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Spacetime coarse grainings in nonrelativistic quantum mechanics

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Sum-over-histories generalizations of nonrelativistic quantum mechanics are explored in which probabilities are predicted, not just for alternatives defined on spacelike surfaces, but for alternatives defined by the behavior of spacetime histories with respect to spacetime regions. Closed, nonrelativistic systems are discussed whose histories are paths in a given configuration space. The action and the initial quantum state are assumed fixed and given. A formulation of quantum mechanics is used which assigns probabilities to members of sets of alternative coarse-grained histories of the system, that is, to the individual classes of a partition of its paths into exhaustive and exclusive classes. Probabilities are assigned to those sets which decohere, that is, whose probabilities are consistent with the sum rules of probability theory. Coarse graining by the behavior of paths with respect to regions of spacetime is described. For example, given a single region, the set of all paths may be partitioned into those which never pass through the region and those which pass through the region at least once. A sum-over-histories decoherence functional is defined for sets of alternative histories coarse-grained by spacetime regions. Techniques for the definition and effective computation of the relevant sums over histories by operator-product formulas are described and illustrated by examples. Methods based on Euclidean stochastic processes are also discussed and illustrated. Models of decoherence and measurement for spacetime coarse grainings are described. Issues of causality are investigated. Such spacetime generalizations of nonrelativistic quantum mechanics may be useful models for a generalized quantum mechanics of spacetime geometry.

I. INTRODUCTION

Conventional Hamiltonian quantum mechanics predicts probabilities for alternatives defined on spacelike surfaces. A decomposition of spacetime into space and time is thus required for a Hamiltonian formulation, however invariantly such elements as its inner product may be expressed [1]. By contrast, Feynman's sum-over-histories formulation of quantum mechanics [2] deals directly with spacetime quantities and in particular with spacetime histories. This direct access to spacetime is a unifying conceptual advantage in special-relativistic field theories where there are many different ways of splitting spacetime into space and time corresponding to the many possible families of spacelike surfaces that can foliate flat spacetime. In quantum gravity spacetime is a dynamical variable and there is no one fixed spacetime to split. As a consequence, a sum-over-histories formulation of the quantum mechanics of spacetime may be essentially different from the generalizations of Hamiltonian quantum mechanics commonly applied to this problem [3].

A spacetime formulation of quantum mechanics cannot be considered complete if it calculates amplitudes by spacetime means, but calculates probabilities only for alternatives defined on spacelike surfaces. To restrict the use of path integrals in quantum mechanics only to calculate probabilities for successions of alternatives defined at definite moments of time [4] would be to elevate again the surfaces of constant time to some special status in the theory. Both means *and* ends must be generalized to provide a complete spacetime formulation.

The use of the sum-over-histories formulation of nonrelativistic quantum mechanics to define probabilities for alternatives defined by spacetime regions rather than on spacelike hypersurfaces was discussed by Feynman in his original paper on the subject [2]. In particular, he offered a sum-over-histories definition of the probability that "if an ideal measurement is performed to determine whether a particle has a path lying in a region of spacetime. . .the result will be affirmative." More recently, sum-over-histories calculations of probabilities for alternatives not restricted to spacelike surfaces have been discussed by Sorkin [5], Yamada and Takagi [6], and the author [7]. However, these discussions were complicated by the perceived need to provide something like a "measurement theory" of such spacetime alternatives. Further, little attention seems yet to have been given to effective procedures for actually calculating the relevant sums over histories.

This paper has two aims: First, it aims to provide a coherent and systematic discussion of the rules for assigning probabilities to spacetime alternatives in the context of a generalized nonrelativistic quantum mechanics of closed systems [8]. Fundamental to this approach to quantum mechanics is the notion of a set of alternative coarse-grained histories of the closed system, that is, a partition of its paths into exhaustive and exclusive classes. Quantum mechanics predicts probabilities for the individual histories in certain coarse-grained sets of histories. The consistency of probability sum rules is the primary criterion that determines which sets may be assigned probabilities rather than any notion of "measurement." The notion of a partition of paths is an intrinsically spacetime one and more general than alternatives defined at a single moment of time. In this generalization of familiar quantum mechanics, coarse grainings by spacetime regions are just as natural and just as straightforward as those defined by spacelike hypersurfaces [7,10]. The general formulation of this quantum mechanics is discussed in Secs. II and III. The second aim of this paper is to develop effective operator techniques for calculating the relevant sums over histories. This is done by considering specific classes of examples in Secs. IV–VII. Issues of causality, measurability, and the utility of spacetime coarse grainings are rudimentarily addressed in Sec. IX.

The sum-over-histories formulation of the quantum mechanics of nonrelativistic systems described in this paper is both more and less general than usual formulations. It is less general in that it is a sum-over-histories formulation that deals only with alternatives defined in terms of configuration-space paths. It is more general in that it allows the assignment of probabilities to sets of alternatives more general than those defined on sets of spacelike surfaces. It is an example of the class of generalized quantum mechanics discussed in Refs. [10] and [11]. In general, there is no equivalent Hamiltonian formulation in the usual sense. In particular, as described in Sec. X, there is no natural notion of "state at a moment of time."

Because it concerns closed systems, the quantum mechanics of nonrelativistic systems described here is a model for a quantum mechanics of cosmology. Because it is a more general spacetime formulation, it addresses the "problem of time," which is a central conceptual problem in the construction of a generally covariant quantum mechanics of spacetime [12,10]. It does not eliminate the preferred status of Newtonian time in the formulation of nonrelativistic quantum mechanics, but does generalize the alternatives considered from the usual ones that require a decomposition of spacetime into space and time for their specification to spacetime alternatives that do not. Indeed, as will be argued in Sec. VIII, the more general spacetime alternatives discussed here are much more like those which may be realistically expected in quantum cosmology than those defined solely on spacelike surfaces. The present paper, therefore, may be thought of as one step in the program of providing a generalized quantum mechanics for cosmology [13].

II. GENERALIZED QUANTUM MECHANICS

We begin by briefly reviewing a very general framework for quantum theories of closed systems such as the universe as a whole [14]. The most general objective of quantum theory is to compute the probabilities of the individual alternative histories of a closed system in an exhaustive and exclusive set of such histories. A characteristic feature of a quantum-mechanical theory is that, because of interference, not every set of histories that may be described can be assigned probabilities. A rule, therefore, is needed to specify both which sets of alternative histories may be assigned probabilities and what their values are. Very generally, a quantum-mechanical theory is one for which that rule is constructed from the following elements:

(1) The possible sets of alternative *fine-grained histories*, of the closed system, $\{f\}$, which are the most refined descriptions allowed in the theory. There may be many such sets.

(2) A notion of *coarse graining* by which the sets of fine-grained histories are partitioned into mutually exclusive classes. Each such set is a set of alternative *coarse-grained* histories of the system, $\{h\}$. Further partitioning of these sets results in further coarse graining so that there is a partial ordering of all the possible sets of coarse-grained histories.

(3) A complex-valued *decoherence functional* $D(h',h)$ defined for each pair of histories in an exhaustive set of alternative histories, either fine grained or coarse grained, with the following properties: (i) hermiticity,

$$D(h,h')=D^*(h',h) \; ; \tag{2.1}$$

(ii) positivity,

$$D(h,h)\geq 0 \; ; \tag{2.2}$$

(iii) normalization

$$\sum_{h,h'} D(h,h')=1 \; , \tag{2.3}$$

where the sum in (2.3) is over all members of the exhaustive set.

(4) A superposition principle with respect to coarse graining for the decoherence functional. If $\{\bar{h}\}$ is a coarse graining of $\{h\}$, then D satisfies

$$D(\bar{h},\bar{h}')=\sum_{h\in\bar{h}}\sum_{h'\in\bar{h}'} D(h,h') \; , \tag{2.4}$$

for all pairs $(\bar{h},\bar{h}')$ in the coarser-grained set. If a set of histories is a coarse graining of two different finer-grained sets, the same decoherence functional must result. In particular, once a decoherence functional $D(f,f')$ obeying (i)–(iii) above is specified for the fine-grained sets $\{f\}$, the decoherence functional for any coarser-grained set may be computed by the principle of superposition,

$$D(h,h')=\sum_{f\in h}\sum_{f'\in h'} D(f,f') \; . \tag{2.5}$$

(5) A *decoherence* condition which determines which sets of alternative histories may be assigned probabilities and what these probabilities are. The most general form of this condition [15] is

$$\mathrm{Re}D(h,h')=0, \quad \text{for } h\neq h' \; , \tag{2.6}$$

for all pairs (h,h') of different histories in a given set [16], although other, more restrictive, conditions are sometimes useful [11]. Sets of alternative histories which satisfy (2.6) are said to *decohere*. They may be assigned the probabilities

$$p(h)=D(h,h) \; . \tag{2.7}$$

As a consequence of the decoherence condition (2.6) and the properties of the decoherence functional (i)–(iii) in (3) above, the numbers $p(h)$ are real, positive, and sum to unity over the set $\{h\}$:

$$0\leq p(h)\leq 1, \quad \sum_h p(h)=1 \; . \tag{2.8}$$

Most importantly, as a consequence of the decoherence condition (2.6) and the principle of superposition (2.4), the numbers $p(h)$ satisfy the probability sum rules. (For a derivation, see Ref. [9] or [10].) In their most general form these relate the probabilities of any further partition of the set of histories, $\{h\}$, into exhaustive and exclusive classes $\{\bar{h}\}$ (that is, any coarse graining of $\{h\}$) to the probabilities of the original set $\{h\}$:

$$p(\bar{h})=\sum_{h\in\bar{h}} p(h) . \tag{2.9}$$

In particular, for the empty set ϕ and for the completely coarse-grained set of all possible histories u, we have

$$p(\phi)=0, \quad p(u)=1 . \tag{2.10}$$

Equations (2.8)–(2.10) are the defining requirements [17] for a probability theory on a sample space which consists of a set of coarse-grained histories, $\{h\}$.

It will not escape the reader that in this statement of the principles of quantum mechanics no mention has been made of "measurements" or "observers" or even of "state of the system at a moment of time." In this formulation these are consequent notions, not fundamental ones. They may be appropriate for some quantum theories of closed systems, but not for others. We shall return to such issues briefly in Sec. X, but for a fuller discussion, see Refs. [9–11].

We take the elements (1)–(5) to be the minimal defining elements of a quantum theory [18]. If one adopts the weakest decoherence condition (2.6), then a quantum theory for a closed system is specified by its fine-grained histories, its notion of coarse graining, and its decoherence functional. Hamiltonian quantum mechanics is one way of specifying these ingredients, but it is not the only one. We will now turn to the sum-over-histories implementation of these requirements in nonrelativistic quantum mechanics.

III. NONRELATIVISTIC SUM-OVER-HISTORIES QUANTUM MECHANICS

We now describe the sum-over-histories quantum mechanics of nonrelativistic systems using the general schema of the preceding section. We consider systems described by a ν-dimensional configuration space $\mathbb{R}^{\nu}$. The *fine-grained histories* are paths in this configuration space parametrized by the physical time t. We denote the paths by $X(t)$ or $(X^1(t),X^2(t),\ldots,X^{\nu}(t))$ when it is necessary to specify the individual coordinates. A defining feature of a nonrelativistic system is that its fine-grained histories are *single-valued* functions of the physical time—one and only one X for each value of t. It is a characteristic feature of sum-over-histories formulations of quantum mechanics that a *unique* fine-grained set of histories is assumed. In this case it is the set of paths in configuration space that are single-valued functions of time. A *coarse graining* of these fine-grained histories is a partition of the paths into exhaustive and exclusive classes. Specific examples will be given below.

The dynamics is specified by an action of nonrelativistic form

$$S[X(t)]=\int_{t'}^{t''} dt\,[\mathcal{T}(\dot{X})-V(X)] , \tag{3.1a}$$

where $\mathcal{T}$ is the kinetic-energy quadratic form

$$\mathcal{T}(V)=\tfrac{1}{2}\sum_{i=1}^{\nu} M_i(V^i)^2 . \tag{3.1b}$$

The remaining quantity to be specified in the quantum framework of the previous section is the decoherence functional for the fine-grained histories. This is to describe a *closed* nonrelativistic system obeying the dynamics summarized by (3.1)—a model quantum cosmology. We may think, if we wish, of the quantum mechanics of a system in a box whose evolution is to be described over a finite time range $[0,T]$. In general, a decoherence functional could be constructed with both specified initial and final conditions [19]. However, to keep the discussion as close to familiar quantum mechanics as possible, let us restrict attention to the realistic final condition of future indifference. The sum-over-histories *decoherence functional* for the fine-grained histories is then

$$D[X(t),X'(t)]=\delta(X_f-X_f')\exp(i\{S[X(t)]-S[X'(t)]\}/\hbar)\rho(X_0,X_0') . \tag{3.2}$$

The path $X(t)$ proceeds from an initial value X_0 at time $t=0$ to a final value X_f at time $t=T$. The path $X'(t)$ proceeds similarly between X_0' and X_f'. The initial condition is represented by the density matrix $\rho(X_0,X_0')$ in the coordinate representation. Explicitly,

$$\rho(X_0,X_0')=\sum_j \psi_j(X_0)p_j\psi_j^*(X_0') , \tag{3.3}$$

for some set of orthonormal wave functions ψ_j and probabilities p_j. The final condition of indifference is represented by the δ function, $\rho_{\text{final}}\propto I$. It is easy to verify that this decoherence functional satisfies conditions (i)–(iii) of element (3) above. The only condition which is not immediate is the normalization (iii), but this is easily demonstrated: The normalization condition (2.3) is a double path integral over the class u of *all* paths between 0 and T. We write this as a path integral over all paths between end points X_0 and X_f and a similar path integral over all paths between end points X_0' and X_f' followed by integrals over these end points:

$$\int_u \delta X\int_u \delta X'\,D[X(t),X'(t)]=\int dX_f\int dX_0\int dX_0'\left[\int_{[X_0X_f]}\delta X\,e^{iS[X(t)]/\hbar}\right]\left[\int_{[X_0'X_f]}\delta X'\,e^{-iS[X'(t)]/\hbar}\right]\rho(X_0,X_0') . \tag{3.4}$$

The path integrals with fixed end points are the propagator or its complex conjugate:

$$\int_{[X_0X_f]}\delta X\, e^{iS[X(t)]/\hbar}=\langle X_f|e^{-iHT/\hbar}|X_0\rangle\ , \tag{3.5}$$

where H is the total Hamiltonian. Using completeness of the states $|X_f\rangle$, the right-hand side of (3.4) collapses to the normalization integral for ρ, which is unity by construction, viz.,

$$\int_u \delta X \int_u \delta X'\, D[X(t),X'(t)]=\int dX_0 \rho(X_0,X_0)=1\ . \tag{3.6}$$

The decoherence functional for coarse-grained sets of histories may be constructed from the decoherence functional for fine-grained histories by the principle of superposition (2.5). Let u denote the set of all paths between time 0 and time T and let $\{c_\alpha\}$ denote a partition of that set into exhaustive and exclusive classes c_α:

$$\bigcup{}_\alpha c_\alpha=u,\quad c_\alpha\cap c_\beta=\phi,\quad \alpha\neq\beta\ . \tag{3.7}$$

The decoherence functional for the set of coarse-grained histories $\{c_\alpha\}$ is then

$$D(c_\alpha,c_{\alpha'})=\int_{c_\alpha}\delta X\int_{c_{\alpha'}}\delta X'\,\delta(X_f-X_f')\exp(i\{S[X(t)]-S[X'(t)]\}/\hbar)\rho(X_0,X_0')\ , \tag{3.8}$$

where the two path integrals are over all paths in the classes c_α and $c_{\alpha'}$, respectively. With this definition the necessary ingredients for a generalized quantum mechanics are complete.

