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SIMPLICIAL QUANTUM GRAVITY AND UNRULY TOPOLOGY

James B. HARTLE

Department of Physics, University of California,
Santa Barbara, CA 93106 USA

The sums over histories which specify the quantum state of the universe may be given concrete meaning with the methods of the Regge calculus. Sums over geometries may include sums over different topologies as well as sums over different metrics on a given topology. In a simplicial approximation to such a sum, a sum over topologies is a sum over different ways of putting simplices together to make a simplicial geometry while a sum over metrics becomes an integral over the geometry's edge lengths. The role which decision problems play in several approaches to summing over topologies is discussed and the possibility that geometries less regular than manifolds - unruly topologies - may contribute significantly to the sum is raised. In two dimensions pseudomanifolds are a reasonable candidate for unruly topologies.

1. INTRODUCTION

The proposal that the quantum state of the universe is the analog of the ground state for closed cosmologies is a compelling candidate for a law of initial conditions in cosmology.[1] Ian Moss and Jonathan Halliwell and Stephen Hawking have reviewed elsewhere in these proceedings how the correlations and fluctuations in this state can explain the large scale approximate homogeneity, isotropy and spatial flatness of the universe as well as being consistent with the observed spectrum of density fluctuations. The model calculations they review explore the wave function of the quantum state of the universe on regions of its configuration space which are close to the observed universe. To explore this proposal further one would like to calculate, and calculate in a systematic way, the wave function on increasingly exotic regions of the configuration space. For example, one would like to know it on the three geometries appropriate to large inhomogeneities, black holes, and complex topologies. It is just as important for the wave function of the universe to be small on these regions as it is for it to be large in the regions near what we observe.

One might also imagine extending the proposal for the quantum state of the universe to ask questions which are not the traditional issues of cosmology but are observable properties of geometry nevertheless. Why is spacetime four dimensional on large scales? Why does it have Euclidean topology locally on all scales which have been experimentally investigated? Are the familiar large scale topologies of the Friedmann universes in some sense preferred in the theory of initial conditions? To ask such questions one must be able to calculate the amplitudes in the quantum state of the universe, not only for different metrics, but also for different manifolds. Bryce DeWitt has stressed the great departure from the framework of traditional field theory needed to ask such questions[2] but I think they are of interest to pursue nonetheless.

In this lecture I would like to describe one approach to making systematic calculations of the sums over histories in quantum cosmology needed to investigate both the more traditional questions described above and the less traditional ones. It is not a new approach. It is the nearly twenty five year old invention of Tullio Regge called the Regge calculus[3].

2. SIMPLICIAL QUANTUM GRAVITY

A simplicial geometry is made up of flat simplices joined together. A two dimensional surface can be made out of flat triangles. A three dimensional manifold can be built out of tetrahedra; in four dimensions one uses 4-simplices and so on. The information about topology is contained in the rules by which the simplices are joined together. A metric is provided by an assignment of edge lengths to the simplices and a flat metric to their interiors. With this information one can, for example, calculate the distance along any curve threading the simplices.

A two dimensional surface made up of triangles is in general curved as, for example, the surface of the tetrahedron in Figure 1. The curvature is not in the interior of the triangles; they are flat. It is not on the edges; two triangles meeting in a common edge can be flattened without distorting them. Rather, the curvature of a two dimensional simplicial geometry is concentrated at its vertices, because one cannot flatten the triangles meeting in a vertex without cutting one of the edges. If one does cut one of the edges and flatten, then the angle by which the separated

