." This can be done if one or three of a, b, and c vanish. This is because the orbits of SU(2) must collapse in dimension. The collapsed orbits are still homogeneous spaces of SU(2). The only possibilities are zero-dimensional points (nuts) for which $a=b=c=0$ or two dimensional subspaces ("bolts of the second kind") for which one of a,b,c vanishes and the other two become equal. The resulting bolt is thus either an S^2 or an RP^2.

Now, let us suppose that a, b, and c are all unequal on Σ. With no loss of generality we may then assume $a-b>a-c>0$. The right-hand sides of (4.8) and (4.9) are thus initially strictly positive. It follows from (4.8) and (4.9) that $\alpha-\beta$ and $\alpha-\gamma$ are both initially positive and for all subsequent times strictly positive. Thus, if a, b, and c start unequal they will remain unequal and no nut or bolt can arise. Thus, we must have $a=b$ or $a=c$ (or both) at $\tau=0$. But then $a=b$ or $a=c$ (or both) will be true for all time and the metric is biaxial.

Biaxial Bianchi type-IX metrics have an extra Killing vector, i.e., they are invariant under a group homomorphic to U(1)$\times$SU(2). If $a=b$ and c vanishes at a bolt with $a=b\neq 0$ and $da/d\tau=0=db/d\tau$ the generator of the U(1) factor has a fixed point set at the bolt; i.e., the bolt is of both the first and the second kind.

C. The Page metric

Page[32] has examined all biaxial or Taub-NUT solutions of (4.6) and (4.7) and finds that the only compact one other than S^4 is given by[38]

$$a^2=b^2=\left[\frac{1-\nu^2\sin^2x}{3+6\nu^2-\nu^4}\right]\left[\frac{3(1+\nu^2)}{\Lambda}\right], \tag{4.12}$$

$$c^2=\left[\frac{3(1+\nu^2)}{4\Lambda}\right]\left[\frac{3-\nu^2-\nu^2(1+\nu^2)\sin^2x}{(3+\nu^2)^2(1-\nu^2\sin^2x)}\right]\cos^2x \,, \tag{4.13}$$

where

$$\left[\frac{3(1+\nu^2)}{\Lambda}\right]^{1/2}dx\left[\frac{1-\nu^2\sin^2x}{3-\nu^2-\nu^2(1+\nu^2)\sin^2x}\right]^{1/2}=d\tau \,, \tag{4.14}$$

and $\nu\sim 0.281\,701\,56$ satisfies

$$\nu^4+4\nu^3-6\nu^2+12\nu-3=0 \,. \tag{4.15}$$

Clearly, the Page metric admits the orientation reversing isometry

$$\theta\colon\; x\to -x \tag{4.16}$$

so that $K_{ij}=0$ on the surface $x=0$. The surface $x=0$ is thus the nucleation surface, Σ, of a real tunneling solution of the Einstein equation for which M_R is the part of the Page geometry bounded by Σ. On Σ, i.e., at $x=0$, we have

$$a^2/c^2=3.742\,759\,8 \tag{4.17}$$

so that Σ is a squashed three-sphere.

To obtain the Lorentzian section of the Page metric, M_L, we take x to be pure imaginary. We obtain an ever expanding universe in which ultimately a and c grow exponentially as $\exp(\tau\sqrt{3/\Lambda})$. The ratio a^2/c^2 (which provides a measure of how squashed the three-spheres are) tends to the limiting value of approximately 0.803 682 49. This is consistent with the general ideas associated with the ideas of cosmic baldness and the cosmic no-hair theorem as we shall show in the next section.

The Page metric exhibits a number of other interesting features. First it does not admit a spin structure which may make it difficult to incorporate into a "no-boundary" sum over histories where integral of realistic spinor matter fields are included. Second, like the round product metric on $S^2\times S^2$, the closed Euclidean Page metric possesses a negative-eigenvalue mode of the Lichnerowiz operator for transverse-traceless metric perturbations.[38] Its action is therefore not a local minimum in a real direction which is distinct from that associated with the conformal mode. Such extrema could be the dominant contributions to a steepest-descents approximations to functional integral (1.1) only if the defining contour $\mathcal{C}$ can be distorted to pass through the extrema in complex directions such that the action is a local minimum. In the absence of a compelling argument fixing this contour,[12] especially on manifolds with nontrivial topology, we shall consider this possibility. Third, the surfaces at constant time in the Page metric do not themselves admit an orientation reversing isometry, or put in more physical terms are not invariant under spatial parity. There are, thus, two distinct Page universes which could be created via tunneling—a right-handed and a left-handed one. This phenomenon is also related to the property of spectral asymmetry: the spectrum of the instantaneous Dirac equation Hamiltonian on surfaces of constant time (in the Lorentzian regime) is not symmetric with respect to the sign of the energy eigenvalues. This might have some relevance to the production of particles after tunneling—more fermions than antifermions might be created, for example. We do not wish to discuss this point further in the present paper but simply remark that a more detailed examination of the Page metric would seem to be called for. We will return to a discussion of tunneling via the Page metric in Sec. V.

