Simplicial minisuperspace. II. Some classical solutions on simple triangulations

James B. Hartle
Department of Physics, University of California, Santa Barbara, California 93106

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The extrema of the Euclidean Regge gravitational action are investigated numerically for some closed, compact, four-dimensional simplicial manifolds with topologies S^4, CP^2, and $S^2 \times S^2$.

I. INTRODUCTION

Sums over geometries such as occur in the Euclidean functional integral approach to quantum gravity may be given a practical meaning through simplicial approximation. In this approximation sums over smooth geometries are replaced by sums over simplicial geometries built up out of flat simplices by the methods of the Regge calculus.[1] Simplicial geometries are specified by the way the simplices are joined together and by the squared lengths of their edges. A sum over different topologies is approximated by a sum over different ways of putting simplices together. A sum over metrics on a given manifold is approximated by a multiple integral over the squared edge lengths of a collection of simplices which triangulate the manifold.

Simplicial approximations to sums over geometries were discussed in general in the first paper in this series[2] (Paper I) where some references to the extensive earlier literature may be found. The expectation value of some physical quantity A in the state of minimum excitation for closed cosmologies provides a typical example of such a sum. This might read

$$\langle A \rangle = \frac{\int_C d\Sigma_1 A(s_i) \exp[-I(s_i)]}{\int_C d\Sigma_1 \exp[-I(s_i)]}, \tag{1.1}$$

where I is the Euclidean Regge gravitational action for a closed compact simplicial geometry. For simplicity we have illustrated only a sum over metrics on a fixed simplicial manifold. Both I and A are functions of the squared edge lengths s_i, $i = 1,...,n_1$. The integral is a multiple integral over the space of squared edge lengths along some appropriate contour C with some appropriate measure. As throughout we use units where $\hbar = c = 1$ and write the Planck length as $l = (16\pi G)^{1/2}$.

In suitable limits the integral (1.1) can be evaluated by the method of steepest descents. This is the semiclassical approximation. The value of the integral is dominated by the contribution near one or more stationary points through which the contour can be distorted to pass. At these,

$$\frac{\partial I}{\partial s_i} = 0. \tag{1.2}$$

These are the simplicial analogs of the Einstein field equations. Even when not quantitatively accurate the semiclassical approximation often yields qualitative insight into the behavior of the integral in a straightforward way.

The important quantities for constructing a simplicial sum over geometries and evaluating it in the semiclassical approximation are the Regge action and its stationary points. Methods for evaluating the action and the Regge equations (1.2) were reviewed in Paper I. In this paper we shall illustrate these methods by numerically evaluating the action and locating its stationary points for a few simple simplicial manifolds. We make no attempt to be exhaustive. We consider only the Regge gravitational action with positive cosmological constant. In the continuum limit this is the action of Einstein's theory. We shall confine attention to compact simplicial manifolds which have no boundary. These are the important nets for evaluating expectation values such as (1.1) (Paper I). We shall consider only real (Euclidean) edge lengths. Even if the contour of integration in (1.1) is complex, the real stationary points seem likely to play a significant role in any semiclassical evaluation of the integral.[3] Within this limited scope, however, we shall be able to illustrate how the Regge action approximates the continuum action, to display its values in a number of interesting cases, and to solve for the stationary points on simple manifolds with differing topologies.

To evaluate the action one must first have a simplicial manifold. That is, one must specify a set of vertices, edges, triangles, tetrahedra, and four-simplices which make up a manifold with the desired topology. The specification of a simplicial manifold is discussed in Sec. II. Quoting largely from the mathematical literature we shall exhibit simplicial manifolds which are triangulations of S^4, CP^2, $S^2 \times S^2$, and $S^1 \times S^3$.

In Sec. III we illustrate the evaluation of the action using families of geometries on S^4 and CP^2. We compare the action of the most symmetric simplicial geometries with that of the most symmetric continuum geometries on these manifolds. In less symmetric cases we illustrate the behavior of the action for homogeneous, anisotropic geometries and for geometries which are conformal deformations from the most symmetric cases. We shall recover features familiar from the continuum theory such as arbitrarily negative actions arising from conformal deformations.

Section IV is concerned with the solution of the Regge equations on S^4 and CP^2. Solutions are found by imposing symmetries. The eigenvalues of the matrix describing the second variation of the action at these stationary points is also calculated. A more systematic approach to solving the Regge equations is discussed in Sec. V and the difficulties for this method arising from the approximate diffeomorphism group of a simplicial geometry are illustrated.

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II. SOME SIMPLICIAL MANIFOLDS

A four-dimensional simplicial manifold is a collection of vertices, edges, triangles, tetrahedra, and four-simplices joined together such that a neighborhood of every point can be smoothly and invertibly mapped into a region of four-dimensional Euclidean space $\mathbb{R}^4$. In more mathematical terminology, a simplicial manifold is a simplicial complex which is a piecewise linear manifold.[4]

A complex may be described by labeling its simplices and specifying how they are contained in one another. (Here and from now on we shall omit the qualification "simplicial" from manifold, complexes, etc., it being understood that the objects of interest in this paper are constructed from simplices.) A complex which is a four-manifold is homogeneously four-dimensional. That is, every simplex of dimension $k<4$ is contained in some four-simplex. Homogeneously four-dimensional complexes may be specified by labeling their k-simplices by integers from 1 up to the total number n_k and then listing the vertices of the four-simplices. Such a list defines the vertex matrix $j_4(i,j)$ which gives the five vertices $j = 1,...,5$ of the ith four-simplex. From this the vertices of the edges, triangles, and tetrahedra of the complex can be computed and in particular the matrices $j_k(i,j)$ which give the vertices $j = 1,...,k+1$ of every k-simplex i.