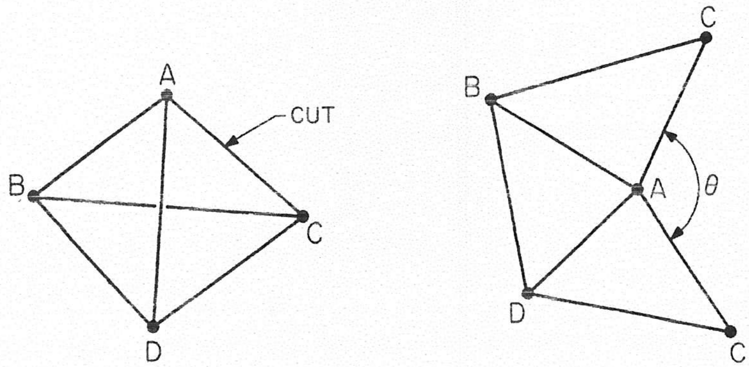

Figure 1. The surface of a tetrahedron is a two dimensional surface whose curvature is concentrated at its vertices. To flatten the three triangles meeting at vertex A one could cut the tetrahedron along edge AC. The angle θ by which the edges AC fail to meet when flattened is a measure of the curvature at A called the deficit angle.

edges fail to meet is a measure of the curvature called the deficit angle. (See Figure 1.) It is the angle by which a vector would be rotated if parallel transported around the vertex. Concretely, the deficit angle is 2π minus the sum of the interior angles of the triangles meeting at the vertex. It can thus be expressed as a function of their edge lengths.

In four dimensions the situation is similar with all dimensions increased by 2. The geometry is built from flat 4-simplices. Curvature is concentrated on the two dimensional triangles in which they intersect. There is a deficit angle associated with each triangle which is 2π minus the sum of the interior angles between the bounding tetrahedra of the 4-simplices which intersect the triangle.

As Regge showed, Einstein's familiar gravitational action may be expressed as a function of the deficit angles and the volumes of the simplices. For example, the Euclidean Einstein action with cosmological constant for a connected closed manifold in n-dimensions is,

$$g_n \ell^{n-2} I_n = - \int d^n x (g)^{1/2} (R-2\Lambda) \tag{1}$$

Here, $\ell = (16\pi G)^{1/2}$ is the Planck length and g_n is a dimensionless coupling. We use units where $\hbar = c = 1$ throughout. On a simplicial geometry (1) becomes exactly

$$g_n \ell^{n-2} I_n = -2 \sum_{\sigma \in \Sigma_{n-2}} V_{n-2}\theta_{n-2} + 2\Lambda \sum_{\tau \in \Sigma_n} V_n \quad . \tag{2}$$

Here, Σ_k is the collection of k-simplices and V_k is the volume of a k-simplex. The deficit angle θ_{n-2} is defined by

$$\theta_{n-2}(\sigma) = 2\pi - \sum_{\tau \supset \sigma} \theta_k(\sigma,\tau) \quad , \tag{3}$$

where the sum is over all the n-simplices τ which meet σ, and the $\theta_{n-2}(\sigma,\tau)$ are their interior angles at σ. Both V_k and $\theta_{n-2}(\sigma,\tau)$ are simply expressible in terms of the edge lengths through standard flat space formulae. By using these expressions in (2) the action becomes a function of the edge lengths. As Hamber and Williams have shown[4] other gravitational actions, such as curvature squared Lagrangians, may be similarly expressed ν in an approximate form which becomes exact in the continuum limit.

Sums over geometries may be given concrete meaning by taking limits of syms of simplicial approximations to them. This is analogous to defining the Riemann integral of a function as the limit of sums of the area under piecewise linear approximations to it. Consider, by way of example, the sum over four geometries which gives the expectation value of physical quantity $A[\mathcal{G}]$ in the quantum state of the universe,[1]

$$\langle A \rangle = \frac{\Sigma_{\mathcal{G}} A[\mathcal{G}]\exp(-I[\mathcal{G}])}{\Sigma_{\mathcal{G}} \exp(-I[\mathcal{G}]} \tag{4}$$

The sum is over compact, closed Euclidean four geometries. We are accustomed to think of a geometry as a manifold with a metric, and one might therefore want to think of the sum in (4) as a sum over closed manifolds and a sum over physically distinct metrics on those manifolds. Simplicial approximation could be used to give a concrete meaning to such a sum as follows: (1) Fix a number of vertices n_0. (2) Approximate the sum over manifolds as the sum over the number of ways of putting together 4-simplices so as to make a simplicial manifold with n_0 vertices. (3) Approximate the sum over physically distinct metrics by a multiple integral over the squared edge lengths s_i. (4) Take the limit of these sums as n_0 goes to infinity. In short, express $\langle A \rangle$ as

$$\langle A\rangle = \lim_{n_0\to\infty} \frac{\sum_{M(n_0)} \int_C d\Sigma_1 A(s_i,M)\exp[-I(s_i,M)]}{\sum_{M(n_0)} \int_C d\Sigma_1 \exp[-I(s_i,M)]} . \tag{5}$$