D. Cosmic baldness

There have been a number of studies of cosmological models driven by a positive cosmological term motivated in part by the inflationary scenario. Two important points have emerged which are relevant to the concerns of this present paper:

First, if the universe continues to expand forever one expects the metric to settle down to an asymptotic de Sitter-like state. However, this settling down is a local result within the cosmological event horizon of any given observer. It is definitely not a global result over all of space since a gravitational-wave perturbation may be shown to freeze in and persist at late times. The Page metric is an explicit illustration of this. In the Lorentzian regime a^2 and c^2 tend to the same universal exponential form $\exp(2\tau\sqrt{3/\Lambda})$ but the ratio a^2/c^2 does not tend to unity, but rather to the ratio 0.803 682 49. This means that the surfaces of constant time never become round. This is also consistent with the general asymptotic analysis of Starobinsky.[39]

The second point concerns static solutions. The only static solution which is nonsingular inside a single cosmological horizon is de Sitter spacetime. (More precisely, the results refer to static metrics for which $\Sigma=\Sigma_+\cap B\cap\Sigma_-$ is topologically S^3, Σ_+ and Σ_- being homeomorphic to a three-ball and B to a two-sphere.) Proofs of this result have been presented by Boucher[40] and Friedrichs.[41] In fact the regularity condition on the cosmological horizon (the constancy of the surface gravity κ) ensures that on Wick rotation we obtain an Einstein metric on S^4 with an hypersurface orthogonal Killing field. It is believed that the only Einstein metric on S^4 is the standard round metric although no rigorous proof is known. Thus, if the surface Σ is homeomorphic to S^3, it seems rather plausible that the only static tunneling solution is de Sitter space. If however we allow different topologies the situation is less clear. We know of at least one other static solution—the Nariai metric for which $\Sigma=S^1\times S^2$.

If one relaxes the condition that the Riemannian section is compact and nonsingular then there are probably very many static solutions containing many black holes in unstable equilibrium inside a cosmological event horizon.[42] It is possible that some of these do have a nonsingular Riemannian section. However, even if such solutions do exist their significance for tunneling is dubious because (as we shall explain in detail in the next section) they must have larger action than the S^4 solution.

V. THE GRAVITATIONAL ACTION AND THE LARGE-SCALE TOPOLOGY OF SPACETIME

As we have described in the Introduction, if the topology and metric of spacetime are quantum variables, then any theory of the initial condition should yield a prediction for the ensemble of geometrices (manifolds + metrics) possible for the late universe. We are now in a position to analyze something of predictions of the "no-boundary" proposal for this ensemble. Recall that an ensemble of possible classical spacetimes is predicted when the Euclidean sum over histories defining the wave function of the universe can be approximated semiclassically and the Lorentzian part of the action at the dominant stationary point varies much more rapidly with three-metric and matter field than the imaginary part [cf. Eq. (1.3)]. If, for a given three-geometry and spatial matter field configuration $(h,\chi,\partial M)$ there are several different stationary points $(\hat{g},\hat{\phi},M)$ contributing to the semiclassical approximation for Ψ, they contribute with the weight

$$\nu(M)\Delta_{\mathrm{WKB}}[h,\chi,\partial M;\hat{g},\hat{\phi},M]\exp(-I_R[h,\chi,\partial M;\hat{g},\hat{\phi},M]). \tag{5.1}$$

Here, Δ_{WKB} is the usual WKB prefactor and we have written out the whole dependence on boundary data and classical solution. By examining (5.1) we can determine which large-scale geometries of the universe the "no-boundary" proposal favors. Such a program, of course, says nothing about the topology of spacetime on scales near the Planck scale where the semiclassical approximation is unlikely to be valid. Computing a distribution just of the possible manifolds M would involve assigning a distribution to $(h,\chi,\partial M)$ and integrating these out of (5.1)—a task which involves some unresolved issues in the theory. Some idea of the distribution of manifolds, however, can be obtained by fixing a fiducial $(h,\chi,\partial M)$, allowing $(\hat{g},\hat{\phi},M)$ to range over the remaining possibilities, and comparing their relative weights according to (5.1). This is the course we shall follow here.