An alternative way of specifying a complex is to give its incidence matrices. We assign numbers $1,...,n_k$ to label the k-simplices of the complex. The incidence matrix[5] $i_k(i,j)$ for $j = 1,2,...$ gives the labels of the k-simplices contained in the $(k+1)$-simplex i. Clearly the incidence matrices can be computed from the vertex matrices and vice versa. Given the matrix $j_4(i,j)$ which specifies the vertices of the four-simplices of a complex we can compute all the other i_k and j_k. It is not true, however, that given the matrix $i_0(i,j)$, which specifies which vertices are connected by edges, one can compute the rest of the complex. For example, there might be a complex with $n_0>5$ vertices in which every vertex is connected to every other (we shall display some subsequently), so that $i_0(i,j)$ is always 1 for $i\neq j$. This i_0 is the same as the i_0 for the (n_0-1)-simplex. To be a four-dimensional complex the five-simplices, which *could* be constructed from the given edges, and the four-simplices in which they intersect must be left out and i_0 does not say which they are.

Not every matrix $j_4(i,j)$ which specifies a four-dimensional complex specifies a manifold. A complex is a manifold if every point (including the interior points of the simplices) has a neighborhood which is homeomorphic to a ball in $\mathbb{R}^4$. A necessary and sufficient condition for a complex to be a manifold may be stated in terms of the star and link of a simplex. The *star* of a simplex σ is the collection of all simplices which have σ as a face together with all of their faces. The *link* of a simplex σ consists of all simplices in its star which do not have σ as a face. (See Paper I for some illustrations.) A complex is a four-manifold if and only if the link of every k-simplex is a $(3-k)$-sphere.[6] This is not a condition which translates very straightforwardly (if at all) into an algorithm for deciding whether a complex is a manifold or not.[7] However, necessary conditions which are easy to test can be derived. For example, for the link of every tetrahedron to be a zero-sphere (two vertices), two four-simplices must intersect in exactly two tetrahedra or not at all. This is the analog of two triangles intersecting in exactly one edge in a two-manifold. In particular this implies that the total number of tetrahedra and four-simplices are related by

$$5n_4 = 2n_3\,. \tag{2.1}$$

Another condition on the total number of simplices may be derived[8] by fixing attention on a vertex σ and considering the collection of simplices $N(\sigma)$ which is the star of σ less its link and less the vertex σ itself. The Euler number of any homogeneously n-dimensional complex with m_k k-simplices is

$$\chi = \sum_{k=0}^{n} (-1)^k m_k\,. \tag{2.2}$$

Since the link of a vertex of a four-manifold is a sphere with $\chi = 2$, and since the star of a vertex is a four-ball with $\chi = 1$, the Euler number of $N(\sigma)$ vanishes. Summing this relation over all vertices of the manifold one finds

$$2n_1 - 3n_2 + 4n_3 - 5n_4 = 0\,. \tag{2.3}$$

This is an example of a Dehn–Sommerville relation.

Neither (2.1) nor (2.3) is sufficient to guarantee that a complex is a manifold. The absence of a straightforward combinatoric check of whether a complex is a manifold means that finding explicit simplicial manifolds with interesting topology is a challenging mathematical problem. In the literature (to quote a review of the three-dimensional problem[9]) "explicit triangulations of topologically nontrivial three-manifolds have been observed only very rarely" and their construction by and large has been by special techniques. Given this situation we cannot attempt a systematic survey of four-dimensional simplicial manifolds. Rather in this section, drawing almost entirely on the mathematical literature, we shall exhibit a few examples. We shall classify them by the customary name of their topological space. The specific complex is then said to be a specific *triangulation* of the space.

A. S^4

The surfaces of the tetrahedron, octohedron, and icosohedron are triangulations of the two-sphere. They are regular in the sense that no vertex or edge is distinguished from any other. The analog regular triangulations of S^4 are the surfaces of the regular solids in five dimensions, which are composed entirely of four-simplices. There are only two.[10] The first is the surface of the five-simplex α_5 obtained by joining each of six vertices in five-dimensional Euclidean space to every other vertex. Thus, $n_0 = 6$, $n_1 = 15$, $n_2 = 20$, $n_3 = 15$, and $n_4 = 6$. The second is the surface of the five-dimensional cross polytope β_5. This may be constructed by taking five orthogonal axes, locating two vertices on each axis on opposite sides of the origin, and connecting each vertex to every other except its opposite. For β_5, $n_0 = 10$, $n_1 = 40$, $n_2 = 80$, $n_3 = 80$, and $n_4 = 32$. The vertex matrices of α_5 and β_5 are given in Table I. The regular nature of the triangulations α_5 and β_5 can be expressed concretely by giving their symmetry groups expressed as operations on the vertices.[11] The symmetry group of α_5 is the permutation group on the six-vertices S_6. In the context of the construction described above the symmetry group of β_5 consists of

TABLE I. Four-simplices of α_5, β_5, and CP_9^2.

α_5 $n_0=6,\ n_4=6$	β_5 $n_0=10,\ n_4=3$	CP_9^2 $n_0=9,\ n_4=36$
1 2 3 4 5	1 2 3 4 5	1 2 4 5 6
1 2 3 4 6	1 2 3 4 6	2 3 5 6 4
1 2 3 5 6	1 2 3 5 7	3 1 6 4 5
1 2 4 5 6	1 2 3 6 7	1 2 4 5 9
1 3 4 5 6	1 2 4 5 8	2 3 5 6 7
2 3 4 5 6	1 2 4 6 8	3 1 6 4 8
	1 2 5 7 8	2 3 6 4 9
	1 2 6 7 8	3 1 4 5 7
	1 3 4 5 9	1 2 5 6 8
	1 3 4 6 9	3 1 5 6 9
	1 3 5 7 9	1 2 6 4 7
	1 3 6 7 9	2 3 4 5 8
	1 4 5 8 9	4 5 7 8 9
	1 4 6 8 9	5 6 8 9 7
	1 5 7 8 9	6 4 9 7 8
	1 6 7 8 9	4 5 7 8 3
	2 3 4 5 10	5 6 8 9 1
	2 3 4 6 10	6 4 9 7 2
	2 3 5 7 10	5 6 9 7 3
	2 3 6 7 10	6 4 7 8 1
	2 4 5 8 10	4 5 8 9 2
	2 4 6 8 10	6 4 8 9 3
	2 5 7 8 10	4 5 9 7 1
	2 6 7 8 10	5 6 7 8 2
	3 4 5 9 10	7 8 1 2 3
	3 4 6 9 10	8 9 2 3 1
	3 5 7 9 10	9 7 3 1 2
	3 6 7 9 10	7 8 1 2 6
	4 5 8 9 10	8 9 2 3 4
	4 6 8 9 10	9 7 3 1 5
	5 7 8 9 10	8 9 3 1 6
	6 7 8 9 10	9 7 1 2 4
		7 8 2 3 5
		9 7 2 3 6
		7 8 3 1 4
		8 9 1 2 5

permutations of the five orthogonal axes and the reflections in each. In more mathematical terminology[12] it is the wreath product of the permutation groups S_2 and S_5 written S_2 wr S_5. Less regular triangulations of S^4 could be obtained by subdividing α_5 or β_5 in a systematic fashion or by subdividing the faces of the only other regular solid in five dimensions—the cube.