There remains the specification of the measure $d\Sigma_1$ and the contour C for the integral over edge lengths. Of course, today we understand little about the convergence of such a process but it is at least definite enough to be discussed.

Figures 2 and 3 show a few simple numerical calculations[5] of the Regge action which enters into the sum (5). In Figures 2 and 3 the manifold is the four sphere, S^4. The simplest triangula-

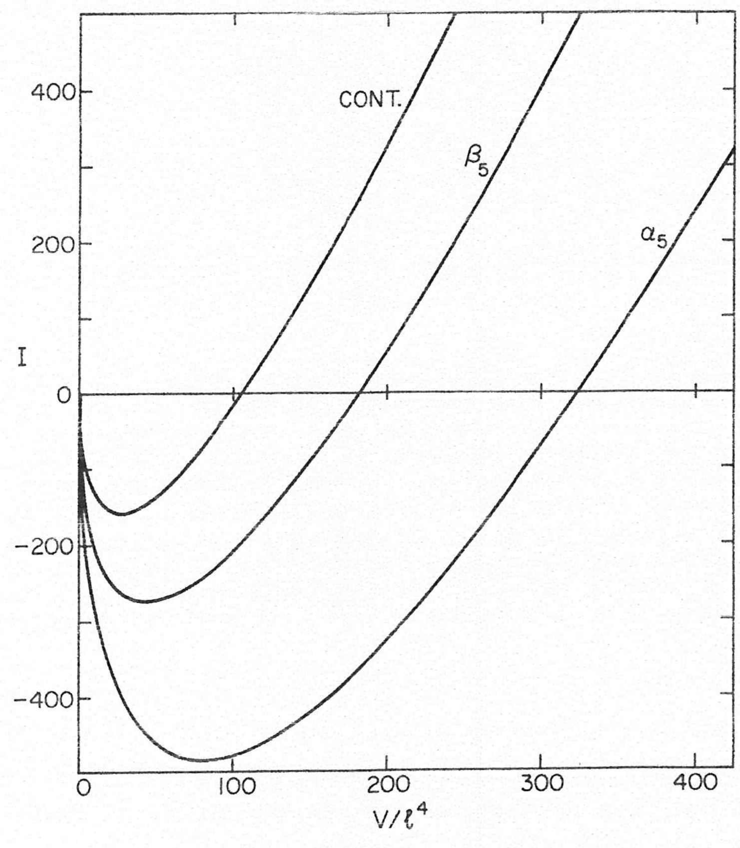

Figure 2. The action for some homogeneous isotropic four geometries as a function of volume. The figure shows the action for the 4-geometries which are the boundary of a 5-simplex (α_5) and the 5-dimensional cross polytope (β_5) (the 5-dimensional generalization of the octohedron) when all of their edges are equal. Also plotted is the "continuum" action for the 4-sphere.