The most familiar type of coarse graining is by regions of configuration space at successive moments of time (see Fig. 1). Suppose, for example, we consider sets of exhaustive nonoverlapping regions of $\mathbb{R}^\nu$, $\{\Delta_{\alpha_1}(t_1)\}$, $\{\Delta_{\alpha_2}(t_2)\},\ldots,\{\Delta_{\alpha_n}(t_n)\}$, at a discrete series of times, $t_1,\ldots,t_n$. At each time t_k,

$$\bigcup{}_{\alpha_k}\Delta_{\alpha_k}(t_k)=\mathbb{R}^\nu\ ,$$
$$\Delta_{\alpha_k}(t_k)\cap\Delta_{\beta_k}(t_k)=\phi,\quad \alpha_k\neq\beta_k\ . \tag{3.9}$$

Since the paths are single valued in time, they pass through one and only one region at each of the instants t_k. The class of all paths may be partitioned into classes corresponding to the different possible ways they cross these regions. Coarse-grained histories are thus labeled by the particular sequence of regions, $\alpha_1,\ldots,\alpha_n$, that are crossed at times $t_1,\ldots,t_n$. We write them as $c_{\alpha_n\cdots\alpha_1}$. The individual coarse-grained history $c_{\alpha_n\cdots\alpha_1}$ corresponds to the particle being localized in region $\Delta_{\alpha_1}(t_1)$ at time t_1, $\Delta_{\alpha_2}(t_2)$ at time t_2, and so forth.

Coarse grainings specified by alternatives at individual moments of time arise naturally in Hamiltonian quantum mechanics as we shall describe more fully below. From the spacetime perspective, however, there is no need to restrict coarse grainings to this special type. We shall discuss more general possibilities in Sec. V, but first we review how to calculate the path integrals in (3.8).

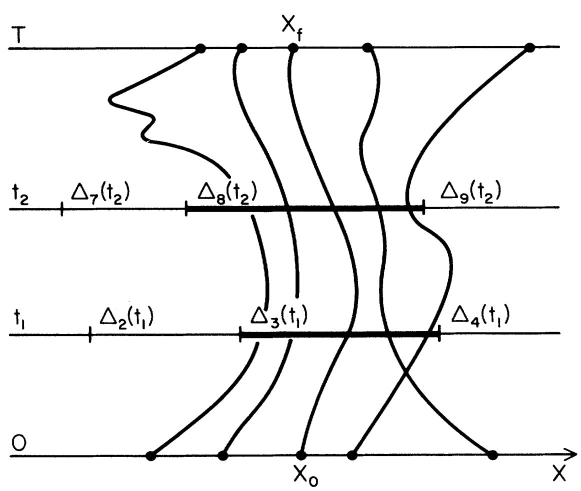

FIG. 1. Coarse graining by regions of configuration space at successive moments of time. The figure shows a spacetime that is a product of a one-dimensional configuration space (X) and the time interval $[0,T]$. At times t_1 and t_2 the configuration space is divided into exhaustive sets of nonoverlapping intervals: $\{\Delta_{\alpha_1}(t_1)\}$ at time t_1 and $\{\Delta_{\alpha_2}(T_2)\}$ at time t_2. Some of these intervals are illustrated. The fine-grained histories are the paths which pass between $t=0$ and T. Because the paths are assumed to be single valued in time, the set of fine-grained histories may be partitioned according to which intervals they pass through at times t_1 and t_2. The figure illustrates a few representative paths in the class c_{83} which pass through region $\Delta_3(t_1)$ at time t_1 and region $\Delta_8(t_2)$ at time t_2.

IV. EVALUATING SUMS OVER PATHS

To calculate the decoherence functional (3.8), we must evaluate Feynman path integrals. This section reviews briefly how to do that.

Introducing an arbitrary complete set of final states $|\phi_i\rangle$ in the Hilbert space $\mathcal{H}$ of square-integrable functions on $\mathbb{R}^\nu$ and using (3.3), the decoherence functional (3.8) may be written

$$D(c_\alpha,c_{\alpha'})=\sum_{ij}p_j K_{ij}(c_\alpha)K_{ij}^*(c_{\alpha'})\ , \tag{4.1}$$

where

$$K_{ij}(c_\alpha)=\int_{c_\alpha}\delta X\,\phi_i^*(X_f)e^{iS[X(t)]}\psi_j(X_0)\ , \tag{4.2}$$

units having been chosen for this and subsequent sections so that $\hbar=1$. Feynman path integrals of the form (4.2) are therefore the most general of interest. How are they defined and how do we compute them?

General arguments [20] show that it is not possible to introduce a complex measure on the space of paths to define the Feynman integral. However, path integrals may be defined and computed by other means [21]. Here we take the point of view, introduced by Feynman [2], that expressions such as (4.2) are to be *defined* by the limits of their values on polygonal (skeletonized) paths on a time slicing of the interval $[0,T]$. Suppose that this interval is divided into N subintervals of equal length $\epsilon=T/N$ with boundaries at $t_0=0,t_1,t_2,\ldots,t_N=T$. A polygonal path is specified by giving the values $(X_0,\ldots,X_N)$ of $X(t)$ on the $N+1$ time slices including the value X_0 at the initial time $t=0$ and the value X_N ($\equiv X_f$) at the final time $t_N=T$. The polygonal paths consist of straight-line segments joining the points $(X_0,\ldots,X_N)$ at the times defining the subdivision. The nonrelativistic action (3.1) is straightforwardly evaluated on polygonal paths when the spacing ϵ is small:

$$S(X_N,\ldots,X_0)\approx\sum_{k=0}^{N-1}\epsilon\left[\mathcal{T}\left[\frac{X_{k+1}-X_k}{\epsilon}\right]-V(X_k)\right] . \tag{4.3}$$

Any partition of continuous paths will also partition the polygonal paths. Let $e_\alpha(X_N,\ldots,X_0)$ be the function which is unity on all polygonal paths in the class c_α and zero otherwise. Then, with these preliminaries, we define an expression such as (4.2) as the limit

$$K_{ij}(c_\alpha)=\lim_{N\to\infty}\int dX_N\int dX_{N-1}\cdots\int dX_0\mu(N)\phi_i^*(X_N)e_\alpha(X_N,\ldots,X_0)e^{iS(X_N,\ldots,X_0)}\psi_j(X_0) , \tag{4.4}$$

where $\mu(N)$ is an N-dependent constant "measure" factor and the integrals are all over $\mathbb{R}^\nu$.

The definition (4.4) is not, by itself, a computationally effective way of evaluating Feynman integrals. Operator methods provide a more efficient tool. If the limit in (4.4) exists, it is clear from its linearity in ψ_j and antilinearity in ϕ_i that it defines a linear operator C_α in the Hilbert space of square-integrable functions on $\mathbb{R}^\nu$ for each history in a set of coarse-grained histories. The matrix elements of this operator are

$$\begin{aligned}K_{ij}(c_\alpha)&=\int_{c_\alpha}\delta X\,\phi_i^*(X_f)e^{iS[X(t)]}\psi_j(X_0)\\&=\langle\phi_i|C_\alpha|\psi_j\rangle .\end{aligned} \tag{4.5}$$

When the class operators $\{C_\alpha\}$ for a coarse-grained set of histories, $\{c_\alpha\}$, can be identified, the calculation of the decoherence functional is immediate. For then

$$D(c_\alpha,c_{\alpha'})=\mathrm{Tr}(C_\alpha\rho C_{\alpha'}^\dagger) . \tag{4.6}$$

One general relation among the C_α follows from (4.5), $u=\cup_\alpha c_\alpha$, and the fact [cf. (3.5)] that the Feynman integral over all paths between $[0,T]$ is the propagator. This general relation is

$$\sum_\alpha C_\alpha=e^{-iHT} . \tag{4.7}$$

It is an interesting question *which* sets of operators satisfying (4.7) can be represented as a family of Feynman integrals of the form (4.5) with $\cup_\alpha c_\alpha=u$. These sets of operators are the *spacetime* analogues of "observables." Sum-over-histories quantum mechanics predicts the probabilities of these classes when the sets of coarse-grained histories to which they correspond decohere.

As was first recognized by Nelson [22], operator-product formulas provide both a way of demonstrating the existence of limits such as (4.4) and of evaluating the class operators C_α to which they correspond. As the most familiar example, consider the propagator which is the path integral (4.2) evaluated on the class u of all paths on the time interval $[0,T]$. Then $e_\alpha=1$. Divide the total Hamiltonian H following from the action (3.1) into a free path H_0 corresponding to the kinetic energy $\mathcal{T}$ and the potential V:

$$H=\sum_{i=1}^{\nu}\frac{P_i^2}{2m_i}+V(X^k)\equiv H_0+V . \tag{4.8}$$

The propagator for the free part of the Hamiltonian is an elementary calculation

$$\langle X''|e^{-iH_0t}|X'\rangle=F(t)\exp\left[it\mathcal{T}\left[\frac{X''-X'}{t}\right]\right] , \tag{4.9}$$

where

$$F(t)=\prod_{i=1}^{\nu}(M_i/2\pi it)^{1/2} . \tag{4.10}$$

It follows that, if the constant μ in (4.4) happens to be $[F(\epsilon)]^N$, then we can write

$$K_{ij}(u)=\lim_{N\to\infty}\langle\phi_i|(e^{-iH_0(T/N)}e^{-iV(T/N)})^N|\psi_j\rangle . \tag{4.11}$$

But if H_0 and V are densely defined, self-adjoint, and bounded from below, the Trotter product formula [23] states

$$\lim_{N\to\infty}(e^{-iH_0(T/N)}e^{-iV(T/N)})^N=e^{-i(H_0+V)T} . \tag{4.12}$$

Thus the limit in (4.4) exists, and the path integral $K_{ij}(u)$ is evaluated as

$$K_{ij}(u)=\langle\phi_i|e^{-iHT}|\psi_j\rangle . \tag{4.13}$$

The relation (4.13) is hardly a surprise. It is the path-integral expression for the propagator originally derived by Feynman [2].

By such methods the class operators C_α for coarse grainings defined by alternatives at a discrete sequence of times may be readily evaluated. These coarse grainings were discussed at the end of the last section. If $\alpha=(\alpha_n,\ldots,\alpha_1)$ denotes the coarse-grained history in which the paths pass through regions $\Delta_{\alpha_1}(t_1),\ldots,\Delta_{\alpha_n}(t_n)$ at times $0\le t_1\le t_2\le\cdots\le t_n\le T$, then

$$C_{\alpha_n\cdots\alpha_1}=e^{-iH(T-t_n)}P^n_{\alpha_n}e^{-iH(t_n-t_{n-1})}\times P^{n-1}_{\alpha_{n-1}}\cdots P^1_{\alpha_1}e^{-iHt_1}\ , \tag{4.14}$$

where $P^k_{\alpha_k}$ is the projection on the configuration-space region $\Delta_{\alpha_k}(t_k)$ at time t_k. The expression is more compact with Heisenberg picture operators

$$C_{\alpha_n\cdots\alpha_1}=e^{-iHT}P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\ . \tag{4.15}$$

This is enough to show that the C_α in general will neither be unitary nor Hermitian. Neither is it true that $C_\alpha C_\beta=0$ for distinct histories.

The relations (3.9) expressing the conditions that the regions of configuration space are exhaustive and exclusive at each time translate into

$$\sum_{\alpha_k}P^k_{\alpha_k}(t_k)=1,\quad P^k_{\alpha_k}(t_k)P^k_{\alpha_k'}(t_k)=\delta_{\alpha_k\alpha_k'}P^k_{\alpha_k}(t_k)\ . \tag{4.16}$$

These are enough to show explicitly that (4.7) is satisfied and further that

$$\sum_{\alpha_n\cdots\alpha_1}C^\dagger_{\alpha_n\cdots\alpha_1}C_{\alpha_n\cdots\alpha_1}=1\ , \tag{4.17}$$

for this particular class of coarse grainings.

Coarse-grained sets of histories defined by alternatives at definite moments of time have been extensively discussed in Refs. [9] and [11]. In fact, the C's of Eq. (4.15) are exactly the C's of Ref. [11] except for an overall factor of $\exp(-iHT)$ whose presence or absence leaves the decoherence functional (4.6) unchanged. In the following we shall find the class operators for a much more general class of coarse grainings by spacetime regions.

Euclidean methods can sometimes be useful in evaluating Feynman integrals for coarse grainings corresponding to partitions of paths which do not discriminate one time from another over a finite time interval. Then the partition is unaffected by a continuation of $t\to -i\tau$. For example, if condition c_α is time independent over the whole of the interval $[0,T]$, we expect the corresponding Feynman integral to be the analytic continuation of the Euclidean path integral over the interval $[0,\tau]$ back to real times. That is, we expect

$$K_{ij}(c_\alpha)=\left[\int_{c_\alpha}\delta X\,\phi_i^*(X_f)e^{-I[X(\tau)]}\psi_j(X_0)\right]_{\tau=iT}\ , \tag{4.18}$$

where I is the Euclidean action

$$I[X(\tau)]=\int_0^\tau dt[\mathcal{T}(\dot X)+V(X)]\ . \tag{4.19}$$

A measure on the space of paths *can* be used to define strictly Euclidean path integrals [24], and there is a close connection with stochastic processes. This connection can be exploited to yield practical ways of evaluating Euclidean path integrals by "Monte Carlo" methods. The variety of methods is enhanced because, as a consequence of the central-limit theorem, there are many discrete time stochastic processes which yield the same path integral in the limit of vanishing time steps. If the partition c_α is piecewise time independent on finite-size subdivisions of $[0,T]$, separate Wick rotations may be carried out on each time interval and the results composed. The use of such Euclidean methods was illustrated in Ref. [6]. We shall do more below.

V. COARSE GRAININGS DEFINED BY SPACETIME REGIONS

In this section we shall consider a very general class of coarse grainings defined by how paths pass through regions of the manifold $M=[0,T]\times\mathbb{R}^\nu$. We shall call this product of configuration space and a time interval "spacetime" at the risk of being misleading. It *is* spacetime in the case of a single particle, but a larger dimensioned space in the case of many particles. We fix a division of M into n disjoint regions R_j:

$$\bigcup_i R_i=M,\quad R_i\cap R_j=\phi\ . \tag{5.1}$$

Figure 2 illustrates the general situation. The regions R_i

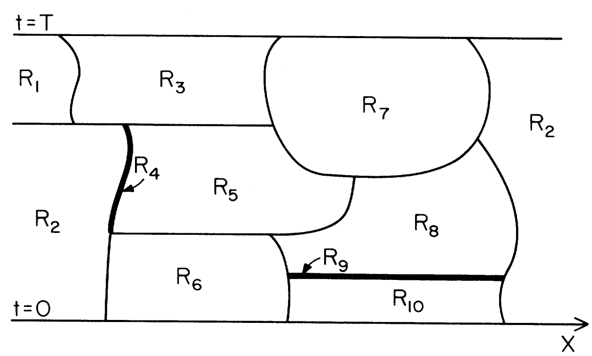

FIG. 2. Division of spacetime into regions. The figure illustrates a spacetime which is a product of a one-dimensional configuration space (X) and the time interval $[0,T]$ (drawn upward). The spacetime has been divided into an exhaustive set of ten spacetime regions $R_1,R_2,\ldots,R_{10}$. Some such as R_1 and R_2 extend to infinite ranges of X. Some such as R_9 are only at one moment of time. Some such as R_4 are degenerate in the spatial direction. Others such as R_2 are disconnected. The set of fine-grained histories, which are the paths passing from $t=0$ to T, may be coarse grained by how they pass through such spacetime regions. For one region the paths may be partitioned into those which never pass through that region and those that pass through at least once. Further partitions of this kind with respect to all of the regions give the spacetime coarse graining discussed in the text.

may have infinite volume as does R_1 in Fig. 2, vanishing volume as does the degenerate R_9, and may be disconnected as illustrated by R_2. If we were only interested in regions of the configuration space of a many-particle system that defined the behavior of each of the particles with respect to three-dimensional *space*, the regions R_i would be unions and intersections of regions of configuration space of the special form of a product of a region $[0,T]\times\mathbb{R}^3$ for one particle and entire $\mathbb{R}^3$'s for all the rest. More general alternatives, however, are defined by regions of M that do not have this special form.