B. CP^2

A highly symmetric triangulation of CP^2 has recently been given by Kühnel and Lassmann[13] and many of its beautiful properties explained in a lucid article by Kühnel and Banchoff.[14] Their triangulation, which they denote by CP_9^2, has $n_0 = 9$, $n_1 = 36$, $n_2 = 84$, $n_3 = 90$, and $n_4 = 36$, so that the Euler number is indeed 3. They found their triangulation by a series of arguments that suggested nine vertices and then a computer search to see how a known list of eight vertex triangulations of the sphere could serve as links of a nine vertex manifold. Their vertex matrix for CP_9^2 is given in Table II. There is an edge connecting every pair of vertices and a triangle filling in every triple of vertices. The symmetry group of CP_9^2 is of order 54 and is generated by the permutations

TABLE II. The action for equal-edged triangulation of S^4 and CP^2.

Manifold	Triangulation	a	$HL_{\rm ext}/l$
	α_5	107.9	4.90
S^4	β_5	81.1	2.80
	round sphere	61.6	···
	CP_9^2	50.4	2.14
CP^2			
	Fubini–Study	37.7	···

$$
\begin{aligned}
\alpha &= (123)(465),\\
\beta &= (147)(258)(369),\\
\tau &= (12)(45)(78)\,.
\end{aligned}
\tag{2.4}
$$

The authors of Ref. 14 denote this by H_{54}.

C. T^4

The most straightforward way to triangulate a two-torus is to represent it as a rectangle with opposite sides identified, divide the rectangle into a sufficient number of smaller rectangles, and triangulate each one. An example is given in Fig. 1(a). In four dimensions, an analogous triangulation of the four-torus T^4 may be constructed by joining together triangulated hypercubes. This construction has been given in detail by Rocek and Williams[15] and used by Hamber and Williams[16] in explicit calculations. The minimum number of hypercubes is 81. This triangulation has 81 vertices and 1944

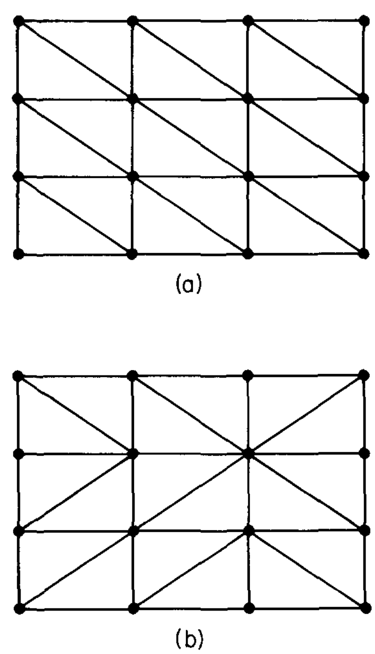

FIG. 1. Two triangulations of a two-torus. The identification of the opposite sides of a rectangle without twist produces a two-dimensional torus T^2. Division of this rectangle into triangles such that the conditions for a simplicial complex are satisfied produces a triangulation of T^2. Two nine-vertex triangulations are shown. The triangulation (a) builds the torus out of standard squares and has two translation symmetries. Viewing the torus as $S^1 \times S^1$ and applying the product construction described in the text to the product of two triangles produces the triangulation shown in (b). It is not as symmetric as (a).

four-simplices. The symmetry group clearly contains the symmetry group of the hypercubic lattice which makes T^4.

The straightforward triangulation of T^2 shown in Fig. 1(a) is not the one with the minimum number of vertices. The minimum number is 7. Similarly, triangulations of T^4 can be found with a smaller number of vertices than the hypercubic triangulation. Kühnel[17] has exhibited a 31 vertex triangulation of T^4. The vertex matrix for its four-simplices is generated by taking the four-simplices (0,1,3,7,15), (0,1,3,11,15), (0,1,5,13,15), and (0,4,5,13,15) and applying the group $x \to x+1$, $x \to -x$, $x \to 2x$ to all the entries considered as elements of $\mathbf{Z}_{31}$. There results a triangulation with 704 four-simplices.

D. $S^2 \times S^2$

The product of two simplices is not another simplex but a cell. For example, the product of two edges is not a triangle but a quadrilateral. Cells, however, can always be divided up into simplices and furthermore in a way which does not introduce any new vertices.[18] In this way a triangulation of a product manifold can be generated from triangulations of its products. The simplest example is the construction of a triangulation of a two-torus $T^2 = S^1 \times S^1$ from the product of two "triangles," which are the simplest triangulations of S^1. The result is a nine vertex triangulation of the torus (e.g., Fig. 1).

The process of triangulating the cells can be systematized as follows[18,19]: Consider the cell $\sigma^m \times \sigma^n$ which is the product of an m-simplex with an n-simplex. Number the vertices of σ^m in some ordered fashion $i_0 < i_1 < \cdots < i_m$ and do similarly for σ^n, $j_0 < j_1 < \cdots < j_n$. The vertices of the cell $\sigma^m \times \sigma^n$ are the pairs (i_α, j_β). The ordering of the vertices establishes a partial ordering on the pairs. We say $(i,j) < (k,l)$ if $i \leqslant k$, $j < l$, or if $i < k$, $j \leqslant l$. A triangulation of the cell $\sigma^m \times \sigma^n$ is given by the k-simplices spanned by vertices $(i_0, j_0),\ldots,(i_k, j_k)$ such that

$$(i_0, j_0) < (i_1, j_1) < \cdots < (i_k, j_k)\,. \tag{2.5}$$

A triangulation of a product manifold may be obtained by triangulating the products of simplices in its factors in this manner. In the case of the torus described above this systematic procedure yields the rather unsymmetric triangulation shown in Fig. 1(b).