All three factors in (5.1) contribute to the overall weight of a manifold. However, because of its exponential dependence, it is a not unreasonable conjecture that for fixed $(h,\chi,\partial M)$ the classical geometry and matter field configurations most favored by the "no-boundary" proposal are those with the least real action, I_R. We shall assume this in the following.

With these assumptions, the problem of predicting the large-scale geometry of the universe reduces to finding those solutions $(\hat{g},\hat{\phi},M)$ that contribute to the steepest-descents approximation with the least real action. Whether a solution contributes or not depends on whether the steepest-descents distortion of the contour $\mathcal{C}$ in (1.1) passes through it.[13] Indeed, since the analytic properties of the action imply that every solution corresponds to two extrema, one with $\mathrm{Re}\sqrt{g}>0$ and the other with $\mathrm{Re}\sqrt{g}<0$, there is, more precisely, the issue of which extra corresponding to a given solution contribute. However, it was argued in Ref. 13 that only solutions with $\mathrm{Re}\sqrt{g}>0$ give physically sensible predictions for field theory in the implied curved classical background. We, therefore, consider only $\mathrm{Re}\sqrt{g}>0$ extrema in what follows and this fixes the sign of the action for a solution.

The predictions of the "no-boundary" proposal for the large-scale geometry of the universe are particularly easy to extract if one restricts attention to the real tunneling solutions discussed in this paper, even if one considers all of them as potential contributors to the steepest-descents approximation. This is because for a real tunneling solution the real part of the action comes entirely from the part of the manifold M_R on which the geometry is Riemannian. For the examples discussed here, I_R is independent of (h,χ) and depends only on M. In this case, assuming the strong form of cosmic censorship which implies that topology cannot change classically, we predict that the large-scale Lorentzian geometry has the topology $\mathbf{R}\times\Sigma$ with a weight proportional to $\exp(-I_R)$.

The unique conception theorem shows that there are no real tunneling solutions for which the Lorentzian spacetime is disconnected if Euclidean matter energy is positive. Were this the case, the first conclusion would, therefore, be that the "no-boundary" proposal predicts that the large-scale topology of spacetime is connected. There would be no "other" universes nucleated by a real tunneling mechanism. One might have thought that the existence of such other universes was beyond the power of observational check. However, Hawking[43] has suggested that, because we cannot distinguish which universe we are in, there might be an effect on the dynamics which is in principle observable at early times.

In the absence of matter, but the presence of a positive cosmological constant, it is possible to identify the real tunneling solution with least real action. For a solution to (4.1) the action is

$$I_R=I[M_R]=\tfrac{1}{2}I[2M_R]=-\frac{\Lambda V}{l^2}\ , \tag{5.2}$$

where V is the total volume of $2M_R$. Clearly, I_R is least when V is greatest. We may now invoke Bishop's theorem[44] which states that if

$$R_{\alpha\beta}V^\alpha V^\beta \geq C V_\alpha V^\alpha \tag{5.3}$$

with $C>0$ the four-volume V is bounded below by

$$V\leq\frac{24\pi^2}{C^2} \tag{5.4}$$

with equality if and only if the metric is the round metric on S^4. This is confirmed by the actions for the explicit solutions we have exhibited in view of (4.1):

$$S^4\colon\ I_R=-24\pi^2/l^2\Lambda\ , \tag{5.5}$$

$$S^2\times S^2\colon\ I_R=-16\pi^2/l^2\Lambda\ , \tag{5.6}$$

$$\text{Page:}\ I_R=-0.955\,344\,86\times16\pi^2/l^2\Lambda\ . \tag{5.7}$$

Thus, among the class of real tunneling vacuum solutions with positive cosmological constant, that on S^4 has the least real action. Were real tunneling solutions driven by positive cosmological constant the dominant extrema of the action, the "no-boundary" proposal would predict $\mathbf{R}\times S^3$ for the most probable topology of Lorentzian spacetime with de Sitter geometry as the most probable metric.

