Simplicial minisuperspace I. General discussion

James B. Hartle
Department of Physics, University of California, Santa Barbara, California 93106

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The use of the simplicial methods of the Regge calculus to construct a minisuperspace for quantum gravity and approximately evaluate the wave function of the state of minimum excitation is discussed.

I. INTRODUCTION

In the search for a conceptually clear and computationally manageable quantum theory of gravity, the sum-over-histories formulation of quantum mechanics has proved to be a powerful tool. For the investigation of conceptual issues, this formulation provides a direct route from classical action to quantum transition amplitude, and in a way which is easily accessible to formal manipulation.[1] Furthermore, the basic elements of the theory are often most clearly formulated in terms of the functional integrals which implement the sum-over-histories formulation. For example, a natural prescription for the wave function describing a closed cosmology in its state of minimum excitation is to take[2–4]

$$\Psi_0[\text{three-geometry}] = \sum_{\text{four-geometries},g} \exp(-I[g]/\hbar). \tag{1.1}$$

Here I is the Euclidean gravitational action including cosmological constant and the sum is over all compact Euclidean four-geometries which have the three-geometry as a boundary. It has been conjectured that this *is* the wave function of our universe.[3,5,6]

To investigate the consequences of a proposal like (1.1) or other consequences of a theory formulated in terms of integrals over four-geometries, one needs to evaluate these sums approximately. One can construct approximations to the sum by singling out a family of geometries described by only a few parameters or functions and carrying out the sum only over the geometries in this family. As the family of geometries is made larger and larger, one can hope to get a better and better approximation to the functional integral.

A restriction on the four-geometries which are included in the sum (1.1) will imply a restriction on the three-geometries which can occur as arguments of the resulting wave function. This is because the three-geometries must be embeddable in the four-geometries. The restriction thus reduces the configuration space on which the wave function is defined from the superspace of all three-geometries to a smaller space of three-geometries—a minisuperspace.[7] For this reason we call such approximations to the functional integral "minisuperspace approximations."

One way of constructing a minisuperspace approximation is to restrict the family of four-geometries to be only those with some symmetry. For example, one might restrict the four-geometries in (1.1) to have four-sphere topology and three-sphere symmetry and the three-geometries to be three-spheres of radius a_0. A single function of a single variable—radius as a function of polar angle—then describes these four-geometries. The sum over geometries reduces to a functional integral over this function. The wave function, which generally is a *functional* of the six components of the three-metric, reduces to a *function* of a single variable a_0. The minisuperspace is just half the real line.

Minisuperspace models based on symmetry have been used to explore quantum cosmology in the canonical theory of quantum gravity.[7,8] Minisuperspace approximations based on symmetry have been applied to the computation of the ground state wave function (1.1) in the functional integral formulation.[2,3,5,6] Minisuperspace approximations based on symmetries are simple to implement and generally easy to interpret. They do not, however, offer the possibility of systematic improvement because a general four-geometry is not well approximated by a symmetric one. It is therefore of interest to consider minisuperspace approximations which are not based on symmetries. The Regge calculus[9] provides an avenue to such approximations.

A general two-surface may be approximated by a surface made up of a net of flat triangles. The net of triangles is itself a two-geometry whose curvature is concentrated at the vertices where the triangles meet. The geometry of the surface is specified by the edge lengths of the triangles. Analogously, a general four-geometry may be approximated by a net of flat four-simplices. This net is a four-geometry specified by the edge lengths of the simplices. Its curvature is concentrated on the triangles in which these four-simplices intersect. Any geometrical quantity such as the curvature or the action can be expressed in terms of the edge lengths of the net. The conditions for the action of general relativity to be an extremum, with respect to variations of the edge lengths, give the simplicial analogs of Einstein's equations.

The simplicial methods adumbrated above were introduced into general relativity in a seminal paper by Regge.[9] These methods are usually called the "Regge calculus." They have been used in a number of interesting investigations in the classical theory of gravity. (See, for example, Refs. 10–19.) As Regge calculus is the natural lattice version of general relativity, it has also been extensively applied to the investigation of quantum theories of gravity (see, in particular, Refs. 20–26).

Simplicial approximation is a natural starting point for constructing a minisuperspace approximation to the functional integrals of quantum gravity. In this approximation one obtains the family of geometries integrated over in two steps. First fix a simplicial net, that is, specify the vertices of the net and the combinations of them that make up the one-simplices (edges), two-simplices (triangles), three-simplices (tetrahedra), and four-simplices. Second, assign lengths to

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the edges and allow these lengths to range over values consistent with their making up the flat simplices of the net. There results a family of four-geometries parametrized by the n_1 edge lengths of the net. Suppose that m_1 of these edges lie in the boundary of the net. These edges define a simplicial three-geometry. The minisuperspace is that portion of $\mathbb{R}^{m_1}$ swept out as the m_1 squared edge lengths of the boundary range over values for which the simplicial inequalities for triangles and tetrahedra are satisfied. The functional integral (1.1) is approximated by an $(n_1 - m_1)$-dimensional multiple integral over the interior edge lengths. Schematically it has the form

$$\Psi_0(s_i, i \in \partial\Sigma_1) = \int_C d\Sigma_1 \exp\left[\frac{-I(s_i)}{\hbar}\right]. \tag{1.2}$$

Here, $\partial\Sigma_1$ denotes the edge lengths of the boundary, I is the Regge action, and $d\Sigma_1$ denotes an integration over the interior edge lengths on a contour C.

Simplicial minisuperspace approximations have a number of significant advantages over those constructed from symmetry.

(1) To represent the sum over the four-metrics on a given manifold there is a different simplicial minisuperspace approximation for each triangulation of the manifold. This is a much larger class than can be generated by symmetry. In particular, as the number of vertices n_0 is increased one expects an arbitrary four-geometry to become closely approximated by some simplicial geometry. Thus the simplicial minisuperspace approximations in principle permit an investigation of the continuum limit.

(2) The simplicial minisuperspace approximation leads directly to a numerical evaluation of the functional integral as a multiple integral. Approximations based on symmetry, by contrast, generally require a further discretization for explicit evaluation.

(3) The simplicial minisuperspace approximation allows a simple and direct discussion of the role topology may play in quantum gravity. Topological information is contained in an elementary way in the simplicial net. The simplicial approximation allows one to investigate different topologies efficiently with simple geometries by investigating different simplicial nets with small numbers of edge lengths. In general, the simplicial minisuperspace approximation permits the investigation of global questions with crude geometries. Moreover, it does this in a way which is accessible to systematic improvement of the approximation. At a time in the development of quantum gravity when qualitative results are often more instructive than precise quantitative calculations, this is an important advantage.

In this paper we shall begin an investigation of the use of simplicial minisuperspace to approximately evaluate the state of minimum excitation in quantum gravity constructed according to the prescription (1.1). The methods we shall discuss are certainly applicable to computations of other quantities in the theory,[27] but we shall focus on this one to obtain concreteness and because of its important role in the theory.

In Sec. II we shall discuss the minisuperspace approximation in greater detail and in particular the form of the action, the issues involved in the choice of the measure, and those issues connected with the choice of integration contour C. The implementation of the Regge calculus to evaluate the action in (1.2) requires a certain amount of algebraic technology. We shall collect and describe the necessary technology in Sec. III. In Sec. IV we shall discuss the semiclassical approximation to the integral in (1.2) and in Sec. V we shall describe how the diffeomorphism group of general relativity should be recovered in the limit of larger and larger simplicial nets. Section VI describes how sums over different topologies might be implemented in the simplicial minisuperspace approximation.