Applied to the product of two tetrahedra, the above procedure yields a triangulation of $S^2 \times S^2$. There are 16 vertices formed by the products (i,j) of the four vertices of each tetrahedron, $i = 0,\ldots,3; j = 0,\ldots,3$. The four-dimensional cells are the products of the form $\sigma^2 \times \sigma^2$. The k-simplices of the triangulation are spanned by all sequences of the form (2.5) in which not all four vertices occur either in the sequence $i_0,\ldots,i_k$ or in the sequence $j_0,\ldots,j_k$. This condition arises because the triangles in the factors of $\sigma^2 \times \sigma^2$ have three vertices so that no more than three different vertices occur in any cell. This triangulation of $S^2 \times S^2$ has $n_0 = 16$, $n_1 = 84$, $n_2 = 216$, $n_3 = 240$, and $n_4 = 96$. Like the triangulation of the torus exhibited in Fig. 1(b), it is not very symmetric. There are 25 independent edge lengths. It is an interesting question whether there are more symmetric triangulations of $S^2 \times S^2$.

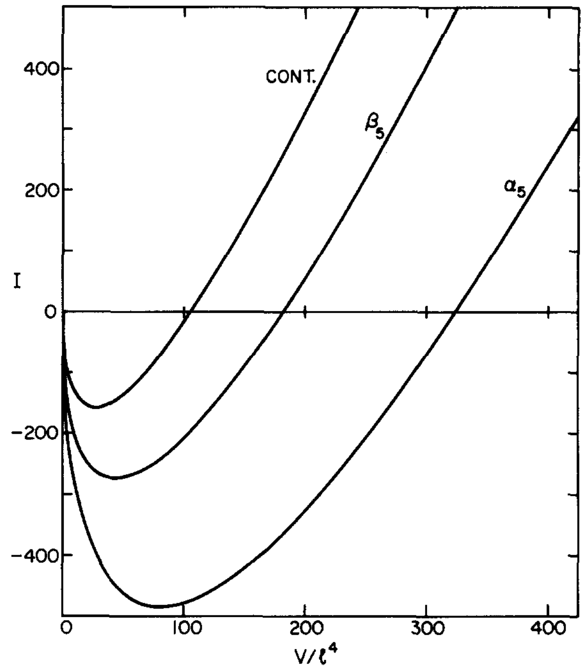

FIG. 2. The action for some homogeneous, isotropic, simplicial four-geometries on S^4. The figure shows the action for the four-geometries which are the boundaries of the five-simplex (α_5) and the five-dimensional cross polytope (β_5) (the five-dimensional generalization of the octohedron). In these triangulations no edge is distinguished from any other. The action for the geometries of highest symmetry with all edges equal is plotted against the total four-volume V for the value of the cosmological constant corresponding to $H^2 = 1$. Also plotted is the "continuum" action for the round four-sphere. The actions are negative for small V but become positive at large V due to the cosmological constant term in the action. At the minimum there is a solution of the Regge equations with all edges equal. The triangulation β_5 is more refined than α_5 and better approximates the continuum action.

E. $S^3 \times S^1$

An 11 vertex triangulation of $S^3 \times S^1$ has been constructed by Kühnel.[20] It is generated by taking the four-simplices (0,2,3,4,5), (0,1,3,4,5), (0,1,2,4,5), and (0,1,2,3,5) and applying the operation $x \to x+1$ to all vertices of each considered as elements of $\mathbf{Z}_{11}$. There result 44 four-simplices.

III. EVALUATING THE ACTION

The Regge action for a simplicial manifold consisting of collections of k-simplices Σ_k, $k = 0,1,\ldots,4$, is

$$l^2 I = -2 \sum_{\sigma \in \Sigma_2} V_2(\sigma)\theta(\sigma) + \frac{6H^2}{l^2} \sum_{\tau \in \Sigma_4} V_4(\tau)\,. \tag{3.1}$$

Here, we have written $3H^2/l^2$ for the cosmological constant, V_k is the volume of a k-simplex, and $\theta(\sigma)$ is the deficit angle of triangle σ. This is defined by

$$\theta(\sigma) = 2\pi - \sum_{\tau \supset \sigma} \theta(\sigma,\tau)\,, \tag{3.2}$$

where the sum is over the four-simplices τ, which contain σ, and $\theta(\sigma,\tau)$ is the dihedral angle between the two tetrahedral faces of τ, which intersect in σ. The volumes V_k and dihedral angles $\theta(\sigma,\tau)$ may all be expressed in terms of the squared edge lengths of the geometry through standard flat space formulas. (How to do this was reviewed in detail in Paper I, Sec. III.) In this way the action becomes a function of the squared edge lengths of the simplicial geometry.

For even the simple triangulations displayed in Sec. II the number of edge lengths is large enough that the functional form of the action can be readily displayed only on slices through the space of edge lengths. The symmetry of the triangulation often suggests suitable slices. In this section we display some numerical calculations of the Regge action on some obvious slices of the triangulations of S^4 and CP^2 described in Sec. II.