Each region defines a partition of the fine-grained histories (the paths) proceeding from $t=0$ to T into the following two classes: (1) the class r_i of all paths that intersect R_i at least once, and (2) the class $\bar{r}_i$ of all paths that *never* cross the region R_i. In symbols,

$$r_i=\{X(t)|X(t)\in R_i \text{ for some } t\in[0,T]\} , \tag{5.2a}$$

$$\bar{r}_i=\{X(t)|X(t)\notin R_i \text{ for all } t\in[0,T]\} . \tag{5.2b}$$

Clearly, r_i and $\bar{r}_i$ define an exhaustive and exclusive partition of the all fine-grained histories u:

$$r_i\cup\bar{r}_i=u,\quad r_i\cap\bar{r}_i=\phi . \tag{5.3}$$

The whole set of fine-grained histories may now be partitioned according to whether they lie in the class r_i or $\bar{r}_i$ for each region R_i. We can enumerate the coarse-grained histories by introducing an index α_i, which is $+1$ if the coarse-grained history is in class r_i and 0 if it is in $\bar{r}_i$. An individual coarse-grained history is thus labeled by a sequence $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i=+1$ or 0. For example,

$$c_{110100\dots1}=r_1\cap r_2\cap\bar{r}_3\cap r_4\cap\bar{r}_5\cap\bar{r}_6\cap\cdots\cap r_n . \tag{5.4}$$

More specifically, consider just three regions R_1, R_2, and R_3. The coarse-grained history $c_{110}=r_1\cap r_2\cap\bar{r}_3$ consists of all paths which intersect regions R_1 and R_2 at least once but never enter region R_3. The whole set u consists of all possible sequences $(\alpha_1,\dots,\alpha_n)$ in which the α's take the values $+1$ or 0. Thus

$$\bigcup_\alpha c_\alpha=u,\quad c_\alpha\cap c_\beta=\phi,\quad \text{any } \alpha_k\neq\beta_k . \tag{5.5}$$

We shall shortly illustrate such coarse grainings with simple examples. First, however, we need to understand how to compute the Feynman integrals over the classes c_α.

We first consider one region R in M and understand how to calculate the sum over paths in the class $\bar{r}$ which never cross R and that over in the class r which cross R at least once. First, consider $\bar{r}$. The Feynman integral over the class $\bar{r}$ is the limit of the integral over polygonal paths in $\bar{r}$ as in (4.4). Each constant-time cross section of R is a region of configuration space $\Delta(t)$. Let $e_{\bar{\Delta}}(X)$ be the indicator function that is 0 when $X\in\Delta$ and unity when $X\notin\Delta$. In the limit of large N, the indicator function $e_{\bar{r}}$ for the polygonal paths in $\bar{r}$ will be increasingly well approximated by

$$e_{\bar{r}}(X_N,\dots,X_0)\approx e_{\bar{\Delta}(T_N)}(X_N)\cdots e_{\bar{\Delta}(T_0)}(X_0) , \tag{5.6}$$

in the sense that the left- and right-hand sides of (5.6) will fail to coincide on a negligible set of paths. [If the boundaries of R are piecewise parallel to the x or t axis (rectangular boundaries), (5.6) is exact for all N.]

By introducing projection operators on the regions $\bar{\Delta}$ at the various times, the integrals over polygonal paths may be expressed as matrix elements of operators. Let $P_{\Delta(t)}$ denote the projection onto the complement of $\Delta(t)$. $P_{\bar{\Delta}(t)}$ is time dependent, not because it is a Heisenberg picture operator, but because the region $\bar{\Delta}(t)$ is time dependent. Clearly,

$$\langle X''|P_{\bar{\Delta}}|X'\rangle=\delta(X''-X')e_{\bar{\Delta}}(X') . \tag{5.7}$$

Using this, (5.6), and the free propagator (4.9), the path integral over the class $\bar{r}$ can be written as the limit

$$K_{ij}(\bar{r})=\lim_{N\to\infty}\left\langle\phi_i\left|P_{\bar{\Delta}(T)}\prod_{k=0}^{N-1}(e^{-iH_0(T/N)}e^{-iV(T/N)}P_{\bar{\Delta}(kT/N)})\right|\psi_j\right\rangle , \tag{5.8}$$

where the product is time ordered, written with the earliest $P_{\bar{\Delta}(t)}$'s to the right.

The projection $P_{\bar{\Delta}}$ can be written in the form

$$P_{\bar{\Delta}(t)}=e^{-E_R(t)\epsilon} , \tag{5.9}$$

where ϵ is an arbitrary positive number and E_R is the *excluding potential* for the spacetime region R, that is,

$$E_R(X,t)=\begin{cases}0, & (X,t)\notin R ,\\ +\infty, & (X,t)\in R .\end{cases} \tag{5.10}$$

Choosing $\epsilon=T/N$, we may then write (5.8) as

$$K_{ij}(\bar{r})=\left\langle\phi_i\left|P_{\bar{\Delta}(T)}\lim_{N\to\infty}\prod_{k=0}^{N-1}(e^{-iH_0(T/N)}e^{-i[V-iE_R(kT/N)](T/N)})\right|\psi_j\right\rangle . \tag{5.11}$$

Again, the operators in (5.11) are time ordered with the earliest on the right.

As a generalization of the Trotter product formula (4.12), we expect

$$\lim_{N\to\infty}\prod_{k=0}^{N-1}(e^{-iH_0(T/N)}e^{-i[V-iE_R(kT/N)](T/N)})$$
$$=\mathbb{T}\exp\left[-i\int_0^T dt[H_0+V-iE_R(t)]\right], \quad (5.12)$$

where $\mathbb{T}$ denotes the time-ordered product [25]. That is, the right-hand side of (5.12) may be interpreted as $U_R(T)$, where $U_R(t)$ is the solution of

$$i\frac{dU_R(t)}{dt}=[H_0+V-iE_R(t)]U_R(t)\ , \quad (5.13)$$

with the boundary condition

$$U_R(0)=I\ . \quad (5.14)$$

Physically, (5.13) represents Schrödinger evolution in the presence of a completely absorbing potential on the spacetime region R. Paths that once cross into the region R do not contribute to the final value of U. We shall discuss more explicitly how to solve Eq. (5.13) in the next section.

Equation (5.12) allows us to identify the class operators for the coarse graining based on a single spacetime region R. There are two coarse-grained histories in the set: r, the fine-grained histories which cross R at least once, and $\bar{r}$, the fine-grained histories which never cross R. For $\bar{r}$ we have

$$C_{\bar{r}}=U_R(T)$$
$$=P_{\bar{\Delta}(T)}\mathbb{T}\exp\left[-i\int_0^T dt[H_0+V-iE_R(t)]\right]\ . \quad (5.15)$$

The operator C_r then follows from the fact that the set of paths r which cross R at least once is the *difference* between the set of all paths u and the set which $\bar{r}$ which never cross R:

$$r=u-\bar{r}\ , \quad (5.16)$$

where, as usual, $a-b\equiv a\cap\bar{b}$. Equation (5.16) is just Eq. (5.3) rewritten. The corresponding relation for the class operators is

$$C_r=e^{-iHT}-U_R(T)\ , \quad (5.17)$$

which is the same as (4.7).

The generalization of this analysis to the general coarse graining based on many spacetime regions is straightforward. In a given coarse-grained history, e.g., (5.4), write all the factors r_i in the form $u-\bar{r}_i$. Expand this using $a\cap(b-c)=(a\cap b)-(a\cap c)$, etc., into differences of intersections consisting exclusively of the $\bar{r}_i$. For example,

$$c_{110}=r_1\cap r_2\cap\bar{r}_3$$
$$=(u-\bar{r}_1)\cap(u-\bar{r}_2)\cap\bar{r}_3$$
$$=[\bar{r}_3-(\bar{r}_1\cap\bar{r}_3)]-[(\bar{r}_2\cap\bar{r}_3)-(\bar{r}_1\cap\bar{r}_2\cap\bar{r}_3)]\ . \quad (5.18)$$

Now a sum over paths in a set which is a difference is the difference of the sums over paths. (A sum over a union of sets is not the sum of the corresponding sums unless the sets are disjoint, because of overcounting.) The class operators corresponding to sets that are the intersections of $\bar{r}_i$ can be calculated according to (5.12) or (5.13) using a potential which excludes the union of the corresponding regions or $\exp(-iHT)$ for the class u.

Thus, for example, corresponding to (5.18),

$$C_{110}=U_{R_3}(T)-U_{R_1\cup R_3}(T)-U_{R_2\cup R_3}(T)$$
$$+U_{R_1\cup R_2\cup R_3}(T)\ . \quad (5.19)$$

In this way the class operators for each history in the spacetime coarse graining based on an exhaustive division of spacetime into exclusive regions can be calculated. They will neither be unitary nor Hermitian as the simplest example of partitions by alternatives at discrete moments of time shows (Sec. IV) and as Eq. (5.13) confirms. They will, however, satisfy the general relation (4.7). With the class operators the decoherence functional for the coarse-grained set may be calculated by (4.6) and probabilities assigned to this very general class of spacetime histories of the closed system when the set of alternative histories decoheres.

VI. EXAMPLES AND THEIR COMPUTATION

The simplest examples of coarse grainings by spacetime regions are provided by exhaustive sets of regions of configuration space at a set of discrete moments $t_1,\ldots,t_n$ as were discussed at the end of Sec. III. Spacetime is partitioned into regions $R_{(\alpha_k t_k)}=\Delta_{\alpha_k}(t_k)$ of negligible extent in time plus the region R_0 exterior to all of them. Many of the coarse-grained histories of the form (5.4) are then empty. For example, the set $\bar{r}_0$ is empty because every path passes through the region R_0 sometime (assuming $t_1\neq 0$ and $t_n\neq T$). Further, a set in which both $r_{(\alpha_k t_k)}$ and $r_{(\beta_k t_k)}$ occur for $\alpha_k\neq\beta_k$ for a given t_k is empty because a fine-grained history crosses a surface of constant time just once. The nonempty coarse-grained histories are then easily seen to be labeled by the sequence $(\alpha_n,\ldots,\alpha_1)$, indicating which region of configuration space the paths cross at times $t_1,\ldots,t_n$. The class operators for these partitions are displayed in Eq. (4.15). Coarse graining by alternative regions of configuration space at discrete instants of time is thus a special case of coarse graining by spacetime regions. We shall now consider some less familiar examples and the computation of their class operators.

A. Simple example

Consider a free particle ($V=0$) in one dimension so that spacetime M is $[0,T]\times\mathbb{R}$. Consider the partition of M into two spacetime regions; R_1 where $X<0$ and R_2 where $X>0$. A partition of the fine-grained paths between $t=0$ and T based on these regions would consist of the four coarse-grained classes $c_{00},c_{01},c_{10},c_{11}$ labeled according to (5.4). The class consisting of paths which never cross either region c_{00} is empty. The description of the remaining classes may be abbreviated as follows: c_{01}, all paths which never cross the region $X<0$; c_{10}, all paths which never cross the region $X>0$; c_{11}, all paths which cross both the regions $X<0$ and $X>0$. Clearly, this is an exhaustive and exclusive set of coarse-grained histories. They are illustrated in Fig. 3.

The class operators for these coarse-grained histories are straightforward to calculate. First, $C_{00}=0$. Next, consider the class c_{01}. The class operator C_{01} is given by the solution to the evolution problem (5.13) with $X<0$ excluded. E_{R_1} is constant in time, equaling 0 for $X>0$ and $+\infty$ for $X<0$. The class operator C_{01} is thus the propagator for free, unitary evolution on $X>0$ calculated with the boundary condition [25] that it vanish for $X=0$. Explicitly,

$$\langle X''|C_{01}|X'\rangle=\theta(X'')\theta(X')\left[\frac{M}{2\pi iT}\right]^{1/2}\left[\exp\left[i\frac{M}{2T}(X''-X')^2\right]-\exp\left[i\frac{M}{2T}(X''+X')^2\right]\right] . \tag{6.1}$$

Here $\theta(X)$ is 1 for $X>0$ and 0 for $X<0$. In a similar way, the matrix elements of C_{10} are

$$\langle X''|C_{10}|X'\rangle=\theta(-X'')\theta(-X')\left[\frac{M}{2\pi iT}\right]^{1/2}\left[\exp\left[i\frac{M}{2T}(X''-X')^2\right]-\exp\left[i\frac{M}{2T}(X''+X')^2\right]\right] . \tag{6.2}$$

The class operator C_{11} is constructed by

$$C_{11}=e^{-iH_0T}-C_{01}-C_{10} , \tag{6.3}$$

so that the sum rule (4.7) is satisfied. Explicitly, the matrix elements of C_{11} are

$$\begin{aligned}\langle X''|C_{11}|X'\rangle=&[\theta(X'')\theta(-X')+\theta(-X'')\theta(X')]\left[\frac{M}{2\pi iT}\right]^{1/2}\exp\left[i\frac{M}{2T}(X''-X')^2\right]\\ &+[\theta(X'')\theta(X')+\theta(-X'')\theta(-X')]\left[\frac{M}{2\pi iT}\right]^{1/2}\exp\left[i\frac{M}{2T}(X''+X')^2\right] .\end{aligned} \tag{6.4}$$

With these class operators the decoherence functional for this set of coarse-grained histories may be computed according to (4.6).

B. Spacetime regions decomposable into rectangles

The techniques illustrated by the simple example above apply more generally in the case that the boundaries of the spacetime regions are piecewise parallel to the time axis or to surfaces of constant time. The general situation is illustrated in Fig. 4. In this case any excluding potential $E_R(t)$ is piecewise constant in time—infinite on the excluded regions of configuration space and zero elsewhere. A propagator $U_R(t)$ "excluding a spacetime region R" is, in each time interval on which E_R is constant, the propagator for unitary evolution on $\mathbb{R}^\nu$ minus the excluded region of configuration space. Specifically, consider the example in Fig. 4. The matrix element of $U_R(T)$, where R is union of all regions shown, is

$$\langle X''|U_R(T)|X'\rangle=\int dX_m\cdots\int dX_1K_m(X''T,X_mt_m)K_{m-1}(X_mt_m,X_{m-1}t_{m-1})\cdots K_0(X_1t_1,X'0) . \tag{6.5}$$

Here each integral is over $\mathbb{R}^\nu$. The propagator $K_k(Xt,X_kt_k)$ is the solution of the Schrödinger equation that is $\delta(X-X_k)$ when t coincides with t_k and that vanishes when X lies on the boundary of the excluded region between t_k and t_{k+1}. The propagator vanishes when X or X_k lies *in* the excluded region R.

The above construction makes clear that, when the spacetime regions R_i are decomposable into rectangular regions, the class operators are well defined and computable by solving the Schrödinger equation in appropriate domains of configuration space. Put differently, Eq. (6.5) is the heart of a demonstration that the Trotter product formula (4.12) can be extended to include singular excluding potentials E_R in the form (5.12) provided the region R is decomposable into rectangles. A mathematically precise route can thus be traced from the path-integral definition to the class-operator result in this case. It is plausible that the class operators for general regions with smooth boundaries can be approximated by regions decomposable into rectangular regions of an increasingly large number of time steps.

The composition of propagators can also be used to reduce the computation for the class operators for general spacetime regions into a series of Schrödinger evolution problems having an excluded region with a moving boundary such as are illustrated in Fig. 2. Solutions can be shown to exist for the analogous parabolic equations, at least, if the boundaries of the regions R_i never become tangent to a constant-t (characteristic) surface [27].