II. THE FUNCTIONAL INTEGRAL FOR THE WAVE FUNCTION

The Euclidean functional integral prescription for the wave function of a closed cosmology in its state of minimum excitation assigns an amplitude Ψ_0 to each possible compact three-geometry. The general compact three-geometry consists of disconnected compact connected three-manifolds $\partial M_1,\ldots, \partial M_n$ each without boundary, each perhaps with nontrivial topology, and three-metrics $h_1, h_2,\ldots,h_n$ on these pieces. The wave function Ψ_0 is a functional of these metrics, given by[2]

$$\Psi_0[h_1,\partial M_1;h_2,\partial M_2;\ldots;h_n,\partial M_n] = \sum_M \nu(M)\int_C \delta g \exp(-I[g,M]). \tag{2.1}$$

The sum is over a class of compact four-manifolds, each with boundary ∂M consisting of the pieces $\partial M_1,\ldots,\partial M_n$ and each contributing with weight $\nu(M)$. The functional integral is over physically distinct metrics on M which induce the metrics $h_1,\ldots,h_n$ on $\partial M_1,\ldots, \partial M_n$. I is the action for general relativity

$$l^2 I[g,M] = -2\int_{\partial M} d^3x\, h^{1/2}K - \int_M d^4x\, g^{1/2}(R-2\Lambda). \tag{2.2}$$

Here and for the rest of the paper we use units where $\hbar = c = 1$. Thus, $l = (16\pi G)^{1/2}$ is the Planck length. To complete the prescription, four further specifications are needed: the class of manifolds summed over, the weight $\nu(M)$ to be given each one, the measure on the space of metrics, and the contour of integration in the space of metrics.

If the transition amplitudes of quantum gravity can be constructed as integrals over Euclidean four-geometries on different manifolds, then the weight given each manifold and the measure on the space of metrics must be consistent with the composition law for quantum amplitudes: $\langle a|b\rangle = \Sigma_c \langle a|c\rangle\langle c|b\rangle$. For example, if one admits disconnected three-geometries, then one expects multiply connected four-geometries will be required for consistency. The composition of an amplitude with two disconnected three-geometries in its final state together with an amplitude with two disconnected three-geometries in its initial state would be represented by a sum over a multiply connected four-geometry. The measure must also be consistent with the composition laws. It may be that the class of manifolds, ν, and the measure are determined by these restrictions and in the case of the measure there are calculations to this effect.[28] The contour C must be chosen so that the integral correctly represents a complete sum over compact geometries and so that the integral is convergent.

To begin a discussion of these specifications in a simplicial approximation to (2.1), let us restrict attention until Sec. VI to the sum over metrics on a fixed manifold M with a single boundary. Suppressing the labels M and ∂M we write

$$\Psi_0[h] = \int \delta g \exp(-I[g]). \tag{2.3}$$

The simplicial approximation to (2.3) as described in Sec. I is

$$\Psi_0(s_i, i \in \partial \Sigma_1) = \int_C d\Sigma_1 \exp[-I(s_i)]. \tag{2.4}$$

Here, we are considering a specific triangulation of M with vertices Σ_0, edges Σ_1, triangles Σ_2, tetrahedra Σ_3, and four-simplices Σ_4. The simplices of the boundary and the interior we denote by $\partial\Sigma_\alpha$ and int Σ_α, respectively.

The action I is the Regge action[9] modified by the boundary term required by the composition law for quantum amplitudes and the classical limit[29]

$$I = -2 \sum_{\sigma \in \partial \Sigma_2} A(\sigma)\psi(\sigma) - 2 \sum_{\sigma \in \mathrm{int}\, \Sigma_2} A(\sigma)\theta(\sigma) + 2\Lambda \sum_{\tau \in \Sigma_4} V_4(\tau), \tag{2.5}$$

where $A(\sigma) = V_2(\sigma)$ is the area of triangle σ and $V_4(\tau)$ is the volume of the four-simplex τ. The angle $\theta(\sigma)$ is the deficit angle for triangle σ. It is given by

$$\theta(\sigma) = 2\pi - \sum_\tau \theta(\sigma,\tau), \tag{2.6}$$

where the sum is over all the four-simplices which contain σ. $\theta(\sigma,\tau)$ is the dihedral angle between the two tetrahedra of τ which have σ as a common face. [For a two-dimensional picture see Fig. 1; for more details on how to compute $A(\sigma), V_4(\tau)$, and $\theta(\sigma,\tau)$ see Sec. III.] The angle $\psi(\sigma)$ necessary for calculating the boundary term is

$$\psi(\sigma) = \pi - \sum_\tau \theta(\sigma,\tau), \tag{2.7}$$

where the sum is over all four-simplices which intersect the boundary triangle σ. The surface term in (2.5) is just that

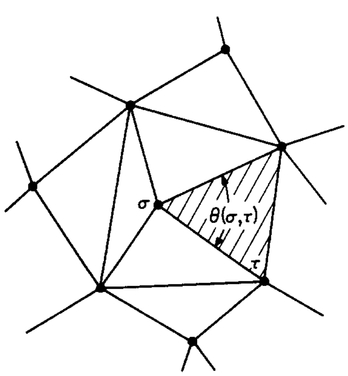

FIG. 1. A two-dimensional simplicial geometry is a net of flat triangles together with an assignment of lengths to their edges. The geometry includes both the interior points of the triangle as well as their edges. The distance between any two points can be determined in terms of the edge lengths. The curvature is concentrated at the vertices and is measured by the deficit angle. The deficit angle at a vertex σ is the 2π minus the sum of the dihedral angles $\theta(\sigma,\tau)$ over all triangles τ which have σ as a vertex.

necessary to ensure that the conditions for the extremum of the action correspond to the Regge equations

$$\sum_{\alpha \in \mathrm{int}\, \Sigma_2} \theta(\sigma) \frac{\partial A(\sigma)}{\partial s_i} = \Lambda \sum_{\tau \in \Sigma_4} \frac{\partial V_4(\tau)}{\partial s_i}, \tag{2.8}$$

which are the simplicial analogs of Einstein's equation.[9]

There are as yet no completely satisfactory arguments for the measure and the contour needed to complete the specification of the integral (2.4). For the measure we shall assume that the integration over edge lengths is restricted to values such that they define possible flat simplices. Thus the appropriate triangle, tetrahedral, and four-simplex inequalities are satisfied. Necessary and sufficient conditions for this are that the squared volumes of all simplices be positive or zero when expressed in terms of the squared edge lengths. To go further we can only proceed by analogy with the continuum case. There a number of possible measures have been put forward. DeWitt,[30] for example, has argued that an appropriate measure is

$$\prod_x \prod_{\mu < \nu} dg_{\mu\nu}(x). \tag{2.9}$$

Since the squares of edge lengths are linearly related to $g_{\mu\nu}$, the corresponding choice in the simplicial approximation is to take

$$d\Sigma_1 = \mu(s_i) \prod_{j \in \mathrm{int}\, \Sigma_1} ds_j, \tag{2.10}$$

where

$$\mu = \begin{cases} 1, & \text{with simplicial inequalities satisfied,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

In particular, this means that the measure does not vanish on zero-volume simplices.