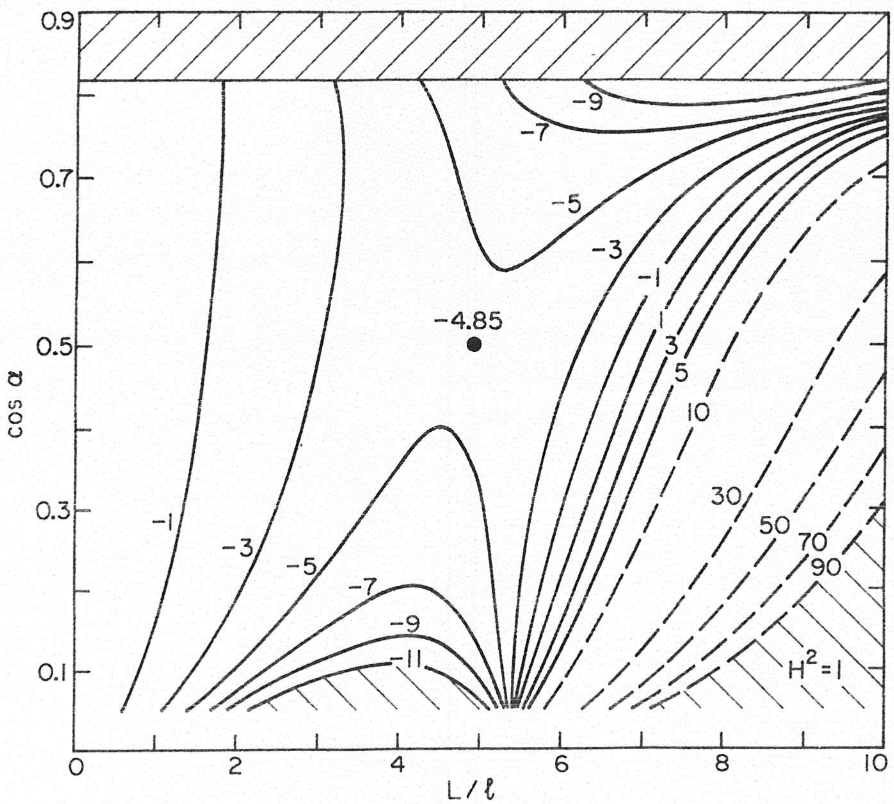

Figure 3. The action for distorted 5-simplices. The figure shows the action (divided by 100) for a two parameter family of 5-simplices in which all the edge lengths are L except for the edges emerging from one vertex which are $L/(2\cos\alpha)$. α near $\pi/2$ corresponds to long thin 5-simplices. α near 0 corresponds to nearly flat 5-simplices. There are no 5-simplices with $\cos\alpha$ greater than .81 because the 4-simplex inequalities would be violated. There is a saddle point corresponding to equal edges of value about 4.9. This is a solution of the Regge equations. The figure displays the negative gravitational action arising from conformal distortions.

tions of S^4 are the four dimensional surface of a 5-simplex (α_5) and the four dimensional surface of the 5-cross polytope (β_5) - the 5 dimensional generalization of the octohedron. These are the only regular solids in five dimensions. The 5 simplex has 6 vertices, 15 edges, 20 triangles, 15 tetrahedra and 6 4-simplices. The cross polytope has 10 vertices, 40 edges, 80 triangles, 80 tetrahedra and 32 4-simplices. Figure 2 shows the action for these triangulations as a function of four volume when all their edges are equal and the cosmological constant is $\Lambda = H^2/3$ with H equal to unity in Planck units.[4] The action is always lower than the "continuum" value corresponding to the round four sphere but becomes closer to it as we move from the coarsest triangulation α_5 to the finer β_5.

Figure 3 shows a family of distorted 5-simplices. All the edges have the value L except those leading to a particular vertex which have the value $L/(2\cos\alpha)$. $\cos\alpha$ near 0 thus corresponds to "long and thin" 5-simplices while large $\cos\alpha$ 5-simplices are "short and squat." $\cos\alpha$ cannot be too large because the analog of the triangle inequality for 4-simplices would not be satisfied. The two parameter family shows the characteristic saddle behavior of Einstein's action. There is an extremum when all the edges are equal to about $4.90\ell/H$. This is a solution of the discrete field equations corresponding to Euclidean de Sitter space. At this solution the action is neither a maximum nor a minimum but a saddle point.

For Figure 4 the manifold is CP^2 in the beautiful triangulation of Kühnel and Lassmann[6], CP^2_9. This has 9 vertices, 36 edges, 84 triangles, 90 tetrahedra and 36 4-simplices. Under the symmetry group of the triangulation the edges fall into two classes - 9 in one class (class I) and 27 in another (class II). Figure 4 shows the action when all the class I edges have a value L_I and all the class II edges a value L_{II}. There is a solution of the Regge equations at the saddle point $L_I = L_{II} = 2.14(\ell/H)$.