The possibilities in competition with $\mathbf{R}\times S^3$ in our catalog of explicit examples would be the following. (1) Nucleation via the Page metric would yield a Lorentzian geometry with a topology of $\mathbf{R}\times S^3$ but a metric different from the de Sitter metric. This metric can be thought of as the de Sitter metric with a gravitational wave of the longest possible wavelength frozen in. (See Ref. 45 for a detailed discussion of this phenomenon.) Because of cosmic baldness it would be difficult to distinguish such a solution locally from de Sitter space, but, were global information available the two cases could be distinguished. (2) Nucleation via the $S^2\times S^2$ solution would yield a Lorentzian spacetime with a geniunely different topology $\mathbf{R}\times S^1\times S^2$ and a different metric. For the present values of the cosmological constant, $l^2\Lambda\sim10^{-118}$ these possibilities of long-wavelength gravitational waves and different spatial topologies are very heavily suppressed despite the modest differences in the numerical coefficients in (5.5)–(5.7).

The real world contains matter. The inclusion of matter will generally make the tunneling solutions complex even in situations with high symmetry.[13] In the absence of an extension of Bishop's theorem to complex cases or a greater catalog of explicit examples, it is difficult to draw general conclusions for the prediction of large-scale topology. However, the fact that the real tunneling solutions are local minima of the real part of the action except for conformal modes[14] lends some support to the idea that the arguments given here may generalize. Were that the case, among the many topologies and metrics which contribute to it, the "no-boundary" proposal may overwhelmingly predict the simplest topology $\mathbf{R}\times S^3$ with the metric of greatest symmetry on the largest scales of the universe.