The remaining specification in (2.4) is the contour C. At a minimum this must be chosen so that the integral (2.4) is convergent. For real edge lengths the action is no more positive in the simplicial approximation than it is in the continuum theory. For example, the action for the closed four-geometry of volume V_4, which is the surface of a five-simplex with all edges equal, is (see Ref. 26 or Paper II)

$$I(V_4) = -107.9(V_4/l^4)^{1/2} + 2\Lambda V_4/l^2. \tag{2.12}$$

For small V_4 and at the extremizing volume, this is negative. In the continuum theory the action can always be made negative by an appropriate conformal deformation of the metric.[31] The same is true in the simplicial approximation.[32] It seems likely that the action can be made arbitrarily negative by considering nets with near-zero four-volume but containing very-large-area triangles whose deficit angles are positive. If this is the case there is the presumption that the integrals in (2.4) will diverge in some large edge-length directions.

In the continuum theory of asymptotically flat spacetimes the functional integral can be made convergent by decomposing the integration over four-geometries into an integration over a conformal factor and one over conformal equivalence classes, and then rotating the conformal factor integration contour to complex values.[28] The same procedure applied in the linear theory of gravity does yield the correct ground state wave function[33] and arises naturally from the parametrization of the Hamiltonian path integral for that theory expressed in terms of the physical degrees of freedom.[34] One can exhibit analogous contours in the simplicial approximation along which the integral is convergent. In the absence of a compelling argument for one or the other, we shall not pursue their discussion further.

The information contained in the wave function Ψ_0 might be determined by carrying out the multiple integral (2.4) for a sampling of the space of edge lengths. In practice, this will be too much data to deal with since the dimension of the space of edge lengths can be very large. Equivalently and more usefully the information in Ψ_0 can be summarized by computing interesting expectation values[35] of Ψ_0:

$$\langle A \rangle = \frac{\int d(\partial\Sigma_1)\Psi_0(s_i)A(s_i)\Psi_0(s_i)}{\int d(\partial\Sigma_1)\Psi_0(s_i)\Psi_0(s_i)}, \tag{2.13}$$

where $d(\partial\Sigma_1)$ is the volume element on the boundary edge lengths analogous to (2.10). Certainly the information contained in Ψ_0 can be extracted from (2.13) by letting A range over appropriate filters sensitive to particular regions of edge lengths. Indeed, one imagines that the interesting physical questions can always be phrased in terms of the expectation value of an appropriate A. Monte Carlo numerical calculations will generally be much more feasible for expectation values of slowly varying A 's than for the wave function itself. For these reasons we shall for the most part be concerned with calculating expectation values in what follows.

Assuming that the choice of measures in $d\Sigma_1$ and $d(\partial\Sigma_1)$ is compatible, Eq. (2.4) may be inserted into (2.13) to give the following expression for $\langle A \rangle$:

$$\langle A \rangle = \frac{\int_C d\Sigma_1 A(s_i)\exp[-I(s_i)]}{\int_C d\Sigma_1 \exp[-I(s_i)]}, \tag{2.14}$$

where the integral is now over the squared edge lengths of the compact, boundaryless, manifold formed by identifying M and a copy of itself at its boundary. At the risk of some confusion we shall also call this M in the following. The boundary term in the action may now be dropped so that

$$l^2 I = -(\mathscr{R} - 2\Lambda\mathscr{V}), \tag{2.15}$$

where $\mathscr{R}$ is the "curvature part" of the action

$$\mathscr{R} = 2\sum_{\sigma\in\Sigma_2} A(\sigma)\theta(\sigma), \tag{2.16}$$

and $\mathscr{V}$ is the total four-volume

$$\mathscr{V} = \sum_{\sigma\in\Sigma_4} V_4(\sigma). \tag{2.17}$$

Here, Σ_1, Σ_2, etc., now refer to the whole compact manifold. If convenient, the contour C may be further distorted so that the s_i on the joining boundary assume complex values.

We have introduced the question of the convergence of the integral (2.4) at large edge lengths. A natural complementary question is its behavior at small edge lengths. In the form (2.13), when the contour runs along real edge lengths, a simple answer to this question can be given. The volume part of the action (2.5) is evidently well behaved at small edge lengths. The curvature part may be bounded as follows. In a flat four-simplex the dihedral angle $\theta(\sigma,\tau)$ always lies between 0 and π. From (2.6) it follows that the deficit angle $\theta(\sigma)$ is bounded by

$$-\pi[k_4(\sigma) - 2] < \theta(\sigma) < 2\pi, \tag{2.18}$$

where $k_4(\sigma)$ is the number of four-simplices containing the triangle σ. Thus for the curvature action we find

$$-4\pi\mathscr{A} \leqslant -\mathscr{R} \leqslant 2\pi K\mathscr{A}, \tag{2.19}$$

where

$$\mathscr{A} = \sum_{\sigma\in\Sigma_2} A(\sigma), \tag{2.20}$$

and

$$K = \max_{\sigma\in\Sigma_2}[k_4(\sigma) - 2]. \tag{2.21}$$

For any particular complex, K is a finite number (for example, for the four-complex which is composed of the faces of a five-simplex, $K = 1$). The curvature action is thus bounded above and below by a multiple of $\mathscr{A}$, and is well behaved as any of the edge lengths go to zero, or at zero-volume simplices where the simplicial inequalities are saturated. This suggests, in particular, that, though one may recover short-distance (ultraviolet) divergences in the continuum limit, they are not present in any finite simplicial approximation.

III. PRACTICAL REGGE CALCULUS

To implement a computation of a functional integral in the simplicial minisuperspace approximation as described in the preceding section, the action must be expressed in terms of the squared edge lengths of the simplicial net. In turn this means that one must be able to express areas, deficit angles, and four-volumes in terms of these squared edge lengths. Further, to evaluate the simplicial analog of the field equations one needs the derivative of areas and four-volumes, with respect to squared edge length. In this section we shall set out the formulas the author has found most useful for these calculations and briefly describe a method for deriving them. By and large both these formulas and the method of derivation have appeared elsewhere in the literature on the Regge calculus[36] and one imagines that they could be tracked down in the older mathematical literature. We collect them here with their derivation in order to have a complete description of the tools with which to attack our problem.