The edges of the triangulations α_5 and β_5 of S^4 are equivalent in the sense that any one edge is transformed into every other by the action of the symmetry group. It is therefore interesting to investigate the action of the triangulations when all their edge lengths are equal; this turns out to be an interesting case for CP_9^2 as well. Equivalently one can quote the action as a function of the total volume of the closed geometry since the total volume of n_4 four-simplices of equal squared edge lengths s is

$$V = n_4(\sqrt{5}/96)s^2 . \tag{3.3}$$

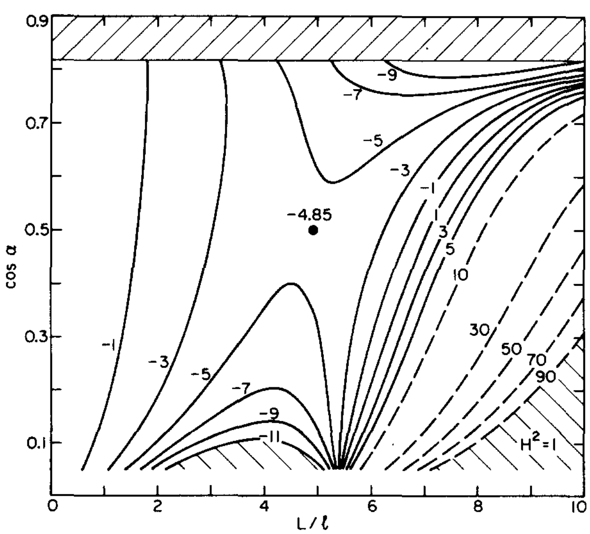

FIG. 3. The action for distorted five-simplices. The figure shows a contour map of the action (divided by 100) for a two-parameter family of five-simplices in which all the edges are of length L except for those emerging from one vertex which have the value $L/(2\cos\alpha)$. The cosmological constant has the value corresponding to $H^2 = 1$. As shown in Fig. 6, values of $\cos\alpha$ near zero correspond to long thin five-simplices while small values of α correspond to nearly flat five-simplices. The solid contour lines are spaced by units of 200 in I while the dotted ones are spaced by units of 2000. The contour lines become too closely spaced for clear display in the hatched areas at bottom and right. Contour lines are not shown for very small values of $\cos\alpha$ because the author's calculation was not very accurate there. There are no five-simplices with a value of $\cos\alpha$ greater than 0.81 because the four-simplex inequalities are not satisfied for larger values. The action is well behaved at this boundary of the space of edge lengths. The contour map shows the negative gravitational action associated with inhomogeneous conformal distortions. There is an extremum corresponding to all equal edges with a value of $4.9l$. This extremum is a saddle point not a maximum or minimum. The action generally becomes positive at large L because of the positive cosmological constant term. For large L and $\cos\alpha$ near 1, however, the action does not become positive but remains negative. These values correspond to large but nearly zero volume four-geometries. They are directions along which the sum over geometries evaluated along a real contour will not be exponentially damped.

On dimensional grounds the action at equal edge lengths must take the form

$$I = -a(V/l^4)^{1/2} + 6H^2V/l^4 . \tag{3.4}$$

Table II shows the dimensionless parameter a for the triangulations α_5 and β_5 of S^4 and the triangulation CP_9^2 of CP^2. For the case of α_5 and β_5 this parameter agrees with that calculated analytically by Hamber and Williams.[16] Its agreement is thus a check of the numerical algorithm.

Table II also shows the continuum value of the parameter a for the metrics of highest symmetry. This is the round sphere metric on S^4 and the Fubini–Study metric[21] on CP^2. The equal edged α_5, β_5, and CP_9^2 may be considered as approximations to these most symmetric continuum geometries. The simplicial actions lie below the continuum action for given V. In the case of S^4, as one proceeds from α_5 to the more refined triangulation β_5, the approximation to the continuum action improves, as is shown more graphically in Fig. 2. The actions are negative for small V as a consequence of the dominance of the curvature term and positive at large V because of the cosmological constant term. The minimum of the action corresponds to a solution of the Regge equations (1.2) as will be discussed in Sec. IV.

Figures 3, 4, and 5 show the Regge action for S^4 evaluated on some two-dimensional slices of the space of edge lengths. Figure 3 shows the action on a family of distorted five-simplices. All edges have the value L except for those emanating from one particular vertex which have the value $L/(2\cos\alpha)$. A two-dimensional analog is shown in Fig. 6. For fixed L, as $\cos\alpha$ increases from zero, the five-simplex ranges from "long and thin" to "short and squat." Beyond a value $\cos\alpha_e = (5/8)^{1/2}$, where it becomes degenerate, it is no longer possible to embed the five-simplex in five-dimension-

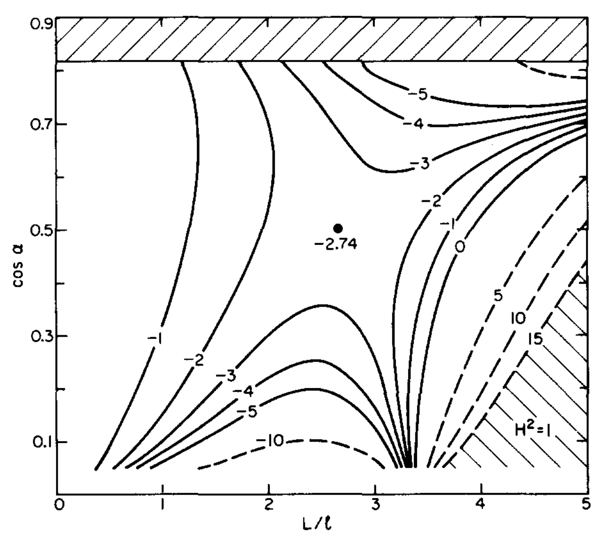

FIG. 4. The action for distorted β_5's. This figure shows the action (divided by 100) for a two-parameter family of geometries which are the surface of a five-cross polytope. All edges have the value L except those emerging from one vertex which have the value $L/(2\cos\alpha)$. The family of geometries is thus essentially the same as that displayed in Fig. 3 but with a more refined triangulation of S^4. The qualitative features of this map are essentially the same as those of Fig. 3 to whose caption the reader is referred for a description.