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[1]See, e.g., J. A. Wheeler, Ann. Phys. (N.Y.) **2**, 604 (1957).
[2]B. S. DeWitt, in *Relativity, Groups and Topology II*, proceedings of the Les Houches Summer School, Les Houches, France, 1983, edited by B. DeWitt and R. Stora (Les Houches Summer School Proceedings, Vol. 40) (North-Holland, Amsterdam, 1984); A. Anderson and B. S. DeWitt, Found. Phys. **16**, 1 (1986).
[3]S. W. Hawking, Commun. Math. Phys. **43**, 799 (1975).
[4]S. W. Hawking, Phys. Lett. **134B**, 403 (1984).
[5]S. Coleman, Nucl. Phys. **B310**, 643 (1988).
[6]S. Giddings and A. Strominger, Nucl. Phys. **B307**, 854 (1988).
[7]The origin of large-scale topology in the initial condition of the Universe was stressed and discussed by L.-Z. Fang and H. J. Mo, in *Observational Cosmology*, proceedings of the 124th Symposium of the International Astrophysical Union, edited by A. Hewitt, G. Burbidge, and L.-Z. Fang (Reidel, Dordrecht, 1987). For related problems, see R. L. Mkrtchyan, Phys. Lett. B **172**, 313 (1986); J. B. Hartle and D. M. Witt, Phys. Rev. D **37**, 2833 (1988).
[8]S. W. Hawking, in *Astrophysical Cosmology*, edited by H. A. Brück, G. V. Coyne, and M. S. Longair (Pontifica Academia Scientarium, Vatican City, 1982); J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[9]Hartle and Hawking (Ref. 8); S. W. Hawking, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).
[10]For the class of manifolds, see, e.g., the discussion and references in J. B. Hartle, Class. Quantum Grav. **2**, 707 (1985).
[11]For the measure, see, e.g., H. Leutwyler, Phys. Rev. **134**, 1155 (1964); B. S. DeWitt, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honour of his 60th Birthday*, edited by J. Klauder (Freeman, San Francisco, 1972); E. Fradkin and G. Vilkovisky, Phys. Rev. D **8**, 4241 (1973); L. Faddeev and V. Popov, Usp. Fiz. Nauk. **111**, 427 (1973) [Sov. Phys. Usp. **16**, 777 (1973)]; M. Kaku, Phys. Rev. D **15**, 1019 (1977); C. Teitelboim, *ibid.* **28**, 310 (1983); H. Hamber, in *Critical Phenomena, Random Systems, and Gauge Theories*, proceedings of the Les Houches Summer School, Les Houches, France, 1984, edited by R. Stora and K. Osterwalder (Les Houches Summer School Proceedings, Vol. 43) (North-Holland, Amsterdam, 1986); M. Bander, Phys. Rev. Lett. **57**, 1825 (1986).
[12]For the contour of integration, see, e.g., the discussion and references in Ref. 13.
[13]J. J. Halliwell and J. B. Hartle, Phys. Rev. D **41**, 1815 (1990).
[14]For a general discussion of decoherence in quantum mechanics, see M. Gell-Mann and J. B. Hartle, in *Proceedings of the Santa Fe Institute Workshop on Complexity, Entropy and the Physics of Information*, edited by W. Zurek (Addison Wesley, Reading, 1990); for a discussion of decoherence of spacetime, see H. D. Zeh, Phys. Lett. A **116**, 9 (1986); C. Kiefer, Class. Quantum Grav. **4**, 1369 (1987); J. Halliwell, Phys. Rev. D **39**, 2912 (1989); T. Fukuyama and M. Morikawa, *ibid.* **39**, 462 (1989); T. Padmanabhan, *ibid.* **39**, 2924 (1989).
[15]There is a large body of literature on the semiclassical approximation. For some convenient recent discussion with references to the earlier literature, see J. B. Hartle, in *Gravitation in Astrophysics*, edited by B. Carter and J. B. Hartle (Plenum, New York, 1986); J. Halliwell, Phys. Rev. D **36**, 3626 (1987); T. Padmanabhan, Class. Quantum Grav. **6**, 533 (1989).
[16]When discussing solutions we use the word "Riemannian" instead of "Euclidean" for signature $(+,+,+,+)$.
[17]A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).
[18]S. Coleman, Phys. Rev. D **15**, 2929 (1977).
[19]L. G. Jensen and P. J. Ruback, Nucl. Phys. **B325**, 660 (1989).
[20]A. Vilenkin, Phys. Lett. **117B**, 25 (1982); Phys. Rev. D **27**, 2848 (1983); D. Atkatz and H. Pagels, *ibid.* **25**, 2065 (1982).
[21]A. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1922).
[22]A. D. Sakharov, Zh. Eksp. Teor. Fiz. **87**, 375 (1984) [Sov. Phys. JETP **60**, 214 (1984)].
[23]See, e.g., Ref. 13 for an explicit demonstration in this case.
[24]R. Schoen and S.-T. Yau, Ann. Math. **110**, 127 (1979); Manuscript Math. **28**, 159 (1979); Proc. Nat. Acad. Sci. **75**, 2567 (1978); M. Gromov and H. B. Lawson, Jr., IHES Publ. Math. **58**, 83 (1983); J. D. Barrow and F. J. Tipler, Mem. R. Astron. Soc. **216**, 395 (1985).
[25]S.-S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).
[26]A. Besse, *Einstein Manifolds* (Springer, Heidelberg, 1987).
[27]N. Woodhouse, Int. J. Theor. Phys. **16**, 663 (1977).
[28]K. Rózga, Rep. Math. Phys. **11**, 197 (1977).
[29]C. F. R. Maunder, *Algebraic Topology* (Cambridge University Press, Cambridge, England, 1980).
[30]T. Frankel, Ann. Math. **83**, 68 (1966).
[31]We thank L. Grischuk for discussions which helped elucidate this point.
[32]D. N. Page, Phys. Lett. **79B**, 239 (1978).
[33]G. Tian, Invent. Math. **89**, 225 (1987); G. Tian and S.-T. Yau, Commun. Math. Phys. **112**, 175 (1987).
[34]Y.-T. Siu, Ann. Math. **127**, 585 (1988).
[35]R. Wald, Phys. Rev. D **28**, 2118 (1983).
[36]T. Eguchi, P. Gilkey, and A. Hanson, Phys. Rep. **66**, 1 (1980).
[37]G. W. Gibbons and C. N. Pope, Commun. Math. Phys. **61**, 239 (1978).
[38]R. E. Young, Phys. Rev. D **28**, 2436 (1983).
[39]A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 55 (1983) [JETP Lett. **37**, 66 (1983)].
[40]W. Boucher, in *Classical General Relativity*, edited by W. Bonner, J. Islam, and M. MacCallum (Cambridge University Press, Cambridge, England, 1983).
[41]H. Friedrichs, J. Geom. Phys. **3**, 101 (1986).
[42]G. W. Gibbons (unpublished).
[43]S. W. Hawking, in *300 Years of Gravitation* (Ref. 9).
[44]R. L. Bishop and J. Crittenden, *Geometry of Manifolds* (Academic, New York, 1964).
[45]L. Grischuk, Zh. Eksp. Teor. Fiz. **69**, 1857 (1975) [Sov. Phys. JETP **42**, 943 (1975)].