An n-simplex is specified by giving its $n+1$ vertices $(0,1,\ldots,n)$ in flat space. Define the n vectors e_i which start with the vertex 0 and proceed to the vertex i. The vectors $e_1,\ldots,e_n$ span the n-simplex. The volume n-form associated with the n-simplex may be defined as

$$\omega_n = e_1 \wedge e_2 \wedge \cdots \wedge e_n. \tag{3.1}$$

The formulas for volumes, areas, angles, etc., become simple to express and simple to derive when these n-forms are manipulated like vectors. To this end, we introduce a scalar product between two n-forms by the definition

$$f\cdot g = (1/n!) f_{\alpha_1\cdots\alpha_n} g^{\alpha_1\cdots\alpha_n}. \tag{3.2}$$

Consider, for example, the squared volume of an n-simplex, V_n^2. The product $\omega_n\cdot\omega_n$ must be proportional to V_n^2 because, up to sign, ω_n is easily shown to be independent of the choice of the perferred vertex 0 and there is no other symmetrical invariant with the correct dimension. By evaluating the constant of proportionality in any special case, one has

$$V_n^2 = \omega_n \cdot \omega_n. \tag{3.3}$$

A more direct derivation may be provided as follows: Consider the n-simplex as made up of the (n-1)-simplex ($0,1,...,n-1$) (the "base") and the vertex n. Evidently

$$\omega_n = \omega_{n-1} \wedge e_n. \tag{3.4}$$

Divide e_n into a part $e_n^\perp$ perpendicular to the base simplex and a part which lies in it. Insert this decomposition into (3.4) and this in turn in (3.2) to compute $\omega_n \cdot \omega_n$. One finds

$$\omega_n \cdot \omega_n = (1/n^2)(e_n^\perp \cdot e_n^\perp)(\omega_{n-1} \cdot \omega_{n-1}), \quad n > 1. \tag{3.5}$$

This formula can be used to prove inductively that $V_n^2 = \omega_n \cdot \omega_n$, because for each n this says $V_n = n^{-1}$ (height) $\times$ (volume of base).

To be computationally effective a formula like (3.3) must be expressed in terms of the squared edge lengths. This can be done by reexpressing the scalar product of two-volume n-forms $\omega_n = e_1 \wedge \cdots \wedge e_n$ and $\omega_n' = e_1' \wedge \cdots \wedge e_n'$ in terms of the scalar products of the constituent vectors as follows:

$$\omega_n \cdot \omega_n' = [1/(n!)^2] \det(e_i \cdot e_j'). \tag{3.6}$$

In particular, for the squared volume one has

$$V_n^2 = [1/(n!)^2] \det(e_i \cdot e_j), \tag{3.7}$$

where the $n \times n$ matrix of scalar products may be expressed in terms of the edge lengths s_{ij} between vertices i and j by

$$e_i \cdot e_j = \tfrac{1}{2}(s_{0i} + s_{0j} - s_{ij}). \tag{3.8}$$

Equations (3.7) and (3.8) express the volumes of the simplices of a net in terms of the squares of the edge lengths. Equations (3.7) and (3.8) do not show all the symmetries of the formula for V_n^2 fully expanded in terms of the edge lengths. This is because their construction involved a preferred vertex 0. Manifestly symmetric formulas can be constructed in terms of $(n+2)\times(n+2)$ bordered determinants (see e.g., Ref. 10) and these can be derived from Eq. (3.7) by a few simple determinantal manipulations. The evaluation of these symmetric formulas, however, involves more operations than does Eq. (3.7), which is therefore preferred for explicit computations.

The deficit angles are the remaining quantities needed to express the action in terms of the squared edge lengths. From Eq. (2.6) the computation of the deficit angle at a given triangle reduces to the computation of the dihedral angles at that triangle of the four-simplices which intersect it. We therefore consider this in detail and in slightly greater generality. Suppose we have an $(n+1)$-simplex which contains two n-simplices intersecting in a common $(n-1)$-simplex (the "hinge"). The dihedral angle θ of the $(n+1)$-simplex at the hinge is the angle between the normal to the hinge which lies in the first n-simplex and the normal to the hinge which lies in the second. It is given by

$$\cos\theta = \omega_n \cdot \omega_n' / (V_n V_n'), \tag{3.9}$$

where ω_n, ω_n' are the volume forms associated with the intersecting n-simplices and V_n, V_n' are the volumes of these simplices (the "lengths" of ω_n and ω_n', respectively). The correct sign for $\cos\theta$ will be obtained if ω_n and ω_n' are constructed as follows: Let ω_{n-1} be the volume form of the hinge. For the appropriate vectors e and e', write

$$\omega_n = \omega_{n-1} \wedge e, \quad \omega_n' = \omega_{n-1} \wedge e'. \tag{3.10}$$

Alternatively, ω_n and ω_n' must be oriented *oppositely* in a consistent orientation for the $(n+1)$-simplex.

Eq. (3.9) may be derived as follows: Decompose e and e' into parts orthogonal to and parallel to the hinge. Insert this decomposition into (3.10) and then into (3.9). One finds the right-hand side of (3.9) is $(e^\perp \cdot e'^\perp)/|e^\perp||e'^\perp|$. This, by definition, is the cosine of the dihedral angle.

Equation (3.9) may be expressed in terms of the edge lengths of the $(n+1)$-simplex through Eq. (3.6). Since each of the vectors e_i and e_i' lies along some edge of the $(n+1)$-simplex, their scalar product may be expressed in terms of the edge lengths through formulas analogous to Eq. (3.8).

A simple formula for $\sin\theta$ may be derived from the identity

$$(n+2)^2\omega^2(\omega \wedge a \wedge b)^2 = (n+1)^2\{(\omega \wedge a)^2(\omega \wedge b)^2 - [(\omega \wedge a)\cdot(\omega \wedge b)]^2\}, \tag{3.11}$$

valid for any n-form ω and one-forms a and b. Here, ω^2 means $\omega \cdot \omega$. Written out with ω_{n-1}, e, and e' as in Eq. (3.10) for ω, a, and b, respectively, decomposing e and e' into components parallel and orthogonal to ω_{n-1}, and using Eq. (3.3), one finds for $\sin\theta$

$$\sin\theta = [(n+1)/n]\,[V_{n-1} V_{n+1}/(V_n V_n')]. \tag{3.12}$$

Here, V_n and V_n' are the volumes of the n-simplices intersecting in the $(n-1)$-hinge, V_{n-1} is the volume of the hinge, and V_{n+1} is the volume of the $(n+1)$-simplex spanned by the two n-simplices. This result for $\sin\theta$ is easily expressed in terms of the edge lengths of the $(n+1)$-simplex by expressing the volumes in terms of the edge lengths. From a computational point of view, however, it is not as useful as (3.9). The dihedral angle θ ranges from 0 to π. From a formula for $\cos\theta$ one can recover the angle itself whereas from a formula for $\sin\theta$ one cannot, and it is the angle which enters in the action.

Equations (3.3) and (3.9) and their expressions in terms of edge lengths are all that are needed to evaluate the action. To evaluate the simplicial field equations [Eq. (2.8)] one also needs as expression for the derivative of the volume of an n-simplex with respect to one of its squared edge lengths, keeping the other edge lengths fixed. One is straightforwardly worked out from Eqs. (3.3) and (3.2) by choosing the preferred vertex 0 so that it is not a vertex of the edge of interest and then considering the variation in the vectors e_i produced by a variation in this edge length. One finds

$$\partial V_n^2/\partial s_{ij} = (1/n^2)\omega_{n-1} \cdot \omega_{n-1}', \tag{3.13}$$

where ω_{n-1} and ω_{n-1}' are the volume forms for the $(n-1)$-simplices formed by the vertices $(0,...,i-1,\, i+1,...,n)$ and $(0,...,j-1,\, j+1,...,n)$, respectively. Equations (3.6) and (3.8) express this in terms of the squared edge lengths.

To find the action or field equations for a simplicial net one would proceed as follows: First, the net must be specified. This means specifying the vertices, edges, triangles, tetrahedra, and four-simplices of the net. One way of doing this is to give a list of all of the simplices in terms of their vertices. A second, and equivalent, way is to give the incidence matrices which specify which $(n-1)$-simplices make up an n-simplex. All the topological information is contained in this specification of the net.