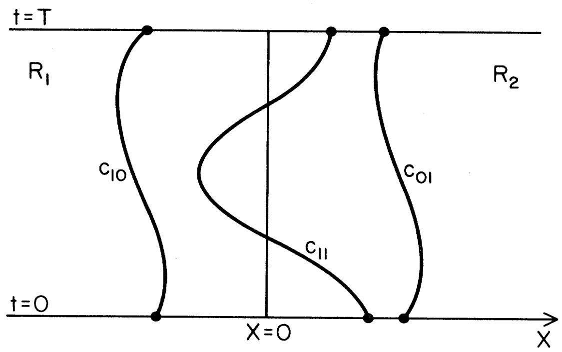

FIG. 3. Simple spacetime coarse graining of the histories of a free particle in one dimension. The regions R_1 and R_2 are $[0,T]\times$ (negative X) and $[0,T]\times$ (positive X), respectively. The paths which pass between $t=0$ and T may be partitioned into the classes c_{01} which pass through R_2 at least once but not R_1, the class c_{10} which pass through R_1 at least once but not through R_2, the class c_{11} which pass through both regions at least once, and the empty class c_{00} of paths which are never in either region. Representative paths from each of the nonempty classes is shown.

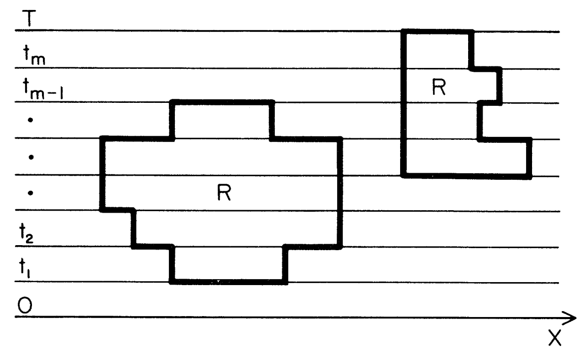

FIG. 4. Region decomposable into rectangles. The propagator that is the sum over all paths *excluding* the spacetime region R may be calculated as the composition of propagators over time intervals in which R is constant according to (6.5). In these time intervals the propagators represent unitary evolution in the presence of a potential constant in time, which is infinite inside R, but zero outside it. These propagators vanish for all points on the boundary of the excluded region. They may be calculated by solving the Schrödinger equation in these time intervals with this boundary condition and a δ-function boundary condition on the initial surface.

C. Euclidean methods

In cases such as those discussed above, where $E_R(t)$ is constant on an interval $[t_1,t_2]$, Euclidean path-integral methods may be used to construct the propagator $U_R(t_2,t_1)$. To keep the notation simple, let us rescale the time so that $t_1=0$, $t_2=T$, understanding that several different rescalings may be required for piecewise constant $E_R(t)$ in the original interval $[0,T]$. If t is continued to $-it$ and T to $-i\tau$, then, as discussed in Sec. IV, there is a measure theoretic notion of the path integral. The product formula (5.12) now becomes

$$\lim_{N\to\infty}\left(e^{-H_0(\tau/N)}e^{-(V+E_R)(\tau/N)}\right)^N=e^{-(H_0+V+E_R)\tau}\ . \tag{6.6}$$

The excluding potential E_R does not rotate in passing from (5.12) to (6.6) because a projection (5.9) excluding paths from the region R has the same expression in terms of E_R in both real- and imaginary-time regimes.

Kato's generalization [26] of the Trotter product formula shows that the left-hand side of (6.6) converges to the right-hand side if H_0 and V are self-adjoint and bounded from below and E_R is positive even when, as here, E_R is not densely defined. Further, by the Feynman-Kac formula, the right-hand side of (6.6) *is* the result of carrying out the Euclidean path integral. Finally, existence and uniqueness can be shown for appropriate boundary-value problem of the parabolic equation which results when (5.13) is rotated to imaginary times [27]. There is thus a happy confluence of mathematical results in the imaginary-time case.

Euclidean-time path integrals may be effectively computed by exploiting their connection with stochastic processes. Consider, for example, the path integral for the Euclidean propagator $\langle X''|C_\alpha|X'\rangle_{\rm Euc}$ between X' at $t=0$ and X'' at $t=\tau$ defined by a class of paths c_α. In a time slicing of N steps, the repeated integrals analogous to (4.4) are

$$K_N(X''\tau,X'0;c_\alpha)=\int dX_{N-1}\cdots\int dX_1\mu(N)e_\alpha(X'',X_{N-1},\ldots,X')e^{-I(X'',X_{N-1},\ldots,X')}\ . \tag{6.7}$$

The path integral is the limit of such expressions as $N\to\infty$, keeping X'', X', and τ fixed. At any intermediate stage, K_N may be thought of as a "probability" distribution in X'' which is the result of an N-step stochastic process with the "probability" of each step proportional to e^{-I} (assuming μ is chosen so this "probability" is normalized). Paths are "absorbed" in those regions of spacetime where $e_\alpha=0$ and propagate according to this "probability" distribution elsewhere. Because of the central-limit theorem, a variety of stochastic processes will give the same results in the limit as $N\to\infty$. This can be useful in explicit computations.

The Euclidean analogue of the one-dimensional example discussed in Sec. VI A above provides a simple example. Consider, specifically, the matrix element $\langle X''|C_{01}X'\rangle_{\rm Euc}$ for the class of paths which never intersects the region $X<0$. By techniques identical to those used in the real-time case, one arrives at the Euclidean analogue of (6.1):

$$\langle X''|C_{01}|X'\rangle_{\text{Euc}}=\theta(X'')\theta(X')\left[\frac{M}{2\pi\tau}\right]^{1/2}\left[\exp\left[-\frac{M}{2\tau}(X''-X')^2\right]-\exp\left[-\frac{M}{2\tau}(X''+X')^2\right]\right]. \tag{6.8}$$

Figure 5 shows the result of a numerical calculation of the same matrix elements using stochastic methods. In the case of a free particle, each time step in (6.7) corresponds to a Gaussian random walk with a distribution

$$\left[\frac{M}{2\pi(\tau/N)}\right]^{1/2}\exp\left[-\frac{M}{2(\tau/N)}(X''-X')^2\right]. \tag{6.9}$$

For fixed $X'>0$, one can compute the path integral by carrying out a large number of N-step random walks with this Gaussian distribution and discarding all walks that cross $X=0$. The distribution in end points after N steps is an approximation to integral which should become better and better as N is taken larger and larger. Figure 5 shows that this is the case.

A modestly more sophisticated example of the use of the same techniques is the direct calculation of the matrix element $\langle X''|C_{11}|X'\rangle_{\text{Euc}}$ in the same example. The class of paths is all those which intersect both the regions $X>0$ and $X<0$ at least once. This matrix element was, in fact, calculated by stochastic methods for $X''>0$ and $X'>0$ in Ref. [7], although in a slightly different notation. We shall not repeat the details of that calculation here, but merely note some essential features. Spacetime was divided into a lattice of points spaced by ϵ in the time direction and η in the X direction. The stochastic process used was not the Gaussian random walk with the "probability" distribution (6.9), but rather a single-step random walk with a "probability" of $\frac{1}{2}$ of moving one lattice spacing to the left or right. The two distributions are equivalent in the limit of large N because of the central-limit theorem. The utility of using a single-step random walk as the stochastic process is that all the relevant "probabilities" may be computed analytically by difference equation methods and the continuum limit studied explicitly [28]. The Euclidean versions of (6.1) or (6.2) for the matrix elements $\langle X''|C_{01}|X'\rangle_{\text{Euc}}$ or $\langle X''|C_{10}|X'\rangle_{\text{Euc}}$, for example, are straightforwardly recovered.

The matrix element $\langle X''|C_{11}|X'\rangle_{\text{Euc}}$ corresponding to the sum over all paths starting at $X'>0$ which cross $X=0$ at least once was computed as follows: First, the "probability" of a *first* crossing of $X=0$ at time t_1 was computed. This is a random walk from X' at $t=0$ to $X=0$ at $t=t_1$ with an absorbing barrier at $X=0$. Next, the "probability" of starting at $X=0$ at $t=t_1$ and arriving at X'' at $t=\tau$ was computed. The total "probability" to arrive at X'' at τ via paths which cross the origin at least once is the product of these two "probabilities" summed over t_1 between 0 and τ. Finally, the limit $\epsilon\to 0$, $\eta\to 0$, keeping $\epsilon/\eta^2=M$, was taken.

The result obtained in Eq. (6.5) of Ref. [7] for the continuum limit of the lattice sum over paths corresponding to the $\langle X''|C_{11}|X'\rangle_{\text{Euc}}$ was (in the present notation) when $X'>0$ and $X''>0$:

$$\langle X''|C_{11}|X'\rangle_{\text{Euc}}=\int_0^\tau dt_1\left[\left[\frac{M}{2\pi(\tau-t_1)}\right]^{1/2}\exp\left[-\frac{MX''^2}{2(\tau-t_1)}\right]\right]\left[\left[\frac{M}{2\pi t_1^3}\right]^{1/2}X'\exp\left[-\frac{MX'^2}{2t_1}\right]\right]. \tag{6.10}$$

The term in the first pair of large square brackets is the continuum limit of the unrestricted random walk from t_1 to τ. That in the second pair is the continuum limit of the random walk with absorbing barrier between $t=0$

The integral in (6.10) can be carried out analytically. The result is

$$\langle X''|C_{11}|X'\rangle_{\text{Euc}}=\left[\frac{M}{2\pi\tau}\right]^{1/2}\exp\left[-\frac{M}{2\tau}(X''+X')^2\right]. \tag{6.11}$$

This is the same result as would be obtained by the indirect method of computation discussed in Sec. VI A using excluded regions and the sum rule (4.7). That is, (6.11) is (6.4) evaluated at imaginary times for $X'>0$ and $X''>0$. The coincidence between these two different methods of calculating $\langle X''|C_{11}|X'\rangle_{\text{Euc}}$ is a reassuring check of the methods under discussion.

VII. FINER AND COARSER GRAININGS

Dividing the coarse-grained histories of a set partitioned by spacetime regions into smaller mutually exclusive classes is an operation of fine graining. Combining coarse-grained histories into larger mutually exclusive classes is an operation of coarse graining [9,11]. If the starting coarse-grained set decoheres, then all coarser grainings of it will decohere because the probability sum rules defining decoherence for the coarser-grained set are already contained in those of the starting set. Finer grainings of the starting set may not decohere. In this section we discuss some simple examples of finer and coarser grainings of sets of histories partitioned by spacetime regions.

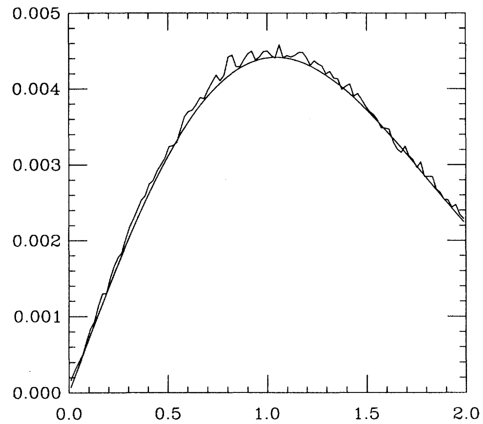

FIG. 5. Numerical calculation by stochastic methods of the sum over histories defining the matrix elements of a class operator. The paths of a free particle in one dimension between points $X'>0$ at $t=0$ and $X''>0$ at $t=\tau$ may be partitioned into the class of all paths that never cross $X=0$ (class c_{01}) and the class of all paths which cross $X=0$ at least once (class c_{11}). The figure shows a numerical evolution of the Euclidean sum over paths defining the matrix element $\langle X''|C_{01}|X'\rangle_{\rm Euc}$ representing the class c_{01}. $X'=0.5$ in units that have been chosen so that $\hbar=M=1$. The sum over paths is defined as the limit of the integral over polygonal (skeletonized) paths specified by their values on N slices of the time interval between 0 and τ. The values of this integral are the same as the "probability" distribution of the outcomes in X'' of an N-step random walk starting at X' with a "probability" distribution for each step defined by the classical Euclidean action. In the present case of a free particle, the random walk is Gaussian with a distribution given by (6.9). To calculate the matrix element of the class operator C_{01}, paths which cross the origin are terminated and do not contribute to the final distribution. The figure shows the final distribution of 1 010 000 random walks starting at $X'=0.5$ of $N=3000$ steps each. The outcomes on the interval $X''=[0,2]$ have been collected in 101 bins of equal size. The smooth curve shows the analytic result (6.8) binned in the same way. The calculated curve and analytic one are close, within statistical errors and the error caused by using a finite value of N to approximate the limit $N\rightarrow\infty$. Indeed, by using larger bins one can smooth the curve further by reducing the statistical fluctuations in each bin. The calculated curve is systematically higher than the exact result by an amount which decreases with larger N. The matrix element $\langle X''|C_{01}|X'\rangle$ could also be written as the composition of N exact Euclidean propagators of a free particle with an infinite potential barrier for $X<0$. These propagators have the form of Eq. (6.8). The two terms in that equation may be thought of as the contributions of the two possible classical paths between X' at $t=0$ and X'' at $t=\tau$. These are a direct path and a path which reflects off the origin. However, the integrals resulting from the Gaussian random walk defined by the free-particle propagator (6.9) contain no such reflected paths. The coincidence of the two results in the limit $N\rightarrow\infty$ is essentially a consequence of the central-limit theorem. What this calculation shows explicitly is that path integrals can be defined as the limits of integrals over polygonal paths whose action [Eq. (4.3)] is computed using the *free*-particle kinetic energy even when there are several classical paths in the limit of arbitrarily small time steps. The action of the free-particle path increasingly dominates the others as the time steps become small.

A. Finer grainings

The natural finer graining of a partition by spacetime regions would be to specify not just whether a history crosses a region at least once or never, but also to specify exactly how many times it crosses. Considering this kind of coarse graining, however, illustrates that it is possible to define coarse grainings which are trivial in the sense that the set of all histories is partitioned so finely that the amplitudes for most classes are zero. This was discussed at length in Refs. [7] and [6]. There, coarse grainings were considered which partitioned paths according to how many times and at what locations they crossed a surface which was partially timelike. Such partitions could be defined on a spacetime lattice, and the relevant amplitudes [e.g., the lattice analogues of (4.2)] were computed using the stochastic methods of Sec. VI. In the continuum limit, however, the amplitudes for any *finite* number of crossings vanished. The physical reason is that, because nondifferentiable paths dominate the sum, the expected number of crossings of a path which crosses a timelike surface at least once is infinity. The amplitude for any finite number of crossings, therefore, is zero. There is nothing incorrect about this result; it only shows that the partition was too fine to be useful.

Of course, if the surface is entirely spacelike (a surface of constant time), then it is possible to specify the number of crossings. The paths, being single valued in time, cross a spacelike surface once and only once.

B. Coarser grainings defining momentum

As an example of a useful coarser graining of a partition defined by spacetime regions, we consider coarse grainings defining momentum.

Momentum is not a fundamental variable on equal footing with position in a sum-over-histories-formulation quantum mechanics that posits paths in configuration space as the unique set of completely fine-grained histories. However, as demonstrated by Feynman and Hibbs [29], momentum can be defined in terms of configuration-space paths by analyzing idealized experiments that determine it. For example, momentum may be determined by time of flight. Such determinations correspond to coarser grainings of coarse-grained sets of histories defined by spacetime regions, as we shall now illustrate by following what is essentially the analysis of Feynman and Hibbs.