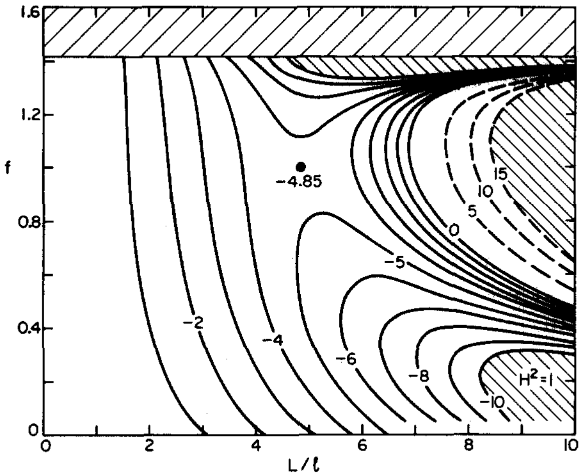

FIG. 5. The action for some homogeneous anisotropic five-simplices. If four of the five edges emerging from any one vertex of a five-simplex are assigned a value L and the remaining edge is assigned the value fL, there results a simplicial geometry which is homogeneous in the sense that any vertex is equivalent to any other, but anisotropic in the sense that not all directions at a given vertex are equivalent. A contour map of the action (divided by 100) for these geometries is shown here for the vlaue of the cosmological constant corresponding to $H^2 = 1$. Solid contour lines are spaced by intervals of 100 in action and dotted lines by 500. In the shaded regions the contours are too close together for clear display. The four-simplex inequalities are violated for sufficiently large f so that there is a boundary to the space of edge lengths. The action has a saddle point extremum at the isotropic (all edges equal) geometry previously shown in Fig. 2 and 3.

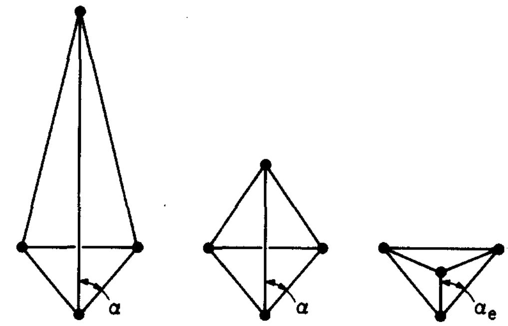

FIG. 6. A family of distorted three-simplices. The figure shows two-dimensional analogs of the distorted five-simplices whose action is displayed in Fig. 3. All the edges are equal except those emanating from the top vertex. The ratio of the two types of edges is controlled by the angle α. As α decreases from $\pi/2$ to 0 one moves from a long and thin three-simplex, through shorter and squatter ones. Eventually at a value α_e the three-simplex degenerates. For values of α smaller than α_e the geometry is no longer embeddable in flat three-dimensional space. It remains a well defined two-geometry, however, since the triangle inequalities are not violated until $\alpha = 0$.

al flat space even as it is impossible to embed the two-dimensional analog of Fig. 6 in three-dimensional flat space beyond $\cos\alpha_e = (3/4)^{1/2}$. Embedability in a higher-dimensional flat space, however, is not a physical requirement for a four-geometry. The physical range of α extends beyond α_e to the value $\alpha_{\rm crit}$ where $\cos\alpha_{\rm crit} = (2/3)^{1/2}$ at which the volume of the four-simplices vanishes and the four-simplex analog of the triangle inequalities are no longer satisfied.

One may think of the sequence of five-simplices generated by varying α as produced by a conformal deformation of the equal-edged five-simplex. Following Rocek and Williams,[16] a conformal transformation of a simplicial geometry may be defined by giving a function Ω_i on the vertices and then transforming the edge lengths as

$$s_{ij} = \Omega_i \Omega_j \tilde{s}_{ij} \,. \tag{3.5}$$

If we take $\tilde{s}_{ij} = L$, for all i and j and $\Omega_i = 1$, on all vertices but one where it equals $1/(2\cos\alpha)$, we recover the sequence of distorted five-simplices.

A contour map of the action for the two-parameter family $(L, \cos\alpha)$ of distorted five-simplices is shown[22] in Fig. 3. A similar family of distorted cross polytopes can be constructed by singling out the edges extending from a particular vertex. The contour map for this family is shown in Fig. 4. The two cases have essentially the same features: The action becomes positive for large L where the cosmological constant term dominates. There is one extremum which has all edges equal ($\cos\alpha = \frac{1}{2}$). It is not a minimum or a maximum but a saddle point. In the directions of conformal deformation away from the extremum the action becomes significantly negative. This is a simplicial example of the nonpositivity of the gravitational action in the continuum theory.[23]

The distorted α_5's and β_5's described above are inhomogeneous in the sense that one vertex is distinguished from among all others. A class of homogeneous but anisotropic simplicial geometries may be produced by treating all vertices equally but by allowing different length edges to emanate from each vertex. For example, we can consider the five-parameter family of surfaces of the five-simplex obtained by allowing the five edges meeting in each vertex to take on different values. A simple example is the two-parameter family in which four edges have equal values and the remaining edge a distinct value. The action for such a two-parameter family is shown in Fig. 5. The familiar saddle point extremum is seen again on this new slice.

While all the vertices of the triangulation CP_9^2 of the manifold CP^2 are equivalent, all the edge lengths are not. The edge lengths fall into two classes. For any pair of edges of a given class there is an element of the symmetry group H_{54} which carries one edge into the other. With the labeling of the vertices used in Table I, one class (class I) consists of the edges (12), (13), (23), (45), (46), (56), (78), (79), and (89), and the other (class II) consists of all the rest. It is therefore interesting to plot the action when all the edge lengths of class I have the value $L_{\rm I}$ and all those of class II the value $L_{\rm II}$. Such a plot is shown in Fig. 7 for $H^2 = 1$. Again the familiar saddle point extrema of the action is observed at a point where $L_{\rm I} = L_{\rm II}$.