With a net in hand one can now proceed to specify the squared edge lengths and calculate the action. Not every assignment of edge lengths is consistent with the simplices having flat interiors. The triangle inequalities and their analogs for tetrahedra and four-simplices must be satisfied. Necessary and sufficient conditions for this are that the squared volumes of all the triangles, tetrahedra, and four-simplices in the net must have a positive squared volume, i.e.,

$$(V_2)^2 \geqslant 0, \quad (V_3)^2 \geqslant 0, \quad (V_4)^2 \geqslant 0, \tag{3.14}$$

for the whole net. To see this in the case of triangles, for example, fix two of the edges and consider $(V_2)^2$ as a function of the remaining squared edge length, s. As s is varied from a value where the triangle inequality is satisfied to one where it is saturated, $(V_2)^2$ ranges from a positive value to zero. Since $(V_2)^2$ is quadratic in s no further regions of positive $(V_2)^2$ exist outside the range where the triangle inequality is satisfied. The generalization to higher simplices is straightforward. The conditions (3.14) are independent. One can find, for example, squared edge lengths for which $(V_3)^2$ is positive but the triangle inequalities are violated.

For an assignment of squared edge length which does satisfy (3.14) one can compute the action and field equations by evaluating the necessary volumes and their derivitives using Eqs. (3.3), (3.9), and (3.13) and expressing these relations in terms of the edge lengths via Eqs. (3.7) and (3.8). From the specification of the net one can compute which four-simplices are incident on a given triangle. Their dihedral angles at the triangle may be found from (3.9) and the deficit angle from the sum in (2.6). By doing this for all triangles and performing the sums in (2.5) and (2.8), one arrives at the action and field equations for the net.

IV. THE SEMICLASSICAL APPROXIMATION

Considerable insight into the qualitative behavior of the ground state wave function and its expectation values may be obtained by evaluating these quantities in the semiclassical approximation. In the simplicial approximation this means carrying out the defining multiple integral (2.4) by the method of steepest descents.

To apply the method of steepest descents, one first locates the stationary points of the action in the space of complex edge lengths by solving the simplicial field equations (2.8). One then attempts to distort the contour of integration in (2.4) so that it runs through one or more of these stationary configurations and elsewhere follows a contour along which $|\exp(-I)|$ decreases as rapidly as possible away from these stationary configurations. The asymptotic behavior of Ψ_0 as $\hbar \rightarrow 0$ is then given by the integral (2.4) in the neighborhood of one or more of the stationary configurations or in the neighborhood of the boundaries of the contour. It may or may not be possible to distort the contour C to pass through any particular stationary point. The stationary point of smallest $\operatorname{Re} I$ therefore does not always give the semiclassical behavior of the wave function.[37] Even if the dominant stationary point can be identified, one should still check whether it or the behavior near a boundary of the contour dominates the integral. In the present case the contour has boundaries because of the simplicial inequalities (see Sec. III). Because of the necessity of a classical limit, however, it is a reasonable expectation that the semiclassical approximation will be given by one or more stationary configurations. If the stationary configurations have complex edge lengths they will contribute to the semiclassical approximation in complex conjugate pairs since the original integral was real.

To see the form of the semiclassical approximation let us consider for simplicity the case when a single real stationary configuration provides the dominant contribution. Let s_i^0 be the squared edge lengths of the stationary configuration. Evaluate the measure on this configuration, expand the exponent in (2.4) to quadratic order in small deviations of the edge lengths from this configuration, and evaluate the resulting Gaussian integral to find

$$\Psi_0 \approx N\mu(s_i^0)\left[\det\left(\frac{\partial^2 I}{\partial s_i\, \partial s_j}\right)_{s_i = s_i^0}\right]^{-1/2} \exp[-I(s_i^0)], \tag{4.1}$$

for some constant N. The expectation value of a quantity A will be $A(s_i^0)$ in this approximation.

To make the further discussion of possible stationary configurations more concrete let us focus on the integrals over the boundaryless simplicial geometries which define the expectation values in the ground state [Eq. (2.14)]. The continuum theory gives some guide as to when one can expect stationary configurations with real edge lengths. There, the analogous problem is to solve the Euclidean Einstein equation

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}, \tag{4.2}$$

for real metrics $g_{\alpha\beta}$ on a compact manifold.[38] For positive Λ there is the four-sphere metric on S^4, the product of equal radii two-sphere metrics on $S^2 \times S^2$, and the Fubini-Study metric on CP^2. For the case of $S^1 \times S^3$, T^4, and $K3$ real solutions with either sign of Λ are ruled out by the inequalities[39]

$$\chi > 0, \quad \chi > \tfrac{3}{2}|\tau|, \tag{4.3}$$

which are necessary conditions for solutions to (4.1) with $\Lambda \neq 0$. $S^1 \times S^3$ and T^4 have $\chi = 0$ while $K3$ has $\chi = 24$ and $\tau = -16$.

One would expect the situation regarding the existence of solutions to the Regge equations to be similar to that for their continuum limit. The Regge equations offer the opportunity for approximately addressing questions still open in the continuum case (e.g., the existence of complex solutions) through an analysis of a finite number of algebraic equations.

V. RECOVERY OF THE DIFFEOMORPHISM GROUP

Diffeomorphisms are an invariance of general relativity. On a given manifold M, if two metrics g and g' are diffeomorphic they have the same physical consequences. The Einstein action which summarizes the theory is preserved by diffeomorphisms. This is analogous to the preservation of the action by the gauge group in a gauge theory[40] and has important and well-known consequences for a formulation of the quantum theory in terms of functional integrals. Consider, for example, the expression for the ground-state expectation value of a physical quantity $A\,[g]$ which is the natural generalization of those in field theories without gauge symmetries

$$\langle A\rangle=\frac{\int_C \delta g A\,[g]\exp(-I\,[g])}{\int_C \delta g\,\exp(-I\,[g])}. \tag{5.1}$$

By $\int \delta g$ is meant the sum over all metrics on the manifold M in some class C. Two diffeomorphic metrics contribute identically in both numerator and denominator of (5.1) since both A and I are invariant under diffeomorphisms. Each integral is therefore the volume of the diffeomorphism group times a sum over physically distinct metrics. Since the volume of the diffeomorphism group is infinite each integral diverges. The divergent factor of the diffeomorphism group formally cancels between the numerator and denominator of (5.1) to give a finite answer for $\langle A\rangle$, but to implement this cancellation in a practical way all the familiar techniques of gauge fixing and ghosts are required.

Simplicial geometries are simplicial manifolds with a metric. By simplicial manifold we shall mean a piecewise linear manifold[41] made up of simplices. We stress that by the manifold we mean the points interior to the four-simplices and on their boundaries, and not only the vertices of the net. A metric is determined by the edge lengths of the net and a flat metric in the interior of each simplex. With this information the distance between two points on any curve threading the simplicial geometry could be computed.