Consider one spatial dimension and the particular division of spacetime into the regions illustrated in Fig. 6. Space at time t is divided into intervals of equal length Δ labeled by an integer k ranging from $-\infty$ to $+\infty$. Similarly, space at time t' is divided into intervals of equal length Δ' labeled by an integer k' ranging from $-\infty$ to $+\infty$. We shall be considering cases where $\Delta'\ll\Delta$. The remaining part of spacetime outside these regions participates trivially in any coarse graining since all paths pass through it. Each coarse-grained history in the partition defined by these regions may be labeled by two integers (k,k'). The coarse-grained history (k,k') consists of paths which pass through the interval k at time t and the interval k' at time t'. To define momentum by time of

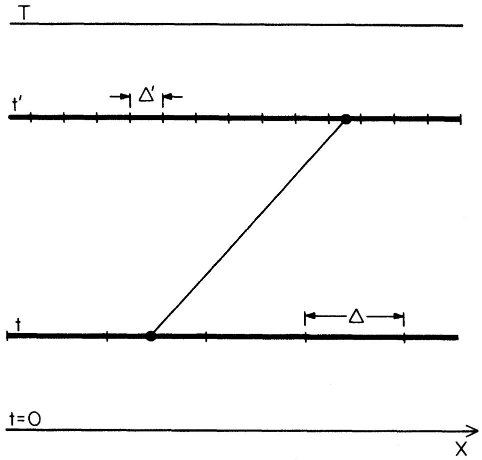

FIG. 6. Coarse graining defining momentum. Classically the momentum of a free particle may be determined from the distance traveled in a known time interval. The figure shows a one-dimensional configuration space divided into intervals of equal length Δ at time t and of equal length Δ' at time t'. A coarse graining defining momentum partitions all paths from t' to t into classes specified by the net distance traversed between these times up to an accuracy specified by the sizes of the intervals. This is a coarser graining than obtained by specifying the *particular* intervals that the paths pass through at these times. By making Δ' small and Δ and $t'-t$ both large in a way that $\Delta/(t'-t)$ becomes small, both classical and quantum-mechanical uncertainties in the determination of momentum may be made arbitrarily small.

flight, we partition the paths, not by the specific position intervals they pass through, but, more coarsely, by the distance between the two intervals traversed in the flight between t and t'. Specifically, let $k\Delta$ be the center of interval k at time t and $k'\Delta'$ the center of interval k' at time t'. The union of all coarse-grained histories (k,k') such that

$$(m-\tfrac{1}{2})\Delta' \le (k'\Delta'-k\Delta) \le (m+\tfrac{1}{2})\Delta' \tag{7.1}$$

defines a coarser-grained set labeled by a single integer m. The coarse-grained history m consists of all paths which have been displaced by $m\Delta'$ to an accuracy of $\pm(\Delta/2+\Delta')$, in their flight from t to t'. If the particle is free between t and t', this corresponds classically to paths with momenta in the range

$$M\left[\frac{m\Delta'-(\Delta/2+\Delta')}{t'-t}\right] \le P \le M\left[\frac{m\Delta'+(\Delta/2+\Delta')}{t'-t}\right] . \tag{7.2}$$

The classical uncertainty in P may be made arbitrarily small by choosing Δ and Δ' small or by making $t'-t$ large. However, quantum mechanically, there is an additional uncertainty in the momentum of order $\hbar/\Delta$. To have a coarse graining which makes both classical and quantum-mechanical uncertainties small, one must take Δ large and $(t'-t)$ large in such a way that $\Delta/(t'-t)$ is small.

To show that this coarse graining does indeed determine momentum in the familiar quantum-mechanical way, let us assume that it decoheres, say, by coupling to a larger decohering system, as in situations where the intervals through which the particle passes at time t and t' are measured [9], and calculate the probabilities for the various values of the momentum P. These are the diagonal elements of the decoherence function (4.6).

The class operators for the partition labeled by m are evidently

$$C_m = \sum_{k'k} e_{k'k}(m) e^{-iHT} P_{k'}(t') P_k(t) , \tag{7.3}$$

where $P_k(t)$ is the projection on interval k at time t, $P_{k'}(t)$ is similarly the projection on interval k' at time t', and $e_{k'k}(m)$ is unity if condition (7.2) is satisfied and zero otherwise. For definiteness assume that the initial state is pure, corresponding to a wave packet $\psi(X,0)$ that under Schrödinger evolution evolves to a wave packet $\psi(X,t)$ at time t which has a characteristic width W. Then the probability for the coarse-grained history labeled m with momenta in the range (7.2) is

$$\begin{aligned} p(m) &= \mathrm{Tr}(C_m|\psi\rangle\langle\psi|C_m^\dagger) \\ &= \sum_{k'k} e_{k'k}(m) \int_{-\Delta'/2}^{+\Delta'/2} d\xi' \left| \int_{-\Delta/2}^{\Delta/2} d\xi \langle k'\Delta'+\xi'|e^{-iH_0(t'-t)}|k\Delta+\xi\rangle \psi(k\Delta+\xi,t)\right|^2 . \end{aligned} \tag{7.4}$$

Fix a value of P. In the limit $\Delta\to\infty$, $\Delta/(t'-t)\to 0$, and $m\to\infty$ keeping P fixed through (7.2), $p(m)$ becomes the probability that the momentum has the value P in the range δP determined by (7.2):

$$\delta P = \frac{Mm\Delta'}{t'-t} . \tag{7.5}$$

To see this explicitly, evaluate (7.4) explicitly. The free-particle propagator is given by (4.9), specifically in this one-dimensional case by

$$\langle k'\Delta'+\xi'|e^{-iH_0(t'-t)}|k\Delta+\xi\rangle = \left[\frac{M}{2\pi i(t'-t)}\right]^{1/2} \exp\left[\frac{iM}{2(t'-t)}[(k'\Delta'-k\Delta)+\xi'-\xi]^2\right] . \tag{7.6}$$

In the limit, since the wave packet has a finite width, only the term with $k=0$ contributes to (7.4) and only that value of k' connected to m by (7.1). In the limit, the integral over ξ' receives a nonvanishing contribution only from the finite width W. Thus the only term in the exponent of (7.6) which is nonvanishing and does not contribute a trivial phase is the cross term between $k'\Delta'$ and ξ'. One finds, making use of (7.6), that the limit of (7.4) is

$$\delta P \left| (2\pi)^{-1/2} \int_{-\infty}^{+\infty} d\xi' \, e^{-iP\xi'} \psi(\xi',t) \right|^2 . \tag{7.7}$$

This is indeed the probability that the momentum at time t is P in the range δP.

VIII. UNSPECIFIED TIMES

In quantum cosmology, even in those epochs when spacetime is approximately classical, we do not expect the universe to exhibit clocks keeping accurate time from the big bang to now. We expect to determine accurate time intervals between events in the present epoch, but to determine their temporal distance from the big bang only crudely, say, plus or minus a fraction of 10^9 yr (or in the absence of cosmological observations not at all). We are therefore necessarily interested in coarse grainings of the universe for which only time differences are specified. An analogous class of coarse grainings in nonrelativistic quantum mechanics would be those which specify the time interval between spacetime regions but not their absolute location in spacetime. We can illustrate the definition of such partitions and the computation of their associated class operators with a simple example.

Consider a free particle in one dimension and the fine-grained histories that proceed from $t=0$ to T. We can partition these paths into the following exclusive classes: c_a, all paths which never cross $X=0$ between $t=0$ and $t=T$; c_b, all paths whose first crossing $X=0$ is between $t=0$ and $T-S$; c_c, all paths whose first crossing of $X=0$ is between $t=T-S$ and T. We can then further partition the class c_b into the subclasses: c_{bX}, all paths that are at a position X in a small interval dX a definite time S after their first crossing of $X=0$ between $t=0$ and $T-S$. Clearly, the classes (c_a, c_{bX}, c_c) are an exhaustive and exclusive set of coarse-grained histories as X ranges over all values. They are an example of the type of coarse graining discussed above. The time of first crossing of $X=0$ is not specified except that it lies somewhere in the interval $[0, T-S]$. However, the position X is determined at a time S after that first crossing, whatever time that takes place.

To calculate the decoherence functional for this coarse-grained set of histories, it is sufficient according to (4.6) to know the class operators C_a, C_{bX}, C_c. We calculate their matrix elements in the position representation using the Euclidean stochastic methods described in Sec. VI and Ref. [7]. Since the problem is clearly symmetric about $X=0$, we only evaluate matrix elements of the form $\langle X_f | C | X_0 \rangle$ for $X_0 > 0$. The rest are obtained by reflecting all X's in $X=0$. The matrix elements for C_a have already been calculated in connection with the example in Sec. VI. They are, from (6.1),

$$\langle X_f | C_a | X_0 \rangle = \theta(X_f) \left[\frac{M}{2\pi i T} \right]^{1/2} \left[\exp\left[\frac{iM}{2T}(X_f - X_0)^2 \right] - \exp\left[\frac{iM}{2T}(X_f + X_0)^2 \right] \right] . \tag{8.1}$$

The evaluation of C_c is likewise already contained in the results of the example of Sec. VI. The sum over all the paths whose first crossing of $X=0$ is between $t=T-S$ and T is

$$\langle X_f | C_a | X_0 \rangle = \int_0^\infty dY \left[\frac{M}{2\pi i S} \right]^{1/2} \exp\left[\frac{iM}{2S}(X_f + Y)^2 \right] \left[\frac{M}{2\pi i \bar{T}} \right]^{1/2} \left[\exp\left[\frac{iM}{2\bar{T}}(Y - X_0)^2 \right] - \exp\left[\frac{iM}{2\bar{T}}(Y + X_0)^2 \right] \right] , \tag{8.2}$$

where we have written $\bar{T}$ for $T-S$. The last two terms in (8.2) are the sum over all paths from X_0 at $t=0$ to $Y>0$ at $t=\bar{T}$ which never cross $X=0$ [cf. (6.1)]. The first term is the sum over all paths from Y at $t=\bar{T}$ to X_f at $t=T$ which cross $X=0$ at least once [cf. (6.4)]. The sum over paths is completed by a sum over Y.

To calculate C_{bX}, however, we must do some work. We introduce a spacetime lattice as shown in Fig. 7. Points on the lattice are labeled by a discrete pair of labels (x,y). The lattice spacing in the spatial, x direction is η. That in the temporal, y direction is ϵ. This is the setup used in Ref. [7] except that the discrete time label t used there has been called y here. Then, as described earlier or in Ref. [7], Euclidean sums over the paths of a free particle may be calculated as the "probabilities" of a single-step random walk in x with time steps ϵ. The matrix element $\langle X_f | C_{bX} | X_0 \rangle$ is the analytic continuation to real time of the continuum limit of the "probability" for the corresponding class of walks on the lattice. That class is all walks that start at x_0 at $y=0$, first cross $x=0$ at some discrete time y_c between $y=0$ and $y_\tau - y_\sigma$, arrive at x a number of steps y_σ later, and proceed on to x_f in a total of y_τ steps. The discrete times y_σ and y_τ correspond to σ and τ (the analytic continuations of S and T), respectively, in the continuum limit and the discrete positions x_0 and x_f to X_0 and X_f, respectively. The "probability" corresponding to this class of random walks is

$$u_{bx} = \sum_{P \in c_{bx}} \left(\tfrac{1}{2}\right)^{|P|} , \tag{8.3}$$

where the sum is over all paths P in the described class and $|P|$ is the number of steps in path P.

Evaluating (8.3) is a purely combinatoric problem. The sum may be broken down into partial sums and composed. In the notation of Ref. [7], it is convenient to introduce some partial sums which can be evaluated explicitly. We denote by $u(x',y';x,y)$ the "probability" for the unrestricted random walk between (x,y) and (x',y'). We denote by $u_-(x',y';x,y)$ the "probability" for a random walk between $(x>0,y)$ and $(x'>0,y')$ with an absorbing barrier at $x=-1$. The "probability" u_{bx} is then

$$u_{bx}=\sum_{y_c=1}^{y_\tau-y_\sigma} u(x_f,y_\tau;x,y_\sigma+y_c)u(x,y_\sigma+y_c;0,y_c)\times u_-(0,y_c;x_0,0)\ . \tag{8.4}$$

The sums corresponding to u and u_- are easily evaluated by standard difference-equation methods [28] and the results are exhibited in Ref. [7] [Eqs. (4.7) and (4.22)]. Here we display only their continuum limits when $\epsilon\to 0$ and $\eta\to 0$, keeping fixed $M=\epsilon/\eta^2$ and appropriate X's and times as $X=\eta x$ and $t=\epsilon y$. They are (Eqs. (4.8) and (4.23) of Ref. [7])

$$(2\eta)^{-1}u(x',y';x,y)\to\left[\frac{M}{2\pi(t'-t)}\right]^{1/2}\exp\left[-\frac{M(X'-X)^2}{2(t'-t)}\right] \tag{8.5}$$

and

$$(2\eta)^{-1}u_-(0,y_c;x_0,0)\to 2\epsilon X_0\left[\frac{M}{2\pi t_c^3}\right]^{1/2}\exp\left[-\frac{MX_0^2}{2t_c}\right]\ . \tag{8.6}$$

The factor of 2ϵ in (8.6) turns the sum in (8.4) into a time integral in the continuum limit. The result for $\langle X_f|C_{bX}|X_0\rangle_{\rm Euc}$ is

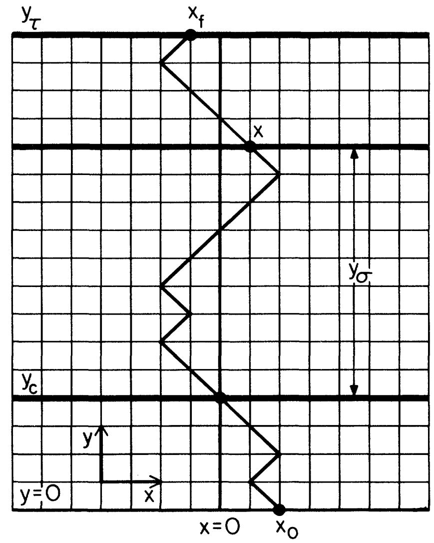

FIG. 7. Euclidean lattice calculation of the class operators for a coarse graining in which an elapsed time interval is specified, but the times of the end points of the interval are unspecified. The figure shows a lattice spacetime. Points on the lattice are labeled by a pair of discrete labels (x,y). The lattice spacing in the spatial direction is η, that in the temporal direction ϵ. Lattice paths, one of which is illustrated, proceed from the initial time $y=0$ to the final time y_τ. Euclidean sums over paths may be calculated as the "probabilities" for random walks on such a lattice. A typical path in the coarse-grained history labeled c_{bX} in the text is illustrated. This consists of paths which start at $x_0>0$ at $y=0$, cross $x=0$ at some unspecified time y_c later, arrive at position x a specified time y_σ later, and end at x_f at the final time y_τ. (The crossing must clearly be before time $y_\tau-y_\sigma$.) The sum over such paths defining the matrix elements of the associated class operator therefore includes a sum over y_c. Such coarse grainings with unspecified times are analogous to those expected to be of interest in quantum cosmology.

$$\langle X_f|C_{bX}|X_0\rangle_{\rm Euc}=dX\int_0^{\bar\tau}dt_c\left[\frac{M}{2\pi(\bar\tau-t_c)}\right]^{1/2}\exp\left[-\frac{M(X_f-X)^2}{2(\bar\tau-t_c)}\right]\times\left[\frac{M}{2\pi\sigma}\right]^{1/2}\exp\left[-\frac{MX^2}{2\sigma}\right]X_0\left[\frac{M}{2\pi t_c^3}\right]^{1/2}\exp\left[-\frac{MX_0^2}{2t_c}\right], \tag{8.7}$$

where we have abbreviated $\tau-\sigma$ as $\bar\tau$. The integral in (8.7) may be carried out analytically and the result continued back to real time. The answer is

$$\langle X_f|C_{bX}|X_0\rangle=dX\left[\frac{M}{2\pi iS}\right]^{1/2}e^{iMX^2/2S}\left[\frac{M}{2\pi i\bar T}\right]^{1/2}e^{-M(|X_f-X|+X_0)^2/2\bar T}\ . \tag{8.8}$$

As an exercise, one can verify that the sum rule (4.7) is satisfied.

This example serves not only to crudely illustrate the kind of coarse grainings with unspecified times that we expect in quantum cosmology, but also to show that reasonably complicated partitions can be calculated explicitly.