IV. SOLUTIONS FOR SIMPLICIAL GEOMETRIES WITH HIGH SYMMETRY

Two things are important for the exploration of the semiclassical approximation to a sum over geometries such as that of Eq. (1.1). First, one needs the extrema of the action, that is the solutions of the Regge equations

$$\frac{\partial I}{\partial s_i} = 0, \quad i = 1,\ldots,n_1. \tag{4.1}$$

Second, one needs the eigenvalues λ_i, $i = 1,\ldots,n_1$ of the second derivative matrix of the action

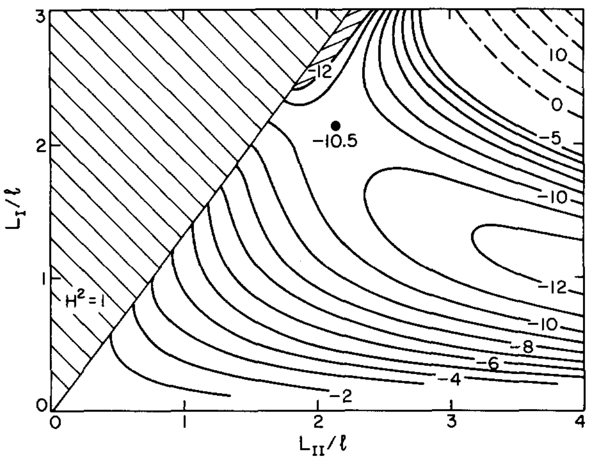

FIG. 7. The action for the triangulation CP_9^2 of the manifold CP^2 plotted as a function of the edge lengths of the two types of edges. Here CP_9^2 has two classes of edges which are carried into each other by the action of its symmetry group. The action (divided by 10) is plotted here for $H^2 = 1$ against the two values of the edge lengths $L_{\rm I}$ and $L_{\rm II}$. Solid contour lines are spaced by units of 10 and dotted lines by units of 50. The simplicial inequalities are violated in the shaded region to the left. Contours become too dense for display in the shaded region at the upper left. There is a saddle point extremum when $L_{\rm I} = L_{\rm II} = 2.14$.

$$I_{ij}^{(2)} = \frac{\partial^2 I}{\partial s_i \, \partial s_j} \tag{4.2}$$

evaluated at the extrema. The extrema determine where the action is to be evaluated in constructing the semiclassical approximation. The determinant of $I_{ij}^{(2)}$ gives the contribution from the integration over quadratic fluctuations about the extremum [see, e.g., Eq. (4.1) of Paper I]. This determinant is the product of the eigenvalues λ_i.

For simplicial geometries in which the typical edge length is small compared to the curvature scale, there will be local regions containing many vertices in which the geometry is essentially flat. Variations in the edge lengths corresponding to those induced by motions of these vertices in flat space will leave the geometry and hence the action approximately unchanged. These variations are the analogs of the diffeomorphisms of the continuum theory.[2,15] Their presence is signaled by small eigenvalues λ_i, which may require special treatment to evaluate the semiclassical approximation accurately. The values of the individual λ_i are therefore of interest.

In constructing the semiclassical approximation, solutions of (4.1) with both real and imaginary s_i are of interest, but the real solutions are easiest to find. One would expect to find real solutions for triangulations of manifolds for which real solutions of Einstein's equation

$$R_{\alpha\beta} = (3H^2/l^2) g_{\alpha\beta} \tag{4.3}$$

exist in the continuum theory. There could also be "spurious" extrema of a simplicial action which do not correspond to continuum solutions; some have been reported.[24] The Euler number of a manifold for which there is a real solution of Eq. (4.3) must satisfy

$$\chi > 0. \tag{4.4}$$

Thus, for the catalog of simplicial manifolds described in Sec. II, we do not expect to find real solutions for triangulations of T^4 and $S^1 \times S^3$ whose Euler number vanishes. We do expect to find solutions for S^4, CP^2, and $S^2 \times S^2$ for which continuum solutions exist.

For simplicial manifolds of high enough symmetry that no edge is distinguished from any other, it is easy to find the solutions of (4.1) with all edges equal if they exist. With all equal edges, all the $\partial I/\partial s_i$ are equal and one can easily calculate this one number and see where it passes through zero. We have carried out such a procedure for the triangulations α_5 and β_5 of S^4. As described in Paper I, $\partial I/\partial s_i$ was computed from

$$f_i \equiv l^2 \frac{\partial I}{\partial s_i} = -2 \sum_{\sigma \in \Sigma_2} \theta(\sigma) \frac{\partial V_2}{\partial s_i} + \frac{6H^2}{l^2} \sum_{\tau \in \Sigma_4} \frac{\partial V_4(\tau)}{\partial s_i} = 0. \tag{4.5}$$

The value of the edge length $L_{\rm ext}$ for which the action is extremized is shown in Table II for α_5 and β_5. These correspond to the extrema located graphically in Sec. III.

Figure 7 suggests that the extrema for CP_9^2 is found when all the edge lengths of the simplicial geometry are equal. We have verified numerically that all the f_i are equal when all the edge lengths are equal by evaluating (4.5). The value of the edge length $L_{\rm ext}$, which extremizes the action, is quoted in Table II.

TABLE III. Eigenvalues and multiplicities of $\partial^2 I/\partial s_i \, \partial s_j$ at the stationary point.

α_5		β_5		CP_9^2	
$l^4H^{-2}\lambda$	ρ_λ	$l^4H^{-2}\lambda$	ρ_λ	$l^4H^{-2}\lambda$	ρ_λ
−0.27	9	−0.39	4	−0.72	6
−0.11	5	−0.36	15	−0.67	2
+0.11	1	−0.34	5	−0.37	5
		−0.30	10	−0.25	6
		−0.15	5	−0.16	6
		+0.23	1	−0.048	2
				−0.047	2
				−0.012	6
				+0.29	1

The matrix $I_{ij}^{(2)}$ at the stationary point can be straightforwardly computed by numerical differentiation of Eq. (4.5) and its eigenvalues and eigenvectors can be computed by standard numerical methods. The resulting eigenvalues λ and their multiplicities ρ_λ are shown in Table III for α_5, β_5, and CP_9^2. These eigenvalues and eigenvectors are classifiable by the irreducible representations of the symmetry group of the triangulation. Not all irreducible representations will occur. Those that do occur and the corresponding multiplicities can be predicted as follows: The matrix $I_{ij}^{(2)}$ may be viewed as the matrix elements of a linear operation on an n_1-dimensional vector space in a basis in which there is a correspondence between the basis vectors and the edges in some standard order. We shall call this the edge vector space. A permutation of the vertices induces a permutation of the edges and thus a linear operation in the edge vector space. Since the symmetry group G of a simplicial complex is a subgroup of the permutation group on n_0 vertices S_{n_0}, its elements p can be represented as matrices on the edge vector space. These matrices give a representation of G which is reducible. The irreducible represenations that it contains are the irreducible representations that label the eigenvalues of $I_{ij}^{(2)}$ and the dimensions of these irreducible representations are the multiplicities with which the eigenvalues occur.