There are piecewise diffeomorphisms of simplicial manifolds exactly as there are diffeomorphisms in the continuum case. They are the one-to-one invertible maps from a simplicial manifold to itself[42] which are smooth on each simplex. Relabeling the vertices and smooth diffeomorphisms of the interior of simplices are two trivial examples. The action of a piecewise diffeomorphism on a metric gives a new metric which is physically equivalent to the old one. For a general curved simplicial geometry one expects a diffeomorphism to leave the edge lengths unchanged or to change them only according to a trivial relabeling of the vertices. This is because a nontrivial reassignment of edge lengths will in general correspond to different curvatures and a different geometry. The example of flat space, however, shows that there can be cases where diffeomorphisms lead to nontrivial reassignment of edge lengths.[43] Imagine a simplicial net obtained by distributing vertices about flat space, connecting them to form a simplicial net, and assigning edge lengths which are the flat distances between them. By moving the location of the vertices in flat space, one can find a different assignment of edge lengths on the same simplicial net which represents the same flat geometry. There is thus a $4n_0$-parameter family of transformations of edge lengths in flat space which leads to different metrics on the simplicial manifold which are piecewise diffeomorphic.

It is not easy to give an algorithm for deciding when two simplicial metrics are piecewise diffeomorphic any more than it is in the continuum case. Necessary conditions are certainly that any curvature invariant be the same, and in a certain sense these conditions are sufficient as well.[44] Intuitively, it seems reasonable to suppose that different assignments of edge lengths correspond to different geometries except in the case of flat space. In this sense there are no gauge transformations in the Regge calculus.[45]

If nontrivially different assignments of edge lengths correspond to different simplicial geometries we would expect the multiple integrals defining the expectation values in Eq. (2.14) not to diverge as do those in (5.1). Thus in the simplicial approximation no additional gauge-fixing machinery should be needed to effect a sum over geometries. This is a considerable convenience. One would expect, however, to recover the divergence of these integrals associated with the diffeomorphism group in the continuum limit, that is, in the limit of large simplicial nets. In the following we shall describe how this comes about.

A given continuum geometry may be approximated by a simplicial geometry on an appropriate net. Consider a family of nets with an increasingly large number of vertices obtained by repeatedly subdividing the original net. As the number of vertices n_0 becomes large there will be more and more simplicial geometries (i.e., more and more assignments of edge lengths) which approximate the given continuum geometry to a fixed level of accuracy. For large n_0 all these simplicial approximations contribute approximately equally to the multiple integrals in (2.14). As n_0 becomes large, both numerator and denominator will, therefore, be approximately a large factor times a sum over physically distinct geometries. The factor should cancel between the numerator and the denominator. Thus, while the numerator and denominator will diverge as n_0 becomes large, $\langle A\rangle$ should tend to a definite value. This behavior can be illustrated more precisely in the semiclassical approximation.

By way of illustration let us suppose that one stationary configuration with squared edge lengths s_i^0 gives the dominant semiclassical contribution to both numerator and denominator. The classical approximation to the denominator of (2.14) would read

$$N\mu(s_i^0)\exp[-I(s_i^0)]\times\int\left(\prod_i d\xi_i\right)\exp\left[-\left(\frac{\partial^2 I}{\partial s_i\,\partial s_j}\right)_{s_i=s_i^0}\xi_i\xi_j\right], \tag{5.2}$$

where the ξ_i are the deviations in the squared edge lengths from their stationary values. Suppose that as n_0 becomes large the stationary simplicial geometries approach a continuum geometry. Then, since we expect many different simplicial geometries which *approximate* a given continuum geometry we should expect to find directions λ_i in the space of edge lengths along which the action is *approximately* stationary,

$$\lambda_i\left[\frac{\partial}{\partial s_i}\left(\frac{\partial I}{\partial s_j}\right)\right]_{s_k=s_k^0}\approx 0. \tag{5.3}$$

In fact, we can identify what these directions are.

The curvature of the stationary configuration must be characterized by the only scale in the Regge equations (2.8), that set by the cosmological constant Λ. As n_0 becomes large the characteristic squared edge length s_i^0 in the stationary configuration will become small compared to Λ^{-1} as $\Lambda^{-1}f^2(n_0)$, where f is some rapidly decreasing function of n_0 dependent on the subdivision process. On scales small com-

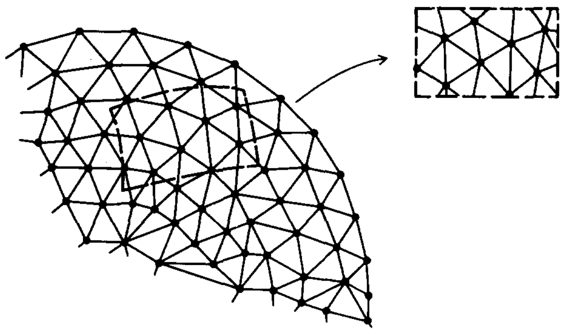

FIG. 2. The origin of the approximate diffeomorphism group. The figure shows a two-dimensional simplicial geometry whose net is sufficiently refined that the characteristic edge lengths are much smaller than the scale of the curvature. Local regions of this geometry will be approximately flat. Variations in the edge lengths which correspond to those induced by motions of the vertices in two-dimensional flat space and which are small on the curvature scale will leave the geometry approximately unchanged. For this net there are thus many different assignments of edge lengths which approximately correspond to the same geometry.

pared to $\Lambda^{-1/2}$ but large eventually compared to $\Lambda^{-1/2}f(n_0)$ the net will approximate flat space. In flat space there is a $4n_0$-parameter family of variations of the edge lengths which leave the geometry flat. In *approximately* flat space there will be a $4n_0$-parameter family of variations of the edge lengths which leave the geometry *approximately* flat (cf. Fig. 2). We thus expcet $4n_0$ directions in which (5.3) is approximately satisfied. Put differently, if a geometry is approximated by increasingly subdivided simplicial nets we expect $4n_0$ of the n_1 eigenvalues of $\partial^2 I/\partial s_i\, \partial s_j$ to be a number near zero times Λ/l^2, while the rest are a number of order unity times Λ/l^2.

The multiple integral in (5.2) is easily carried out along the $4n_0$ directions in which the action is approximately stationary. While the action is approximately stationary in these directions in the vicinity of the stationary configuration, we expect it to remain stationary only for deviations ξ^i which are of order of the curvature scale Λ^{-1}. Beyond that the deviations represent physically distinct geometries. Thus, for increasingly subdivided simplicial nets we expect the semiclassical approximation (5.2) to behave as

$$N(n_0,s_i^0)\exp[-I(s_i^0)] \times (l^{-2}\Lambda^{-1})^{4n_0}\left\{\det{}'\left[\left(\frac{\partial^2 I}{\partial s_i\,\partial s_j}\right)_{s_k=s_k^0}\right]\right\}^{-1/2}, \tag{5.4}$$

where $N(n_0,s_i^0)$ is a slowly varying function of n_0. Here det′ denotes the determinant over the n_1-4n_0 directions in which the action is not approximately zero, i.e., the product of the nonsmall eigenvalues of $\partial^2 I/\partial s_i\,\partial s_j$.

While the above discussion has been illustrated using the integral in the denominator of (2.14) the situation is similar with the numerator. If A is a quantity which is not sensitive to scales much smaller than the curvature scale, the integral for the numerator may be divided into a sum of pieces, each one of which locally behaves like (5.2).