IX. DECOHERENCE, CAUSALITY, MEASUREMENT, UTILITY

A. Decoherence

Probabilities can be assigned to those sets of alternative coarse-grained histories that decohere. A set of coarse-grained histories decoheres when the initial ρ is such that the real part of the "off-diagonal" terms in the decoherence functional are sufficiently small according to (2.6). In the preceding sections we have shown how a decoherence functional for spacetime coarse grainings may be defined and calculated in nonrelativistic quantum mechanics. We illustrated these calculations in simple single-particle models. However, even for alternatives defined at one moment of time, such models have far too few degrees of freedom to exhibit realistic mechanisms of decoherence [30,9]. The situations in which spacetime coarse grainings (not consisting solely of alternatives defined at discrete moments of time) decohere have not been explored in any quantitative detail [31]. In this section we shall offer some rudimentary qualitative remarks on the decoherence of spacetime coarse grainings.

Alternatives defined at one single moment of time decohere automatically. This follows from the cyclic property of the trace and orthogonality of chains C_α consisting of a single projection operator. Decoherence is an issue only for coarse grainings defined at several moments of time. The initial conditions ρ that lead to the decoherence of alternatives defined over a continuous range of time can be expected to be more restricted than those leading to the decoherence of similar alternatives defined only at a few discrete moments of time. This expectation may be illustrated qualitatively by considering the coarse graining discussed in Sec. VI A.

Let us consider decoherence in the model described in Sec. VI A assuming, for simplicity, a pure initial state specified by an initial wave function $\psi(X_0)$, viz.,

$$\rho(X_0,X_0')=\psi(X_0)\psi^*(X_0') \ . \tag{9.1}$$

There are three nonempty, coarse-grained histories: c_{01}, c_{10}, and c_{11}. The off-diagonal element $D(c_{01},c_{10})$ vanishes identically because there are no paths in c_{01} and c_{10} with common end points as required by the final condition in (3.8). The only off-diagonal elements of the decoherence functional that might be nonvanishing are therefore $D(c_{01},c_{11})$ and $D(c_{10},c_{11})$. Consider the first of these. It may be expressed in terms of path integrals over all paths u or path integrals which exclude certain regions:

$$D(c_{01},c_{11})=D(c_{01},u)-D(c_{01},c_{01})-D(c_{01},c_{10}) \ . \tag{9.2}$$

The last term vanishes as already discussed. Denote by $\psi_{01}(X)$ the evolution of the initial wave function $\psi(X)$ over a time T with the region $X<0$ excluded. This is the solution of the free Schrödinger equation with initial condition $\psi(X_0)$ and the boundary condition $\psi_{01}(0)=0$. Similarly, denote by $\psi_u(X)$ the unrestricted evolution of the same initial condition. Then from (4.1), (9.1), and (9.2),

$$D(c_{01},c_{11})=\langle\psi_{01}|\psi_u\rangle-\langle\psi_{01}|\psi_{01}\rangle \ , \tag{9.3}$$

and in a similar manner and notation,

$$D(c_{10},c_{11})=\langle\psi_{10}|\psi_u\rangle-\langle\psi_{10}|\psi_{10}\rangle \ . \tag{9.4}$$

As an illustration, consider an initial wave function $\psi(X)$ that is a wave packet with an initial position $X_0>0$ and momentum P_0 defined to accuracies consistent with the uncertainty principle. If the time T is short compared with the time the packet takes to spread, position, and momentum will continue to be defined, and their expectation values will obey classical equations of motion. The expectation values for ψ_u will evolve as a free particle; those for ψ_{01} will evolve as a free particle with a reflecting wall at $X=0$.

With respect to the values of $D(c_{01},c_{11})$ and $D(c_{10},c_{11})$ and decoherence, two situations may be distinguished. If X_0 and P_0 are such that the center packet remains well away from $X=0$ during the time T, then $\psi_{01}(X)\approx\psi_u(X)$ and $\psi_{10}(X)\approx 0$. Both (9.3) and (9.4) approximately vanish, and the coarse-grained set of histories decoheres. By contrast, if the initial position and momenta are such that the center of the wave packet crosses $X=0$ during the time T, then at time T the center of ψ_u will be located at a position $X_T<0$, while that of ψ_{01}, which has reflected off the wall, will be located at $-X_T>0$. The wave functions ψ_u and ψ_{01} will be nearly orthogonal, and $D(c_{01},c_{11})\approx -1$. With such an initial condition the spacetime coarse-grained set of histories will not decohere. The decoherence of the coarse graining defined by the behavior of particle paths with respect to spacetime regions $(X>0,\ 0<t<T)$ and $(X<0,\ 0<t<T)$ for an initial wave packet state thus depends crucially on whether that wave packet can or cannot cross the origin in the time T.

Contrast this coarse graining by whether the particle is to the left or right of the origin or both over an extended range of time T with a coarse graining by whether the particle is to the left (L) or right (R) of the origin at a sequence of definite time $0<t_1<\cdots<t_n<T$. To each possible history (e.g., $RRLR\ldots LR$) there corresponds a wave function at T that is the initial ψ evolved unitarily between the times t_i and by projections onto $X>0$ or $X<0$ at the times t_i. Unless these projections happen at a time when the wave packet is crossing the origin, they will not much disturb its unitary evolution. In this case only the wave function corresponding to the unique history specified by the classical evolution will be significantly different from zero. The overlap of any pair of differing histories will be negligible. The set of histories therefore decoheres even in the situation that the wave packet crosses the origin in the time period $0<t<T$—in contrast with the coarse graining by spacetime regions discussed above. Of course, as the number of definite times t_i in the interval $0<t<T$ becomes larger, it becomes increasingly difficult to meet the condition that the wave packet be clear of the origin at these times. That fact, however, only supports this conclusion that there is a more restricted class of initial conditions that lead to decoherence for alternatives defined over a continuous range of times than for alternatives defined at a few definite moments of time.

As instructive as the above example is, it does not contain enough degrees of freedom to discuss the decoherence of spacetime coarse grainings by the familiar mechanisms that the effect of decoherence of alternatives at definite moments of time—e.g., the carrying away of phases by variables ignored in the coarse grainings [30,9]. A model in which these effects can be considered is that studied by Feynman and Vernon [32] and by Caldeira and Leggett [33]. The model consists of single oscillator interacting linearly with a large number of others. Coarse grainings are considered which follow the coordinates of the distinguished oscillator and ignore the others. Initial conditions are considered in which the density matrix ρ factors into a product of a density matrix of the distinguished oscillator, $\bar{\rho}$, and another for the rest. Under these conditions the integrals over the rest of the oscillators in (3.8) may be carried out, giving rise to an influence functional describing the interactions of the distinguished oscillator with the rest. The resulting decoherence function can be explicitly exhibited in the special case when there is a uniform cutoff continuum of other oscillators initially in a thermal state with a temperature T_B. The result is especially simple in the limit when kT_B is much higher than the cutoff energy, itself much higher than the characteristic energy quantum of the distinguished oscillator (the Fokker-Planck limit). Let x be the coordinate of the distinguished oscillator, M its mass, $S_{\rm free}$ its free action with frequency ω_R renormalized by the interactions with the others, and $\bar{\rho}(x,y)$ its initial density matrix. The decoherence functional for coarse grainings which partition only the paths of the distinguished oscillator into classes c_α then takes the form

$$D(c_\alpha,c'_\alpha)=\int_{c_\alpha}\delta x\int_{c_{\alpha'}}\delta y\;\delta(x_f-y_f)\exp(i\{S_{\rm free}[x(t)]-S_{\rm free}[y(t)]+W[x(t),y(t)]\}/\hbar)\bar{\rho}(x_0,y_0)\;. \tag{9.5}$$

This is (3.8) when the sum over the rest of the oscillators has been carried out. Their effect is summarized by the Feynman-Vernon influence functional [32], $\exp(iW[x(t),y(t)]/\hbar)$. In the Fokker-Planck limit, Caldira and Leggett find

$$W[x(t),y(t)]=-M\gamma\int_0^T dt(x\dot{x}-y\dot{y}+x\dot{y}-\dot{x}y)+i\frac{2M\gamma kT_B}{\hbar}\int_0^T dt[x(t)-y(t)]^2\;, \tag{9.6}$$

where γ summarizes the interaction strengths of the distinguished oscillator with the rest.

The path integrals in (9.5) may be calculated for coarse grainings based on spacetime regions as described in Sec. V. While the full range of techniques described in Secs. V and VI for calculating such path integrals is available on the full Hilbert space describing all oscillators, they are not all necessarily available on the reduced Hilbert space describing the distinguished oscillator. In particular, because the influence functional in (9.5) couples paths on both sides of the decoherence functional, it is no longer possible to calculate separate path integrals (4.2) for each class c_α and combine them to give the decoherence functional as in (4.1). For the same reason, it is no longer possible to use Schrödinger evolution (5.13) on the reduced Hilbert space to define class operators. We can, however, use master equations which evolve the reduced density matrices on the distinguished oscillator Hilbert space in a similar way. The matrix elements of the reduced density matrix at time T associated with the coarse-grained histories c_α and $c_{\alpha'}$ is defined by

$$\langle x_f|\bar{\rho}(c_\alpha,c_{\alpha'},T)|y_f\rangle=\int_{c_\alpha}\delta x\int_{c_{\alpha'}}\delta y\exp(i\{S_{\rm free}[x(t)]-S_{\rm free}[y(t)]+W[x(t),y(t)]\}/\hbar)\bar{\rho}(x_0,y_0)\;. \tag{9.7}$$

The path integrals defining $\bar{\rho}(c_\alpha,c_{\alpha'},T)$ are the same as those defining the decoherence functional (9.5) except for the trace over final end points of the paths. Thus

$$D(c_\alpha,c_{\alpha'})=\mathrm{tr}[\bar{\rho}(c_\alpha,c_{\alpha'},T)]\;, \tag{9.8}$$

where tr denotes a final sum over $x_f=y_f$. When the classes $c_\alpha,c_{\alpha'}$ are of paths which are excluded from a region of spacetime, it is possible to write a master equation [33–35] for the evolution of $\bar{\rho}(c_\alpha,c_{\alpha'},t)$ involving the appropriate excluding potential and by solving this equation with boundary conditions appropriate for the classes to calculate $D(c_\alpha,c_{\alpha'})$ through (9.8). Furthermore, the bilinearity of the decoherence functional allows all of its elements for a coarse graining based on spacetime regions to be expressed as a linear combination of such integrals over excluded regions as in (9.2).

The differential equation for $\bar{\rho}$ may be solved analytically for the spacetime coarse graining of the model described in Sec. VI A and discussed above by the method of images. The elements of the decoherence functional can be reduced to quadratures of these solutions and the initial wave function $\psi(X_0)$. However, it is not especially instructive to exhibit these analytic expressions. The important point is the effect of the interaction of the distinguished oscillator with the rest on its classical equation of motion. That interaction leads to effective dissipative terms in the equation of motion [36] characterized by a dissipative time scale $1/\gamma$ read off of the real part of (9.6). This may be compared with the time scale $t_{\rm decoh}$ for the decoherence of position alternatives at one moment of time that differ by a characteristic distance d on opposite sides of the decoherence functional. This time scale may be read of the imaginary part of (9.6) and is [37]

$$t_{\rm decoh}\sim\frac{1}{\gamma}\left[\frac{\hbar^2}{2MkT_B}\frac{1}{d^2}\right]\;. \tag{9.9}$$

When M is of order of grams, d of order of centimeters, and T_B of order of K, t_{decoh} is enormously smaller than $1/\gamma$. There is thus a regime

$$t_{\mathrm{decoh}} \ll T \ll 1/\gamma \ , \tag{9.10}$$

where the interactions with the rest of the oscillators effect rapid decoherence of alternatives at definite moments of time spaced by time intervals greater than t_{decoh}, but the effect of these interactions on the classical equations of motion is negligible. The arguments given above for the decoherence or lack of it of the model spacetime coarse grainings with a single wave packet state are therefore essentially unchanged in this regime by the interactions of the distinguished oscillator with the rest.

While this discussion of decoherence has concerned only special model spacetime coarse grainings and special initial states, these examples, serve to emphasize a familiar lesson: Sufficient and particular coarse graining are required for a quantum-mechanical system to decohere and exhibit classical behavior. Even though a coarse graining partitions the classical paths, it may be too fine to decohere quantum mechanically. Consider, for example, the Earth's orbit around the Sun and the alternatives that in an elapsed time of 1 month its center of mass (1) has never moved more than 10^6 km from its present position or (2) has at least once had an excursion more than 10^6 km from its starting position. Classically, the earth moves about 10^8 km in 1 month, and so the classical probability would be zero for alternative (1) and unity for alternative (2). Yet, extrapolating from the simple example discussed in this section, we do not expect these alternatives to even decohere quantum mechanically despite the very large, "macroscopic" scales of both space and time that are involved. The point is that the requirement of alternative (1) that the center of mass of the Earth *never* quantum-mechanically suffer an excursion of more than 10^6 km from its present position is too strong for decoherence. The classical motion of the Earth is more than adequately described by a coarse graining that distinguishes alternative positions of the Earth's center of mass, say, once every millisecond to a macroscopic accuracy. Such coarse grainings will decohere, and their probabilities will exhibit the expected classical correlations in time.

In quantum mechanics a system can be said to behave quasiclassically when histories exhibiting patterns of classical correlation in time have a high probability in a decohering set of alternative coarse-grained histories. The present discussion shows that the coarse grainings used to define classical behavior must be chosen with care.

B. Causality

We are working with a formulation of quantum mechanics which is not time neutral. At one end of the histories in the decoherence functional (3.2) or (3.8), there is the initial density matrix ρ. At the other end there is the δ function enforcing the coincidence of the end points of the histories at the time T. The same asymmetry may be seen in the operator form of the decoherence functional. Consider, for example, the decoherence functional for a set of histories defined by the alternatives $\alpha=(\alpha_1,\ldots,\alpha_n)$ at a sequence of times $t_1,\ldots,t_n$ whose class operators are given by (4.15). Then

$$\begin{aligned} D(c_\alpha,c_{\alpha'}) &= \mathrm{Tr}(C_\alpha \rho C^\dagger_{\alpha'}) \\ &= \mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha'_1}(t_1)\cdots P^n_{\alpha'_n}(t_n)] \ . \end{aligned} \tag{9.11}$$

At one end of the sequence of projections, there is the density matrix ρ; at the other end is the trace, and the projection operators are time ordered in between. In either the sum-over-histories or operator versions, an arrow of time has been built into this formulation of quantum mechanics. An absolute direction of time is not singled out; the expressions for the decoherence functional would be rewritten in the opposite time order by making use of the *CPT* invariance of field theory. It is by convention that we call the direction with the density matrix "the past." This convention, however, should not obscure the fact that the formalism treats the ends of histories in two different ways [38].

The arrow of time built into quantum mechanics is an expression of causality. We know something of the past; we are ignorant of the future. This connection to causality is most easily seen by using a generalized quantum mechanics employing both initial and final conditions represented by density matrices ρ_0 and ρ_f, respectively [40]. In this generalization the decoherence functional would be written

$$\begin{aligned} D(c_\alpha,c_{\alpha'}) &= N\int_{c_\alpha}\delta X\int_{c_{\alpha'}}\delta X'\,\rho_f(X'_f,X_f)\exp(i\{S[X(t)]-S[X'(t)]\}/\hbar)\rho_0(X_0,X'_0) \\ &= N\,\mathrm{Tr}(\rho_f C_\alpha \rho_0 C^\dagger_{\alpha'}) \ , \end{aligned} \tag{9.12}$$

where $N^{-1}=\mathrm{Tr}(\rho_0\rho_f)$. Interchanging ρ_0 and ρ_f merely Hermitian conjugates the decoherence functional, leaving decoherence conditions and probabilities unchanged. There is thus no built-in arrow of time in this formulation of quantum mechanics—the future and the past are treated in the same way. A physical time asymmetry will emerge if ρ_0 and ρ_f are different. The condition of future indifference, $\rho_f \propto I$, will reproduce the usual formulation of quantum mechanics and its arrow of time. The asymmetry produced, however, is best viewed as the asymmetry between specific initial and final conditions in a framework of quantum mechanics that itself treats future and past on an equal footing.