3. SUMMING OVER TOPOLOGY

Summing over metrics is only one of two parts of a sum over geometries even as the metric is only one of two parts in the specification of a geometry. The other part might be loosely called the "topology" and it is therefore of interest to investigate sums over topologies. Simplicial approximation is a natural framework in which to do this, because the topological and metrical aspects of a simplicial geometry are very clearly separated. The topological information is contained in the rules by which the simplices are joined together. The metrical information is contained in the assignment of edge lengths. In particular, it is possible to have geometries with complicated topologies but with relatively few edges. With the Regge calculus one can study topology cheaply.

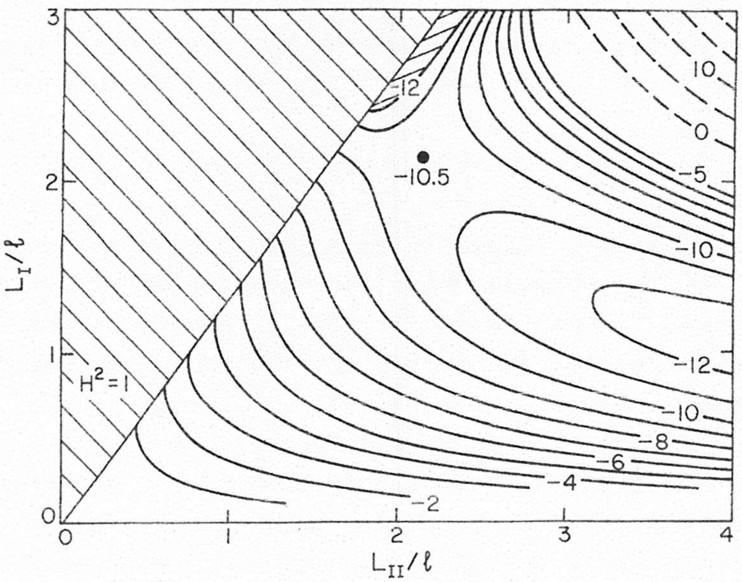

Figure 4. The action on a triangulation of CP^2. The nine vertex triangulation CP^2_9 has two classes of edges which transform among themselves under the action of its symmetry group. The figure shows a contour map of the action (divided by 100) when all the edges of the first class have the value L_I and all of the second class have the value L_{II}. The cosmological constant has the value specified by $H^2 = 1$. In the shaded region at upper left the simplicial inequalities are violated. There is a saddle point extremum and a solution of the Regge equations when $L_I = L_{II} = 2.14\ell/H^2$.

To sum over the topologies of simplicial geometries with n_0 vertices is to sum over some collection of simplices with a total of n_0 vertices. The widest reasonable framework in which to discuss such collections is provided by the connected simplicial complexes. A connected simplicial complex is a collection of simplices such that if a simplex is in the collection then so are all its faces, and such that any two vertices can be connected by a sequence of edges. What connected complexes should be allowed? A natural restriction is to sum only over complexes which are manifolds - that is, such that each point has a neighborhood which is topologically equivalent to an open ball in $\mathbb{R}^n$. In classical general relativity, geometries on manifolds are the mathematical implementation of the principle of equivalence. That principle tells us that locally spacetime is indistinguishable from flat space, and this is the defining characteristic of a manifold. It would, therefore, seem reasonable to consider geometries on manifolds in the quantum regime although it is less clear that on the scale of the Planck length the principle of equivalence should be enforced in this strong way.

It is not straightforward to carry out a sum over manifolds. To do so in the framework of simplicial quantum gravity there must be a rule for stating which simplicial complexes with n_0 vertices should be included in the sum (5) and with what weight. One might think that one should first classify all manifolds and include one triangulation of each different type in the sum. The sum would then include one S^4, one T^4, one $S^2 x S^2$, one $S^3 x S^1$, one K3, and so on. However, in four dimensions the classification problem for manifolds is unsolvable[7]. That is, roughly speaking, there does not exist a computer program which given <u>any</u> two simplicial manifolds will run, halt and print "yes" if the two manifolds are the same and "no" if they are not.