Equation (5.4) implies that for large n_0 the integrals in both numerator and denominator of (2.14) will diverge as $(l^{-2}\Lambda^{-1})^{4n_0}$. ($l^{-2}\Lambda^{-1}$ is greater than unity when the semiclassical approximation is valid.) This is the degree of divergence associated with the diffeomorphism group—four divergent factors for each point. These divergent factors cancel between the numerator and denominator of (2.14) to give a nondivergent expression for $\langle A\rangle$ for those A's which are not sensitive to arbitrarily small scales. Thus in the limit of large n_0 we recover the behavior of the functional integrals arising from the diffeomorphism group of general relativity.

VI. SUMMING OVER TOPOLOGIES

Our discussion up to this point has proceeded as though the Euclidean prescription for the ground state wave function were to take a fixed and particular compact manifold and sum exp(− action) over the possible geometries on this manifold. This has been convenient for the exposition, but there is no compelling physical reason to make such a restriction and none is proposed. Indeed, to have the laws of physics fix the topology of the manifold but allow all possible geometries on it would seem to be assigning a very different status in physics to two closely related elements of geometry. Further, there are attractive physical reasons for considering four-geometries with different topology. In the set of ideas evoked by the words "space-time foam"[46] one would ask for quantum transition amplitudes between states specified by disconnected as well as connected three-geometries and multiply connected as well as simply connected ones. Unitarity would then suggest that the Euclidean functional integral prescription for these amplitudes contain a sum over the topologically nontrivial four-geometries into which these topologically nontrivial three-geometries can be embedded.

In this section we shall discuss how sums over different topologies might be implemented in the simplicial minisuperspace approximation. What we want to give practical meaning to might be written schematically as

$$\langle A\rangle = \frac{\Sigma_M \nu(M)\Sigma_{g \text{ on } M} A[g,M]\exp(-I[g,M])}{\Sigma_M \nu(M)\Sigma_{g \text{ on } M}\exp(-I[g,M])}. \tag{6.1}$$

The first sum is over some class of compact four-manifolds. A weighting $\nu(M)$ which depends on the topological invariants of the manifold M, e.g., its Euler number, signature, fundamental group, etc., would be part of the prescription. The second sum is over physically distinct (i.e., nondiffeomorphic) metrics on M with the action I for general relativity.

The first step in turning the schema (6.1) into a computable procedure is to restrict the sum over four-manifolds to a sum of simplicial manifolds (i.e., simplicial complexes which are piecewise linear manifolds). The second step is to implement the sum over metrics by an integral over edge lengths. The latter sum has been the subject of the preceding parts of this paper. Including only simplicial manifolds is a restriction because not every four-manifold is triangulable. However, the author is not aware of anything physically interesting lost by this. We now discuss the problems involved in implementing a sum over compact simplicial four-manifolds. We shall not be able to answer every question, but we will be able to provide a framework for discussion and some practical proposals to try out.

To specify a sum over simplicial manifolds we must effectively have a procedure for listing those to be included in the sum. One way would simply be to list famous compact four-manifolds, e.g., $S^4, S^2 \times S^2, \tilde{S}^1 \times S^3, T^4, CP^2, K3$, etc., to find triangulations of these, and to add together sums over geometries on these of the kind we have been discussing. This can hardly be the basis for a general principle. A better approach is to sum over all simplicial four-manifolds or some generally specified subclass.

It is not possible to classify *all* four-manifolds. That is to say roughly, an algorithm for deciding when two four-manifolds are the same does not exist.[47] Subclasses of four-manifolds may be classifiable. For example, simply connected four-manifolds with spin structure are classified by their Euler number and signature up to a finite number of connected sums.[48] That all four-manifolds are not classifiable does not mean that they cannot be enumerated. One can imagine a procedure (and we shall describe one below) which would generate an exhaustive list of four-manifolds. A given four-manifold would occur on the list more than once but there would be no universal algorithm to decide when two entries on the list are the same manifold. To implement the sum over simplicial manifolds one therefore has the following choice: One can sum over a class of four-manifolds which are classifiable by virtue of having more structure and assign a weight to each. Alternatively, one can devise a procedure for listing all four-manifolds and accept the weighting implied by the procedure.

To describe concretely how a list of manifolds can be prepared let us first consider how a single manifold is specified.[41] A simplicial complex is a collection of simplices such that whenever a simplex lies in the collection then so does each of its faces, and whenever two simplices of the collection intersect they do so in a common face. The dimension of a complex is the largest dimension of a simplex in it. A four-dimensional simplicial complex may be specified by giving the vertices of each four-simplex in the complex. From the list of four-simplices the vertices of the edges, triangles, and tetraheda may be computed. These must be such that the conditions of the definition are satisfied. Figure 3 shows a two-dimensional example.

Not every simplicial complex is a piecewise linear manifold (i.e., such that every point has a neighborhood which is piecewise linearly homeomorphic to an open subset of R^4 or a half-space of R^4). (From now on we shall omit the qualification "piecewise linear" when referring to manifolds, homeomorphisms, etc.) A two-dimensional example is shown in Fig. 4. There is a necessary and sufficient condition for a complex to be a manifold. To state it we shall need the notions of the *star* and the *link* of a simplex in the complex. They are illustrated in two dimensions in Fig. 5. The star of a simplex σ is the collection of simplices which contain σ together with all their faces. The link of a simplex σ is the collection of simplices in the star of σ which do not meet σ. The necessary and sufficient condition that an n-dimensional complex be an n-manifold is that the link of every simplex of dimension k be a triangulation of (i.e., homeomorphic to) an $(n-k-1)$-sphere.[41] Given a complex, one might imagine checking it to see if it is a manifold. Were space-time

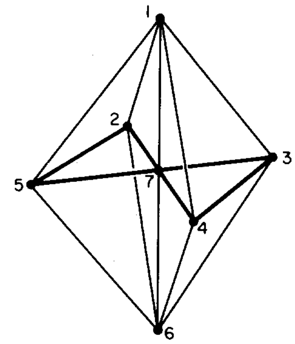

FIG. 4. A two-dimensional simplicial complex which is not a manifold. This complex consists of the triangles

125	147	346	134	256	467
127	157	367	137	267	567 .

This complex is not a manifold because local neighborhoods of vertices 1, 6, and 7 as well as the edges (1,7) and (6,7) are not homeomorphic to a region of R^2. Neighborhoods of vertex 7, for example, are homeomorphic to regions in the intersection of two planes. The link of vertices 1 and 6 is the heavy curve and is not topologically a circle. If edge lengths were assigned to the complex by embedding it in a flat three-dimensional space as suggested by the figure, then the deficit angle of vertex 7 would be -2π. The two-dimensional gravitational action defined by $-2\pi l^{-2}\Sigma$(deficit angles) is $-4\pi/l^2$ for this complex. This is larger than the value $-8\pi/l^2$ for manifolds which are topologically two-spheres such as the example in Fig. 3.

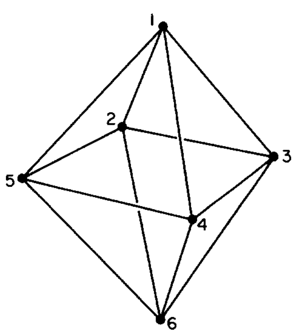

FIG. 3. A two-dimensional simplicial manifold. The six vertices are connected as for an octohedron. The simplicial complex can be specified by giving a list of vertices of the triangles. In the present case the list would be

123	134	236	346
125	145	256	456 .