As has been stressed by Sorkin [5], just using a final condition on the histories representing indifference is not a complete expression of causality. That condition is imposed at a fixed future time T, and one must also check that the predicted probabilities do not depend on the value of T, provided it is chosen sufficiently late. Independence of T is manifest in the operator form of the decoherence functional for sequences of alternatives at definite moments of time [Eq. (9.11)], because that expression nowhere depends on T. Independence of T is also easy to check for the more general spacetime coarse grainings [29,5]. Consider a coarse graining by spacetime regions of $M=[0,T]\times\mathbb{R}^\nu$. This can be considered a coarse graining on $\tilde{M}=[0,\tilde{T}]\times\mathbb{R}^\nu$ built from a longer time interval $\tilde{T}>T$. The one added spacetime region $[T,\tilde{T}]\times\mathbb{R}^\nu$ results in a trivial coarse graining because all paths pass through it. It is therefore easily verified that the class operators $\tilde{C}_\alpha$ for the coarse-grained histories on the extended time interval are related to the corresponding ones on the old interval by

$$\tilde{C}_\alpha=e^{-iH(\tilde{T}-T)}C_\alpha \tag{9.13}$$

[cf. (5.15)]. But then, from the cyclic property of the trace,

$$\tilde{D}(c_\alpha,c_{\alpha'})=\mathrm{Tr}(\tilde{C}_\alpha\rho\tilde{C}^\dagger_{\alpha'})=\mathrm{Tr}(C_\alpha\rho C^\dagger_{\alpha'})=D(c_\alpha,c_{\alpha'})\ , \tag{9.14}$$

and so decoherence and probabilities are independent of any T that is chosen sufficiently late.

Sorkin [5] has discussed a sort of causality violation that could occur if one were attempt to assign probabilities to coarse-grained sets of histories that do not decohere. That violation results from the inconsistency of the probability sum rules which characterize the absence of decoherence. Consider partitioning the paths by two *sets* of spacetime regions $\mathcal{R}_2$ and $\mathcal{R}_1$, where every region which is a member of $\mathcal{R}_2$ lies to the *future* of each region of $\mathcal{R}_1$. Denote by $\{\alpha_1\}$ the set of alternatives arising from the partition by $\mathcal{R}_1$ and by $\{\alpha_2\}$ the set of alternatives arising from $\mathcal{R}_2$. Let probabilities be assigned to histories by the diagonal elements of the decoherence functional, irrespective of whether the histories decohere. The probability sum rules

$$\sum_{\alpha_2}p(\alpha_2,\alpha_1)=p(\alpha_1) \tag{9.15}$$

would not in general be satisfied, and Sorkin [5] provides a specific example when they are not. Such examples cannot be constructed from cases where $\{\alpha_2\}$ refers to alternatives at single latest moment alone, for then (9.15) *is* satisfied, as a glance at (9.11) will show, because of the cyclic property of the trace and the orthogonality of the projection operators corresponding to different alternatives. However, the class operators for spacetime coarse grainings are not projections and they are not orthogonal [cf. the discussion following Eq. (4.15)]. Thus, were the diagonal elements of the decoherence functional interpreted as the probabilities of individual histories when the set of alternative histories does not decohere, violations of (9.15) could be constructed using the kind of spacetime coarse graining that we have been discussing.

Sorkin would assign the probability $\sum_{\alpha_2}p(\alpha_2,\alpha_1)$ to the alternatives $\{\alpha_1\}$ when both sets of alternatives $\{\alpha_2\}$ and $\{\alpha_1\}$ have been "measured" and $p(\alpha_1)$ to the same set of alternatives when only the set $\{\alpha_1\}$ had been "measured." He, therefore, interprets a violation of (9.15) as posing a dilemma: *either* there is a violation of causality—later "measurements" have influenced the probabilities of the outcomes of earlier ones—*or* ideal measurements of the $\{\alpha_2\}$ are not physically realizable.

However, in the formulation of quantum mechanics used in this paper, probabilities can only be assigned to sets of histories that decohere, that is, to exactly those sets of histories for which the probability sum rules such as (9.15) *are* satisfied. A violation of causality by the failure of the probability sum rules thus cannot occur. The failure of a sum rule such as (9.15) signals the absence of decoherence, an inconsistent set of probabilities, and a violation of the theory's rule for assigning probabilities, not a violation of causality [41]. Further, in this formulation of quantum mechanics, probabilities can be assigned to the alternatives $\{\alpha_2\}$ if they decohere whether or not they describe a measurement situation. There is thus no issue of whether they are "physically realizable" either.

Decoherence is a property of coarse-grained sets of histories, and future fine grainings can result in the loss of decoherence as in experiments in which interference between two previously decohering alternatives is recovered. In this sense actions taken today can influence which alternatives in the past can be assigned probabilities by the theory, but one would not interpret this as a violation of causality [42].

C. Measurement

Measurement is not a fundamental notion in the post-Everett formulation of the quantum mechanics of closed systems. Probabilities can be assigned to sets of alternative coarse-grained histories that decohere whether or not these histories describe measurement situations. However, measurement situations can certainly be described within the post-Everett framework as special types of sets of decoherent histories and idealized models of measurement situations can be constructed.

A decoherent set of histories exhibits a measurement situation when there exists a nearly full correlation between range of values of some quantity and another quantity that is part of quasiclassical domain of familiar experience [9,11]. Then, from a knowledge of the value of the quasiclassical quantity, the value of the other may be inferred. We use the term "measurement situation" rather than "measurement" for such correlations to stress that nothing as sophisticated as an "observer" need be present for them to exist.

Idealized models of measurement situations in quantum mechanics have been widely discussed [43]. Typically, these consider an idealized closed system which consists of two parts: a subsystem to be studied and the rest which may be organized into various types of "measuring apparatus" or "observers." Corresponding to this division, the Hilbert space is assumed to be a tensor product $\mathcal{H}_s \otimes \mathcal{H}_r$ of a Hilbert space $\mathcal{H}_s$ for the system and a Hilbert space $\mathcal{H}_r$ for the rest. The "initial condition" for the closed system is likewise assumed to be a tensor product

$$\rho = \rho_s \otimes \rho_r \ . \tag{9.16}$$

An interaction is assumed which couples the subsystem and the rest only at a discrete sequence of times $t_1, \ldots, t_n$. The result of the interaction is assumed to be an exact correlation between each alternative $S^k_{\alpha_k}(t_k)$ of a set k of "measured" alternatives for the subsystem at time t_k and the values of "records" of these measurements $R^{(k,t_k)}_{\beta_k}(t)$ defined on the rest. These records are assumed to persist. That is, an exact correlation is assumed to hold between a particular α_k at time t_k and the appropriate β_k for all times t subsequent to the "time of measurement t_k."

The set of histories of "measured" alternatives of the subsystem can be seen to decohere as a consequence of the existence of persistent records of their outcomes. To see this denote by $[S_\alpha]$ the individual coarse-grained history corresponding to a particular sequence of measurement outcomes $\alpha = (\alpha_1, \ldots, \alpha_n)$. The decoherence functional for the set is

$$D([S_\alpha],[S_{\alpha'}]) = \mathrm{Tr}(S^n_{\alpha_n}(t_n) \cdots S^1_{\alpha_1}(t_1)\rho S^1_{\alpha'_1}(t_1) \cdots S^n_{\alpha'_n}(t_n)) \tag{9.17}$$

[cf. (4.6) and (4.15)]. This decoherence functional is unaffected by the insertion under the trace of partitions of unity of the form

$$\sum_{\beta_k} R^{(k,t_k)}_{\beta_k}(T) = 1 \ , \tag{9.18}$$

at a time T such that $T > t_n > \cdots > t_1 > 0$. However, only one term in each of these sums survives because the R's are exactly correlated with the S's. The R's are commuting projection operators at a final time. They, therefore, decohere because of the cyclic property of the trace in the decoherence functional. This decoherence of the correlated records thus accomplishes the decoherence of the measured alternatives. The "off-diagonal" terms in (9.17) vanish as a consequence of the assumptions of the model.

A further assumption usually made is that the interaction does not disturb the values of the "measured" quantities. In the present language this is the assumption that for the diagonal elements of the decoherence functional the projections $S^k_{\alpha_k}(t_k)$ may be replaced by projections $s^k_{\alpha_k}(t_k)$ acting only on $\mathcal{H}_s$ and evolved by the Hamiltonian of the system alone. Then

$$p([S_\alpha]) \equiv D([S_\alpha],[S_\alpha]) = \mathrm{tr}[s^n_{\alpha_n}(t_n) \cdots s^1_{\alpha_1}(t_1)\rho_s s^1_{\alpha_1}(t_1) \cdots s^n_{\alpha_n}(t_n)] \ , \tag{9.19}$$

where tr denotes the trace on $\mathcal{H}_s$. With this special assumption on the nature of the measurement interactions, the probabilities for the outcomes of a sequence of measured alternatives is expressed entirely in terms of quantities referring only to the subsystem. There is unitary evolution in between measurements expressed by the Heisenberg equations of motion of the $s^k_\alpha(t)$ and "reduction" at a measurement expressed by the action of the appropriate projection.

This is how the usual ideal measurement model would be discussed in post-Everett quantum mechanics. We do not expect the idealizations of the model to hold *exactly* in realistic measurement situations. We cannot expect *exact* correlation of measured values and their records, and certainly records often persist only imperfectly. Especially, we do not expect typical interactions of subsystems producing measurement situations to occur over arbitrarily short time scales or to leave the values of "measured" quantities undisturbed. Indeed, it is known [44,45] that only for very special quantities $S^k_{\alpha_k}$ that commute with all additive, conserved quantities could the assumptions of the model be exactly satisfied even given *arbitrary* lattitude in the form of interaction Hamiltonian and the initial ρ_r. For such reasons ideal measurements cannot have a fundamental status in the formulation of quantum mechanics. The value of the ideal measurement model lies rather in its role as a schema for the approximations that represent realistic measurement situations.

The question naturally arises as to whether the alternatives in a coarse graining defined by spacetime regions can be "measured" in a way similar to alternatives defined at single moments of time. More specifically, there is the question of whether the alternatives in a spacetime coarse graining of an isolated subsystem, not themselves decohering, can be made to decohere by coupling the subsystem to a larger system in such a way that the values of the "measured" quantities are not disturbed. We shall now show that this cannot be done in the sense that the corresponding ideal measurement model is inconsistent for general spacetime coarse grainings.

Consider a closed system divided into a subsystem to be measured corresponding to a Hilbert space $\mathcal{H}_s$ and the rest as described earlier in this section. Consider initial conditions of the tensor product form (9.16). Consider a coarse graining of the histories of the subsystem between $t=0$ and T that divides them into exhaustive and exclusive classes c_α, which have corresponding class operators $\hat{c}_\alpha$ acting on $\mathcal{H}_s$. We assume that these become correlated with records in the larger system at the latest time T in the coarse graining. (Elementary considerations of causality show that it could not be earlier.) Denoting the projections corresponding to alternative values of the records by $R_\beta(T)$, this means

$$\mathrm{Tr}[R_\beta(T)C_\alpha\rho C^\dagger_{\alpha'}R_{\beta'}(T)] \propto \delta_{\alpha\beta}\delta_{\alpha'\beta'} , \tag{9.20}$$

where

$$C_\alpha=\hat{c}_\alpha\otimes I_r . \tag{9.21}$$

Since $\sum_\beta R_\beta(T)=1$, Eqs. (9.16), (9.20), and (9.21) imply

$$D(c_\alpha,c_{\alpha'})=\mathrm{Tr}(C_\alpha\rho C^\dagger_{\alpha'})=\delta_{\alpha\alpha'}\mathrm{tr}(\hat{c}_\alpha\rho_s\hat{c}^{\,\dagger}_\alpha) . \tag{9.22}$$

However, this equation is inconsistent in general cases. The numbers $\mathrm{tr}(\hat{c}_\alpha\rho_s\hat{c}_\alpha)$ are not probabilities. They do not sum to unity unless either (1) the coarse-grained histories of the subsystem already decohere or (2) the class operators satisfy $\sum_\alpha \hat{c}^{\,\dagger}_\alpha\hat{c}_\alpha=1$ as they do for sequences of alternatives at definite moments of time [cf. (4.17)] or for the coarse grainings defining momentum discussed in Sec. VII B. Thus, except for special cases such as (1) and (2), spacetime coarse grainings do not give rise to a natural notion of the probability of an individual history in a set of *nondecohering* histories of a subsystem. There is no natural analogue of the right-hand side of (9.19). There is, therefore, no natural notion of a measurement of spacetime coarse-grained alternatives for a nondecohering subsystem that leaves the values of measured quantities undisturbed because there is no probability distribution of "values" in the subsystem to be left undisturbed.

The physical reason for this situation lies in the fact that the conditions determining a coarse graining based on spacetime regions are extended over time. An interaction with a larger system that is to detect such classes of paths must necessarily act over the corresponding extended period of time. It is difficult to imagine interactions which could do this and, at the same time, leave the evolution of the subsystem undisturbed in all respects. Thus, for example, it is quite possible to imagine a detector located at the origin of the axis of motion of a particle in one dimension which detects whether a particle crosses the origin or does not during an extended interval of time. Such a detector would register the kind of spacetime coarse graining discussed in Sec. VI A. However, we can expect the interaction by which the particle is detected to play a non-negligible role in the quantum dynamics in that interval and in the calculation of the decoherence functional. In general, the dynamics of the entire experimental situation must be taken into account when calculating the probabilities of sets of alternative histories coarse-grained by spacetime regions.

The absence of an ideal measurement model for general spacetime coarse grainings does not mean that the probabilities for such coarse grainings are somehow inaccessible or not useful. Indeed, we shall argue below that spacetime coarse grainings may supply more realistic models of typical measurement situations. The absence of an ideal measurement model merely means that one of its idealizations, probabilities of individual histories in a set of nondecohering histories of a *subsystem,* is too strong in the most general cases.

The possibility of probing the quantum dynamics of isolated subsystems by "measuring" alternatives defined at one moment of time may be one reason for the focus on such alternatives in laboratory science. However, as the analysis of the preceding sections shows, a generalized nonrelativistic quantum mechanics in which "measurement" is not a fundamental notion makes predictions about a more general type of spacetime coarse-grained alternatives even when these are not participants in any sort of "measurement" situation. Coarse grainings of closed systems which distinguish alternatives at one moment of time presume the existence within the system of a clock to measure that time. When spacetime is itself a quantum variable, there may not exist, especially in the early universe, variables of any kind which would be interpreted as clocks [46]. Then, as argued in Sec. VIII, coarse grainings analogous to the more general spacetime coarse grainings discussed in this paper may be of more use and interest than those defined by alternatives at particular moments of time.

D. Utility: measurements and clocks

Is the generalization of quantum mechanics that is concerned with spacetime alternatives necessary? Do these alternatives enable us to describe more accurately the realistic alternatives with which we deal, or are those adequately described by alternatives at definite moments of time? We have mentioned several times the expected utility of the analogues of spacetime coarse grainings in quantum theories of spacetime where there is no well-defined notion of time. However, we shall now argue that, even in the nonrelativistic quantum mechanics of a closed system, spacetime coarse grainings may give a more accurate description of realistic measurement situations than is provided by alternatives at a precise moment of time.