This mathematical result does not rule out the proposed sum as a testable law of physics. After all to compare with observations we will only need the sum (5) to some accuracy ϵ. It could be the case that to evaluate (5) to an accuracy ϵ only complexes with $n_0 < N(\epsilon)$ vertices are needed. Nothing in the unsolvability of the classification problem prevents one from devising an algorithm for deciding whether two simplicial manifolds with $n_0 < N$ vertices are the same. The theorem only shows that a universal algorithm which will work for <u>any</u> N does not exist. What the result <u>does</u> show is that ever more novel mathematical ideas will be needed to devise algorithms to carry out the sum (5) to even smaller levels of accuracy. In this respect, this enterprise in theoretical physics would be more like those of experimental physics.

Were each manifold to contribute once in the sum over topologies it seems likely that there would be measurable numbers of the theory (the expectation values which result) which would not be computable in the mathematical sense. Although, as described above a theory with this property could still be tested, it would be of a radically different type than those encountered previously in physics[8]. This perhaps is motivation enough for asking whether the sum over topologies might be reasonably defined on another class of simplicial complexes in such a way that there is a universal algorithm for carrying out the sum. There are several possibilities[9,10]. Of these the most radical is the idea that one should abandon manifolds and, therefore, the principle of equivalence at the Planck scale and sum over "unruly topologies".

In the sum over histories formulation of quantum mechanics we are familiar with the idea of "unruly histories". These are histories which contribute significantly to the sums for quantum amplitudes but which are less regular than the classical histories. For example, in particle quantum mechanics the dominant paths are non-differentiable while the classical path is always differentiable. In the quantum theory of spacetime we expect to sum over unruly metrics. It seems only reasonable to suppose that we shall have to sum over unruly topologies as well.

The majority of complexes are not manifolds. Some two dimensional examples can be seen later in this paper in Figures 5b, 6 and 7. Some idea of the numbers in two dimensions can be gained from Table 1. A suitable class of simplicial complexes for defining a sum over topologies in quantum geometry must be such that

(1) the action for general relativity can be defined,

(2) there is an algorithm for listing the members of the class,

(3) manifolds are the dominant contribution to the sum over histories in the classical limit.

The last condition is the important one . It guarantees that the principle of equivalence is recovered in the classical limit.

In the laboratory of two dimensional quantum gravity the question of a suitable class of unruly topologies is easily addressed. This is because two dimensional Einstein gravity has no metric degrees of freedom. It is not, however, topologically trivial.

The Regge action extends naturally to any simplicial complex in two dimensions. Recall that

$$g_2 I_2 = -2 \sum_{\sigma\in\Sigma_0} \theta(\sigma) + 2\Lambda \sum_{\tau\in\Sigma_2} V_2(\tau) \quad , \tag{6}$$

where the first sum is over the vertices and the second is over the triangles. Insert the definition (3) in this expression, interchange orders in the resulting double sum over vertices and triangles and note that the sum of the interior angles of a triangle is π. One finds

Table 1. Two Dimensional Labeled Simplicial Complexes*

n_0	Number of Complexes	Spheres $\chi = 2$	RP^2's $\chi = 1$	Tori $\chi = 0$
4	16	1	0	0
5	1,024	10	0	0
6	1,048,576	195	12	0

*This table shows the number of complexes of homogeneous dimension 2 which can be made with n vertices and the number of manifolds among them. The complexes are neither necessarily connected nor closed. No attempt has been made to eliminate redundancies so that two complexes which differ only by permutations of the vertices are both counted.

$$g_2 I_2 = -4\pi(n_0 - n_2/2) + 2\Lambda A \quad , \tag{7}$$

where n_0 is the number of vertices, n_2 the number of triangles and A is the total area. The curvature part of the action is independent of the edge lengths and is therefore metrically trivial. The action, however, does depend on how the simplices are joined together, that is, on the topology. This clean separation of metric and topology makes two dimensional Einstein gravity less interesting than the higher dimensional cases but it also makes topological questions easier to analyze.