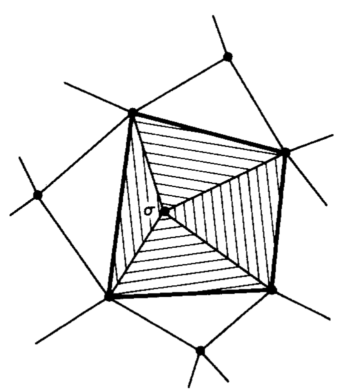

FIG. 5. Link and star of a vertex. The figure shows a portion of a two-dimensional simplicial manifold. The star of a given vertex consists of the interior, edges, and vertices of those triangles which intersect the given vertex. The star of vertex σ consists of the four shaded triangles. The link of a vertex consists of those simplices of its star which do not themselves intersect the vertex. The link of σ is the heavily drawn edges and their vertices in the figure. The link of σ is topologically S^1.

three-dimensional, this would in principle be possible to do. The one- and two-dimensional manifolds are classifiable by their homology groups and these are in principle computable. Were space-time five-dimensional, it would be in principle impossible to decide whether a complex was a manifold. In particular, one would need to decide whether the link of a vertex was a four-sphere and this problem is known to be unsolvable (i.e., roughly, it can be shown that there is no algorithm to do it).[47] In four dimensions the difficult problem would be to decide whether the link of a vertex is a three-sphere. Even assuming the Poincaré conjecture one would still need a procedure for deciding whether the link was simply connected. It seems that at the time of writing both of these decision problems are still open.[49] Thus one cannot list a family of manifolds to sum over by listing all the simplicial complexes with, say, a fixed number of vertices, and then discarding those which are not manifolds. There is no way to check. We must create the list by a different route.

A natural way to generate a list of manifolds is to create simplicial complexes from smaller units, "building blocks," which are locally and explicitly known to be manifolds. We begin with a fixed number of vertices n_0 with the idea of eventually allowing n_0 to become large. In view of the approximate recovery of the diffeomorphism group discussed in Sec. V there is no purpose to be served in considering lists with different numbers of vertices. One expects the manifolds with the largest n_0 to dominate both the numerator and denominator of (6.1) for large n_0. To create the list we start by enumerating all the four-dimensional simplicial complexes with n_0 vertices. This is a matter of enumerating all possible lists of four-simplices and checking that when two intersect they do so in a common face. We now discard from the list all complexes for which the link of every vertex is not one of a finite list of known triangulations of a three-sphere. For example, one might require that the link of every vertex be either the boundary of a four-simplex or of a 600-cell. The complexes remaining on the list are therefore known to be manifolds. If a sufficient number of basic building blocks is taken, lists containing all manifolds can be generated in this way. For example, in two dimensions it suffices to require the links to be five-, six- or, seven-gons.

A procedure such as described above would generate a list of manifolds with which to define a sum over topologies. For each member the sum over geometries would be carried out as described in the previous sections of this paper. One could assign relative weights to the different members of the list on the basis of some computable topological invariant, but the simplest assignment would be the weighting generated by the procedure itself. It then becomes an interesting mathematical question to ask with what multiplicity a given manifold occurs on the list. In particular, for the procedure to make sense, the large multiplicities should be independent of the particular basic building blocks chosen out of some general class.

We have described a procedure for summing over four-dimensional manifolds which at least can be tried out. One could ask: "Why restrict attention to four dimensions?" or "Why consider only manifolds?" The familiar answers that space-time seems to be four-dimensional and that a manifold is the mathematical implementation of the principle of equivalence are possibly too unadventurous. For example, the Regge action extends naturally to simplicial complexes which are not manifolds. (See the example in Fig. 4.) It also generalizes naturally to higher dimensions. One is thus invited to sum over complexes which are not manifolds and over other dimensions. The simplicial minisuperspace methods described here provide a framework for investigating such questions. Before embarking on such a journey, however, it would be useful to know if there is a way back to the familiar four-dimensional space-time on large scales.

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[31] G. W. Gibbons, S. W. Hawking, and M. Perry, Nucl. Phys. B **138**, 141 (1978).
[32] For the definition of conformal transformations in simplicial nets, see Refs. 22. For an explicit numerical example see Paper II in this series.
[33] J. B. Hartle, Phys. Rev. D **29**, 2730 (1984).
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[35] In order to avoid introducing yet another obscure term we have called the result of (2.13) an "expectation value" even though it is not an expectation value in the sense of the traditional canonical theory. These would be calculated on a hypersurface of superspace corresponding to a fixed "time." They are, however, the expectation values in the approach of C. Teitelboim in Ref. 1.
[36] See, for example, Refs. 10, 11, 14, 17, and especially J. Cheeger, W. Müller, and R. Schrader, Commun. Math. Phys. **92**, 405 (1984), where a method essentially the same as used here is described. The author would like to thank Dr. R. Williams for providing these references.
[37] See Ref. 2 for an example.
[38] For a summary of the present status of the existence of Euclidean Einstein metrics on compact manifolds see S–T. Yau, in *Seminar on Differential Geometry*, edited by S–T. Yau (Princeton U. P., Princeton, 1982).
[39] See, e.g., S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge U. P., Cambridge, 1979).
[40] The analogy, however, especially viewed in the Hamiltonian framework is not complete. See J. B. Hartle and K. Kuchar, J. Math. Phys. **25**, 57 (1984); in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1984); C. Teitelboim, Phys. Rev. D **25**, 3159 (1982); Ann. Phys. (N.Y.) **143**, 127 (1982).
[41] See, e.g., C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise Linear Topology* (Springer, Berlin, 1972).
[42] One could consider piecewise diffeomorphisms between different simplicial manifolds. Those between a simplicial manifold and its subdivisions would provide examples. The present restriction will, however, simplify our discussion.
[43] This has been stressed by Rocek and Williams, Ref. 22.
[44] See, e.g., the discussion for the continuum case in A. Karlhede, Gen. Relativ. Gravit. **12**, 693 (1980).
[45] Rocek and Williams (Ref. 22) use the term "gauge transformation"for four-parameter transformations of the edge lengths which leave the action invariant. Infinitesimally in the continuum theory this would correspond to variations in the metric for which

$$\delta I = \int d^4x (g)^{1/2} G_{\mu\nu}(x) \delta g^{\mu\nu}(x) = 0, \tag{*}$$

while in the simplicial theory this would correspond to variations in the squared edge lengths for which

$$\delta I = \sum_i \frac{\delta I}{\delta s_i} \delta s^i = 0. \tag{**}$$

In this paper we use the term "gauge transformation" to mean transformations which not only preserve the action but all other physical quantities as well. For example, electromagnetic gauge transformations preserve the field as well as the action. For simplicial nets by gauge transformations we would mean transformations of the edge lengths which preserve the geometry as well as the action. Distinct geometries may have the same action. Thus in general there will be many more transformations which preserve the action than which preserve the geometry. This can be seen directly from (*) and (**) where it is not difficult to find parametric families of $\delta g^{\mu\nu}$ and δs^i which satsify these relations.
[46] See, e.g., J. A. Wheeler in *Battelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).
[47] See, e.g., W. Haken in *Word Problems*, edited by W. W. Boone, F. B. Cannonito, and R. C. Lyndon (North-Holland, Amsterdam, 1973).
[48] See, e.g., C. T. C. Wall, J. London Math. Soc. **39**, 131 (1964).
[49] See, however, W. Haken (to be published).