To find the irreducible representations of G, which are contained in the reducible representation on the edge vector space, one can analyze the characters of the reducible representation $\pi(p)$ into the characters of the irreducible representations $\chi^{(i)}(p)$ of G. That is, one forms

$$\langle \pi, \chi^{(i)} \rangle = \frac{1}{g} \sum_{p \in G} \pi(p) \chi^{(i)}(p), \tag{4.6}$$

where g is the order of G. An irreducible representation i occurs $\langle \pi, \chi^{(i)} \rangle$ times in π.

The characters $\pi(p)$ are easily seen to be

$$\pi(p) = (\text{number of edges left unchanged by } p). \tag{4.7}$$

The characters $\chi^{(i)}$ are determined by the group G. For α_5 the symmetry group is S_6. For β_5 it is S_2 wr S_5 as discussed in Sec. II. Character tables for these groups can be found in Ref. 12. A character table for the group H_{54} of CP_9^2 was very generously computed for the author by Dr. J. Saxl. The results of the above analysis are as follows: The 15-dimensional reducible representation of S_6 splits as $15 = 1 + 5 + 9$, where the factors are the dimensions of the irreducible representations. The 40-dimensional reducible representation of S_2 wr S_5 splits as $40 = 1 + 4 + 5 + 5 + 10 + 15$. The 36-dimensional reducible representation of H_{54} splits as $2(1 + 1 + 1) + 4 \cdot 6 + 2 + 2 + 2$ where multiplication indicates an irreducible representation which occurs more than once. The multiplicities of the eigenvalues calculated numerically shown in Table III are consistent with this analysis although there is an unaccounted for degeneracy among five of the eigenvalues for CP_9^2.

In each of the cases α_5, β_5, and CP_9^2, one of the eigenvalues is positive and all the rest are negative. The eigenvector of the positive eigenvalue shows that it corresponds to a uniform increase or decrease in the lengths of all edges. This reflects the fact that the stationary configuration is a minimum of (3.4). In all other principle directions the action is a maximum. Thus with these small number of vertices there are not enough degrees of freedom to represent the true physical degrees of freedom of the continuum theory.

V. ITERATIVE SOLUTIONS

Simplicial manifolds with a large number of vertices should not be expected to also possess high symmetry. Only for very special manifolds, therefore, can one expect to be able to use symmetry to find extrema of the action. In general one must simply solve the n_1 algebraic equations

$$f_i = l^2 \frac{\partial I}{\partial s_i} = 0 \tag{5.1}$$

for the sets of n_1 squared real or complex edge lengths which extremize the action.

The numerical problem of extremizing the action is a difficult one. One cannot use the familiar algorithms to search for maxima or minima because, as the examples in Sec. III show, the extrema are, in general, saddle points. There appears to be no better way of locating saddle points than solving Eqs. (5.1) directly.

The Newton–Raphson method is conceptually the simplest technique for solving a system of algebraic equations. Introducing vector notation in the edge vector space, one chooses an assignment of edge lengths $\mathbf{s}$ and attempts to solve for the displacement $\Delta\mathbf{s}$ to an assignment which will make $\mathbf{f}(\mathbf{s} + \Delta\mathbf{s}) = 0$. Expanding this requirement to first order in $\Delta\mathbf{s}$ one finds

$$\Delta\mathbf{s} = -[\mathsf{I}^{(2)}(\mathbf{s})]^{-1} \cdot \mathbf{f}(\mathbf{s}), \tag{5.2}$$

where $\mathsf{I}^{(2)}$ is the matrix of first derivatives of the field equations or second derivatives of the action

$$I_{ij}^{(2)} = \frac{\partial f_i}{\partial s_j} = \frac{\partial^2 I}{\partial s_i \, \partial s_j}. \tag{5.3}$$

In the usual Newton–Raphson method one picks intelligently a starting $\mathbf{s}$, iterates Eq. (5.2), and hopes to converge to a solution.

For simplicial manifolds with small numbers of vertices the Newton–Raphson method works well. For example, we have located the equal-edged extremum of α_5 by starting with significantly differing edge lengths and iterating Eq. (5.2) less than ten times.

For simplicial manifolds with larger number of vertices, the Newton–Raphson method is doomed to work poorly. It requires the inversion of the matrix $\mathsf{I}^{(2)}$. As has been discussed in Sec. IV, one expects the approximate diffeomorphisms of a manifold with a large number of vertices to mean that the matrix $\mathsf{I}^{(2)}$ will have near-zero eigenvalues corresponding to the directions along which the action is approximately constant. In the limit of large n_0 it is increasingly difficult to invert $\mathsf{I}^{(2)}$ and increasingly less likely to find a predicted $\Delta\mathbf{s}$ of reasonable size which does not violate the simplicial inequalities. We have verified this the hard way by attempting to solve the field equations for the 16 vertex triangulation of $S^2 \times S^2$ described in Sec. II. This triangulation has 25 inequivalent edges. Evaluated at a typical point, approximately four of the 25 eigenvalues were near zero. We were unable to locate an extremum in a short time.

Of course, there are many better algorithms for solving algebraic equations than the naive one (5.2) and some have been applied to the Regge calculus with success by Sorkin.[25] It would be of interest to apply them here.

The difficulty encountered in the Newton–Raphson method is generic. For large n_0, the extrema of the action lie in long "troughs" in the space edge lengths along which the action is nearly constant. There *is* an extremum, but it will be difficult to distinguish it from other configurations in the trough. This is no surprise and is in fact a familiar problem in general relativity. Einstein's equation does not possess a unique continuum metric for a solution but rather a family of metrics equivalent under diffeomorphisms. To pick out a unique solution one must specify a coordinate system or "fix the gauge." There are no exact diffeomorphisms of the general simplicial geometry but approximate diffeomorphisms produce an approximate ambiguity in the solution of the Regge equations in approximately the same way.

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