Conventional Copenhagen formulations of quantum mechanics employ the fiction that "measurements" occur at definite, precise moments of time. Realistic measuring apparatus, of course, interacts with a measured subsystem over a finite interval of time. However, since Copenhagen formulations are concerned with the probabilities of alternatives of measured *subsystems,* it is always possible to describe the action of measurement apparatus more realistically by imagining it and the system measured are together a subsystem of an even larger system. The combined system is assumed to be probed by measurements in the larger system which *do* occur at precise moments of time. By this device it can be verified in suitable cases that outcomes registered by the realistic apparatus are distributed with probabilities that approximately coincide with those obtained by assuming that the measurement took place at a definite moment of time. Put more physically, in a theory of subsystems it is possible to posit the existence of an arbitrarily precise, external clock that times measurements at definite moments of time. It is somewhat unsettling that in many situations of interest no such clocks exist. Further, many typical measurement situations do not involve very precise determinations of time. Certainly, human cognition occurs on time scales which are long compared to the atomic time scales historically of interest in quantum-mechanical prediction. However, except for precision in the basic formulation, nothing seems lost by these types of idealizations in familiar cases of laboratory measurement situations.

In quantum-mechanical theories of closed systems, however, there is no room to posit external clocks or larger systems which perform measurements at precise moments of time. Measurement situations must be described realistically including, in particular, the finite time over which they take place, and this leads us naturally to consider spacetime coarse grainings. A measurement that localizes a particle to a position interval Δ over a time δt may be more accurately described by alternatives defined by the associated spacetime region than any one precise moment of time. Measurement situations involving the use of a mechanical clock inside the closed system to determine time may be more accurately described by alternative correlations between the clock variables and measured variables at unspecified moments of time than by alternatives at some particular moment of time.

X. CONCLUSION: HAMILTONIAN AND SUM-OVER-HISTORIES QUANTUM MECHANICS

Conventional, nonrelativistic Hamiltonian quantum mechanics is concerned with the probabilities of sequences of "measured" alternatives of a subsystem defined at precise moments of a preferred time. These probabilities may be calculated making use of the concept of the "state of the subsystem at a moment of time." The state evolves unitarily between "measurements" and by the reduction of the wave packet at them. In the quantum mechanics of closed nonrelativistic systems, it is also possible to introduce a notion of state of the system at a moment of time [9] provided attention is restricted to *decoherent* histories defined by sequences of alternatives at *definite moments of time.* This state summarizes present data for the calculation of future probabilities. For example, in a decoherent set of histories defined by sets of alternatives $(\alpha_1, \ldots, \alpha_n)$ at times $t_1, \ldots, t_n$ the conditional probability that $\alpha_{k+1}, \ldots, \alpha_n$ happen in the future given that $\alpha_1, \ldots, \alpha_k$ have already happened may be written

$$p(\alpha_n t_n, \ldots, \alpha_{k+1} t_{k+1} | \alpha_k t_k, \ldots, \alpha_1 t_1) = \mathrm{Tr}[C_{\alpha_n \cdots \alpha_k} \rho_{\mathrm{eff}}(t_k) C^\dagger_{\alpha_n \cdots \alpha_k}] , \quad (10.1)$$

where

$$\rho_{\mathrm{eff}}(t_k) = \frac{C_{\alpha_k \cdots \alpha_1} \rho C^\dagger_{\alpha_k \cdots \alpha_1}}{\mathrm{Tr}(C_{\alpha_k \cdots \alpha_1} \rho C^\dagger_{\alpha_k \cdots \alpha_1})} , \quad (10.2)$$

and $C_{\alpha_k \cdots \alpha_1}$ denotes the chain of projections at the individual times, as in Eq. (4.15).

As observers of the universe, we are interested in the conditional probabilities for events in the future given data that we know. As we acquire new data, new conditional probabilities become relevant. Suppose we make predictions of the future at a sequence of times $t_k, t_{k+1}, t_{k+2}, \ldots$. Different ρ_{eff} summarize the available data at each of these times. It might be loosely said that there is one $\rho_{\mathrm{eff}}(t_k)$ that evolves "unitarily" between these times (constant in this Heisenberg picture) and is "reduced" as projections are added to the ends of the chains (10.2). That "reduction," however, does not correspond to a physical process. It just means that, as new data are acquired, we choose to focus on a new set of conditional probabilities and a new ρ_{eff} is needed to summarize the new data on which they are conditioned.

However, when, as in this paper, spacetime coarse grainings are considered in which histories are partitioned by their behavior over extended time intervals, it is no longer possible to introduce a notion of "state at a moment of time" that evolves either by unitary evolution or reduction for times in these intervals. Sum-over-histories quantum mechanics, however, predicts probabilities for these spacetime coarse grainings even though the process of prediction cannot be organized in terms of states, their unitary evolution, and reduction.

As mentioned in the Introduction, the sum-over-histories quantum mechanics of nonrelativistic systems described in this paper is both more and less general than usual formulations. It is less general because it is a sum-over-histories formulation that starts from a unique set of fine-grained histories—paths in configuration space. The only alternatives at one moment of time are those defined in terms of the coordinates of that configuration space. The alternatives corresponding to other Hermitian operators can only be represented approximately by devices such as described for momentum in Sec. VII B. It is an open question whether such a sum-over-histories formulation is adequate for physics. However, it is likely that the present discussion of spacetime coarse grainings based on configuration space can be generalized in three ways. First, phase-space path integrals can be used to define partitions of phase space paths as in [47] for alternatives not just restricted to one moment of time. Second, partitions by the values of functionals of paths can be used to define more general classes of coarse grainings [47,48]. Third, it is possible that the Trotter product formula (5.21) can be used with time-dependent families of projections to define even yet more general sets of alternatives. We shall not, however, explore these generalizations here.

In another respect the nonrelativistic sum-over-histories quantum mechanics considered in this paper is more general than the usual Hamiltonian quantum mechanics. It assigns probabilities to more general classes of alternatives—alternatives defined by the behavior of histories with respect to spacetime regions and not just by their behavior at distinct moments of time. With this generalization the sum-over-histories formulation of nonrelativistic quantum mechanics may be said to be fully in spacetime form. Spacetime sums over histories are used to compute the amplitudes for spacetime partitions of those histories. A preferred time is thus no longer prerequisite for defining the alternatives to which the theory attaches probabilities. Such generalizations may be useful in considering realistic descriptions of measurements situations. Their analogues may be essential in constructing a generally covariant quantum mechanics of spacetime which does not single out a preferred set of spacelike surfaces.

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[1] For some examples of recent work on coordinate-invariant formulations of Hamiltonian quantum mechanics in the context of quantum gravity, see K. Kuchař, J. Math. Phys. **13**, 768 (1972); A. Ashtekar, *Lectures on Nonperturbative Canonical Gravity* (World Scientific, Singapore, 1991); C. Rovelli, Phys. Rev. D **43**, 442 (1991); S. Carlip, *ibid.* **42**, 2644 (1990).

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[3] This has been advocated in various ways by C. Teitelboim, Phys. Rev. D **25**, 3159 (1983); **28**, 297 (1983); **28**, 310 (1983); R. Sorkin, in *History of Modern Gauge Theories,* Proceedings of the Conference, Logan, Utah, 1987, edited by M. Dresden and A. Rosenblum (Plenum, New York, 1989); J. B. Hartle, in *Gravitation in Astrophysics (Cargese 1986),* Proceedings of the NATO Advanced Study Institute, Cargese, France, 1986, edited by J. B. Hartle and B. Carter, NATO ASI, Series B: Physics, Vol. 156 (Plenum, New York, 1986); Phys. Rev. D **37**, 2818 (1988); *ibid.* **43**, 1434(E) (1991); **38**, 2985 (1988); and in *Quantum Cosmology and Baby Universes,* Proceedings of the 7th Jerusalem Winter School, Jerusalem, Israel, 1989, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991).

[4] Path integrals allow a compact expression of the standard rules of quantum mechanics for such probabilities. See, e.g., C. Caves, Phys. Rev. D **33**, 1643 (1986); **35**, 1815 (1987); J. Stachel, in *From Quarks to Quasars,* edited by R. G. Colodny (University of Pittsburgh, Press, Pittsburgh, 1986), p. 331ff.

[5] R. Sorkin, in *Conceptual Problems of Quantum Gravity,* edited by A. Ashtekar and J. Stachel (Birkhauser, Boston, 1991). See also, R. Sorkin, in Ref. [3]; S. Sinha and R. Sorkin (unpublished).

[6] N. Yamada and S. Takagi, Prog. Theor. Phys. **85**, 985 (1991).

[7] J. B. Hartle, Phys. Rev. D **37**, 2818 (1988); **43**, 1434(E) (1991).

[8] For an exposition of the formulation of quantum mechanics we shall employ, see Refs. [9] and [10]. For generalized quantum mechanics, see Refs. [10] and [11].

[9] M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy, and the Physics of Information,* edited by W. Zurek, SFI Studies in the Science of Complexity, Vol. VIII (Addison-Wesley, Reading, MA, 1990); in *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology,* edited by S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura (Physical Society of Japan, Tokyo, 1990).

[10] Hartle, in *Quantum Cosmology and Baby Universes* [3].

[11] M. Gell-Mann and J. B. Hartle, in *Proceedings of the XXVth International Conference on High Energy Physics,* Singapore, 1990, edited by K. K. Phua and Y. Yamaguchi (World Scientific, Singapore, 1991).

[12] For classic discussions of the "problem of time," see J. A. Wheeler, in *Problemi dei fondamenti della fisica,* Scuola internazionale di fisica "Enrico Fermi," Corso 52, edited by G. Toraldo di Francia (North-Holland, Amsterdam, 1979); K. Kuchař, in *Quantum Gravity 2,* edited by C. Isham, R. Penrose, and D. Sciama (Clarendon, Oxford, 1981). For more recent treatments see the articles in *Conceptual Problems of Quantum Gravity,* edited by A. Ashteckar and J. Stachel (Birkhauser, Boston, 1991).

[13] For a review of this program, see Ref. [10].

[14] See, e.g., Refs. [10] and [11] for a more complete discussion. There and here the author has referred to this framework as "generalized" quantum mechanics to avoid any possible confusion that it *necessarily* coincides with Hamiltonian quantum mechanics, which is one way of implementing the general principles discussed in this section, but not the only one.

[15] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984).

[16] In realistic situations involving single systems, two probabilities are indistinguishable if they are closer together than some standard that depends on the use to which they are put in the process of prediction. That circumstance makes it useful to consider approximate decoherence in which (2.6) is enforced only to a related standard. For simplicity we have restricted attention only to *exact* decoherence in this exposition of general principles. For further discussion of approximate decoherence, see [10], Secs. II.1, II.2, and II.11.

[17] See, e.g., W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1957). We have here a somewhat more general structure consisting of families of sample spaces which have common elements.

[18] Even this is not the most general formulation. For example, one might consider enforcing the conditions (i)–(iii) in Eqs. (2.1)–(2.3) only on decoherent sets. However, a *very* wide class of theories is encompassed by the elements as stated.

[19] See, e.g., Ref. [15], Ref. [10], Sec. IV.2, and M. Gell-Mann and J. B. Hartle, in Proceedings of the First International

Sakharov Conference (unpublished).
[20] R. H. Cameron, J. Math. Phys. **39**, 126 (1960).
[21] For a lucid review, see C. DeWitt-Morette, A. Maheshwari, and B. Nelson, Phys. Rep. **50**, 255 (1979).
[22] E. Nelson, J. Math. Phys. **5**, 332 (1964).
[23] H. F. Trotter, Proc. Am. Math. Soc. **120**, 887 (1959).
[24] See, e.g., B. Simon, *Functional Integration in Quantum Physics* (Academic, New York, 1979).
[25] The author knows of no proof of the general product formula (5.12) at the time of this writing. The mathematical issues in the extension concern the time dependence of $E_R(t)$ and the fact that it is not self-adjoint because its domain is not dense in $\mathcal{H}$. However, the special case of imaginary time and regions independent of time has been proved by Kato in Ref. [26]. A demonstration of (5.12) or an exploration of the limits of its validity is thus an interesting mathematical problem. A mathematically precise demonstration of a formula such as (5.12) must necessarily specify the domains of the operators on both sides of the relation. Since operators such as E_R are improper, in the sense that their domains are not dense in the Hilbert space, such questions must be treated carefully. A precise discussion for imaginary time and regions R independent of time is given by Kato in Ref. [26], but the more general cases of real-time and time-dependent regions are also of considerable interest. The answers to such questions are important, for example, in giving a precise definition of the exponentials in (5.12), but more immediately in determining the boundary conditions under which the differential equations expressing the evolution in (5.13) are satisfied. For example, consider the one-dimensional case to be discussed in Sec. VI in which E_R excludes the half-line $X<0$. Solving (5.13) amounts to solving the Schrödinger equation for a specific wave function $\psi(X)$ on the half line $X>0$. The boundary conditions which fix this solution correspond to the domain on which the operator on the right-hand side of (5.12) is defined. This domain is fixed by the product formula. The corresponding boundary condition is that $\psi(X)$ vanish at $X=0$ and at $X<0$. This follows from Ref. [26], but is plausible because continuity at the origin is a necessary condition for $(H_0+V+iE_R)\psi$ to be included in the Hilbert space $L_2(-\infty,+\infty)$ since H_0 is a differential operator.
[26] T. Kato, in *Topics in Functional Analyses,* edited by I. Gohberg and M. Kac (Academic, New York, 1978).
[27] See, e.g., A. Friedman, *Partial Differential Equations of Parabolic Type* (Prentice-Hall, Englewood Cliffs, NJ, 1968); O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type* (American Mathematical Society, Providence, RI, 1968).
[28] See, e.g., W. Feller, *Introduction to Probability Theory and Its Applications,* 3rd ed. (Wiley, New York, 1979), pp. 354ff.
[29] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[30] See, e.g., E. Joos and H. D. Zeh, Z. Phys. B **59**, 223 (1985); W. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982); Ref. [9].
[31] An exception is the model explored by N. Yamada and S. Takagi, Prog. Theor. Phys. **86**, 599 (1991). That model, however, deals with only a very special class of initial conditions fine-tuned to the coarse grainings considered.
[32] R. P. Feynman and J. R. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963).
[33] A. Caldeira and A. Leggett, Physica A **121**, 587 (1983).
[34] W. Unruh and W. Zurek, Phys. Rev. D **40**, 1071 (1989).
[35] B.-L. Hu, J. P. Paz, and Y. Zhang, University of Maryland report, 1991 (unpublished).
[36] See, e.g., [33] or M. Gell-Mann and J. B. Hartle (unpublished).
[37] W. Zurek, in *Non-Equilibrium Quantum Statistical Physics,* edited by G. Moore and M. Scully (Plenum, New York, 1984).
[38] Reference [39] is a classic discussion of the arrow of time in quantum mechanics. For a more complete discussion in the present context, see Refs. [9] and [19].
[39] Y. Aharonov, P. Bergmann, and J. Lebovitz, Phys. Rev. **134**, B1410 (1964).
[40] Reference [39] and, in the cosmological context, the references in [19].
[41] Of course, if one deals with approximate probabilities and approximate decoherence, then there may be causality violations of the sort Sorkin has discussed, but in interesting cases one expects the magnitude of the violation to be undetectably small. See, e.g., the discussion in Ref. [10], Sec. II.11.
[42] See, e.g., Ref. [10], Secs. II.2.3 and II.3.2.
[43] Some of the classic references are J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932) [English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955)]; F. London and E. Bauer, *La théorie de l'observation en mécanique quantique* (Hermann, Paris, 1939); E. Wigner, Am. J. Phys. **31**, 6 (1963). A useful collection of papers is in *Quantum Theory of Measurement,* edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1983). For the present discussion, see Ref. [10], Sec. II.10.
[44] E. P. Wigner, Z. Phys. **131**, 101 (1952).
[45] H. Araki and M. Yanase, Phys. Rev. **120**, 622 (1960).
[46] See the discussion in J. B. Hartle, Phys. Rev. D **38**, 2985 (1988); and especially in H. Salacker and E. Wigner, Phys. Rev. **109**, 571 (1958).
[47] M. Gell-Mann and J. B. Hartle (unpublished).
[48] J. B. Hartle (unpublished).