Let us start with simplicial complexes which are two manifolds and enlarge the class by giving up as little as possible until a larger class is found which satisfies our criteria (1), (2) and (3). If a complex is going to fail to be a manifold it must fail on some collection of points. We give up least if we allow failure only at some discrete number of vertices of the complex and do not permit failure along the edges. This means we require every edge to be the face of exactly two triangles as in the complex in Figure 5a. We thus exclude complexes like Figure 5b which branch on an edge but permit those like Figure 6 which fail at vertices. For non-branching complexes, $3n_2 = 2n_1$ and the action is

$$g_2 I_2 = -4\pi\chi + 2\Lambda A \tag{8}$$

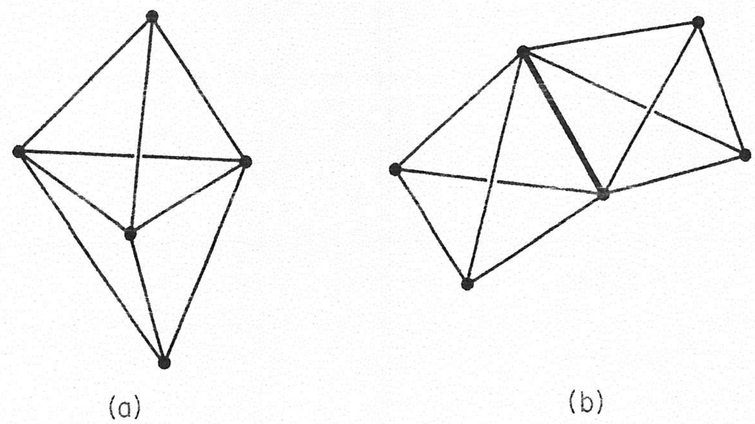

Figure 5. Branching and non-branching complexes. Non-branching two dimensional complexes like that in (a) have exactly two triangles intersecting at any one edge. The complex in (b) has four triangles intersecting along the more heavily drawn edge and is therefore a branching complex. Branching complexes fail to be manifolds at the edges on which they branch.

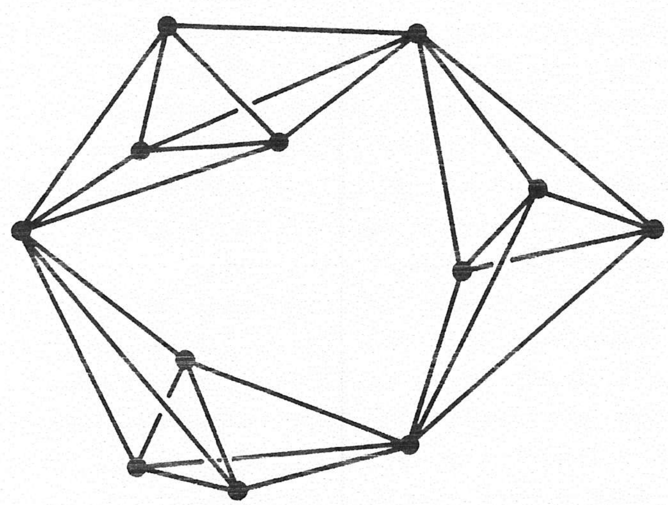

Figure 6. A two dimensional non-branching complex which fails to be a manifold at three vertices. This complex is not strongly connected and is thus not a pseudomanifold. It has Euler number $\chi = 3$.

where $\chi = n_0 - n_1 + n_2$ is the Euler number, a topological invariant.

If we were to stop here we could easily violate our criterion that a manifold have the smallest action. Compare the sphere in Figure 5a which has $\chi = 2$, with the complex in Figure 6. It has $\chi = 3$ and so a smaller action. This is because it consists of almost disconnected pieces. To prevent this we require that the complexes be strongly connected in the sense that any pair of triangles can be joined by a sequence of triangles connected along edges. The resulting complexes are called pseudomanifolds[11]. The complex in Figure 7 is a pseudomanifold whereas the one in Figure 6 is not. In two dimensions, pseudomanifolds have $\chi \leq 2$, and the pseudomanifold with $\chi = 2$ is the sphere. Thus the pseudomanifold with the smallest action is a manifold and we recover manifolds in the classical limit.

Most importantly for us, however, pseudomanifolds are easily enumerable. Their defining properties in n dimensions are

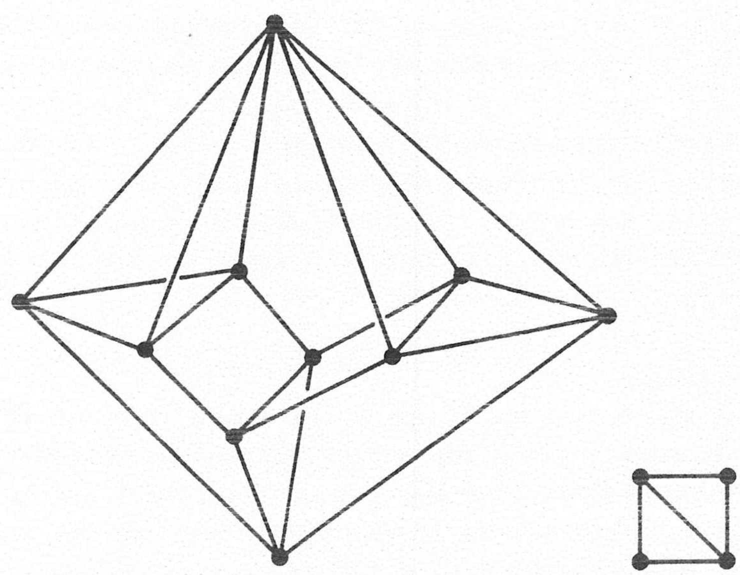

Figure 7. A pseudomanifold which fails to be a manifold at one vertex. The complex is two dimensional, non-branching and strongly connected. It is thus a pseudomanifold. It may be thought of as a sphere with two points identified. The complex has Euler number $\chi = 1$ so that its action is larger than a sphere of equal area. For pictorial clarity some of the edges triangulating quadralaterals have been omitted but they should be imagined as in the example at lower right.

(1) Homogeneous dimension - a simple of dimension $k < n$ is contained in some n-simplex.
(2) Nonbranching - an (n-1)-simplex is the face of exactly two n-simplices.
(3) Strongly connected - any two n-simplices can be connected by a sequence of n-simplices connected along (n-1)-simplices.

These defining properties are essentially combinatorial. Given n_0 vertices one can imagine listing all the possible collections of n-simplices and checking to see which are pseudomanifolds and which are not in a finite number of steps.

In two dimensions pseudomanifolds satisfy all three criteria for a class of complexes with which to define a sum over topologies. The Regge action is defined for them, there is an algorithm for enumerating them, and the pseudomanifold of least action is a manifold. In higher dimensions, finding a class which meets these criteria is a deeper question. Finding the configurations of least action, is now not only a question of topology but also of metric, that is, of solving the Regge equations. The possibilities for pseudomanifolds are so varied in higher dimensions that one must restrict the class of complexes further in order to have manifolds dominate in the classical limit. If however a suitable class can be found then by relaxing the principle of equivalence at the quantum level we will have an attractive class of geometries with which to define a sum over topologies in quantum gravity.

4. CONCLUSIONS

In familiar physical problems the state of a system is determined by dynamical evolution and by initial conditions. Evolution is fixed by dynamical laws applied to the system and the initial conditions by observations of it and the rest of the universe. Cosmology, however, requires a law of initial conditions. If this law is a specification of the quantum state of the universe, then both dynamical evolution and initial conditions are fixed by this state. The problems of finding dynamical laws and their initial conditions become one. In such a law we hope to find an explanation for the large scale regularities of cosmological spacetime. It is also possible that explanations can be found for some of the smaller scale familiar features of spacetime such as the topological properties we have discussed. Simplicial techniques provide a general but concrete approach to such questions. It will be interesting to see how far one can go.

ACKNOWLEDGMENT

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