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Unruly topologies in two-dimensional quantum gravity

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Abstract. A sum over histories formulation of quantum geometry could involve sums over different topologies as well as sums over different metrics. In classical gravity a geometry is a manifold with a metric, but it is difficult to implement a sum over manifolds in quantum gravity. In this difficulty, motivation is found for including in the sum over histories, geometries defined on more general objects than manifolds—unruly topologies. In simplicial two-dimensional quantum gravity a class of simplicial complexes is found to which the gravitational action can be extended, for which sums over the class are straightforwardly defined, and for which a manifold dominates the sum in the classical limit. The situation in higher dimensions is discussed.

1. Introduction

Classical general relativity is a theory of the geometry of spacetime. The classical theory considers geometries which are manifolds with a metric. It does so because manifolds are the mathematical implementation of the principle of equivalence. That principle tells us that locally spacetime is indistinguishable from a small region of flat space ($\mathbb{R}^4$) and this is the defining characteristic of a manifold.

A quantum theory of gravity should assign amplitudes to all possible geometries. Because classical geometries are manifolds with a metric it is as reasonable to assign amplitudes to different manifolds as it is to different metrics†. By allowing manifolds with different topologies one is led to attractive physical pictures such as those evoked by the words 'spacetime foam' [3].

The sum over histories formulation of quantum mechanics provides a direct and powerful framework on which to construct a quantum theory of geometry. Quantum amplitudes are specified by sums over geometries with different amplitudes being constructed by summing over different classes of geometries. Expectation values in quantum states can be expressed as a sum over geometries. For example, in the quantum mechanics of closed cosmologies, the expectation value of a physical quantity A in the state of minimum excitation could be represented schematically as [4]

$$\langle A\rangle = \frac{\Sigma_{\mathscr{G}} A[\mathscr{G}]\exp(-I[\mathscr{G}])}{\Sigma_{\mathscr{G}}\exp(-I[\mathscr{G}])}. \tag{1.1}$$

Here, I is the Euclidean gravitational action. The sum is over compact, connected, four-geometries, $\mathscr{G}$, without boundary. Among these are the classical geometries—metrics on manifolds.

† For a very different view on the reasonableness of considering different topologies in quantum geometry see [1, 2].

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In the sum over histories formulation of quantum mechanics we are familiar with 'unruly histories' [5]. These are histories which contribute significantly or even dominantly to the sums for quantum amplitudes but are less regular than the classical histories. An example is provided by the quantum mechanics of a single particle. The dominant contribution to the path integral for the particle's propagator comes from paths which are nowhere differentiable. By contrast the classical paths which extremise the action and satisfy the classical equations of motion are everywhere differentiable.

In a sum over geometries such as (1.1) one would expect, by analogy with field and particle quantum mechanics, to sum over unruly metrics. It is also possible that one should sum over unruly topologies—geometries defined on objects more general but less regular than manifolds. This paper explores this question in the context of two-dimensional quantum gravity†.

Some motivation for considering a wider class of geometries than those defined on manifolds can be obtained by considering various possible procedures for concretely implementing a sum over manifolds in an expression like (1.1). Simplicial approximation provides a concrete way to define sums over geometries [6-11]. A simplicial geometry is a collection of simplices together with an assignment of lengths to their edges. A sum over geometries is a sum over collections of simplices together with a multiple integral over their edge lengths. A sum over manifolds is a sum over collections of simplices which are locally topologically equivalent to a region of Euclidean space.

The most direct approach to defining a sum over all manifolds would be to list them, find a simplicial representation of each, evaluate the integral over the edge lengths for each, and add the results up. However, in four and higher dimensions it is not possible to construct a list on which every manifold appears just once because the problem of deciding when two manifolds are topologically equivalent is unsolvable in these dimensions (see, e.g., [12]). Manifolds are not classifiable.

A solution of the classification problem is not necessary to carry out a sum over manifolds. Another approach, discussed more fully in § 2, might be to sum over all collections of simplices which *are* manifolds. One thereby counts some manifolds more than others with a weight defined implicitly by the procedure. However, in five dimensions and higher, and possibly in four dimensions, it is unlikely that there exists an algorithm for deciding whether a given collection of simplices is a manifold or not. Simplicial manifolds are not identifiable from among all simplicial collections.

The above examples show that the problem of summing over manifolds is not an easy one. They do not, however, demonstrate that it is impossible. Other approaches have been proposed [11] and more ingenious ones may yet be found. The existence of a universal algorithm may not even be necessary for physics [13]. The difficulty of the problem of summing over manifolds, however, does motivate an examination of the question of whether the quantum sum over geometries might be better defined on geometries for which a procedure like one of those described above *can* be carried out.

There are classes of geometries which are *smaller* than the collection of all manifolds and which are classifiable [14]. For example, certain simply connected manifolds are likely to be classifiable [15, 16]. As yet, however, there seems to be no compelling physical reason for restricting the geometries contributing to the sum over histories in this way.

† Already in classical gravity the notion of manifold breaks down at spacetime singularities. Typically, these singular points are banished from the manifold to the lower dimensional boundary although they are real enough physically. The idea of spacetime manifold is replaced by spacetime manifold with boundary. Our question is whether even this class should be enlarged.

In this paper we shall pursue a second approach. We shall examine whether the sum over geometries may be reasonably defined on a *larger*, less regular class of geometries than those built on manifolds in such a way that there is a straightforward procedure for carrying out the sum in the simplicial approximation. Including geometries in the sum over histories which are not manifolds means relaxing the principle of equivalence at the quantum level. The larger class would be the unruly topologies of the geometrical sum over histories. To identify such a class one must not only identify a class of geometries but also construct a suitable gravitational action. Together they define the sum over histories. Two conditions must then be satisfied. (1) To have a straightforward procedure, there must be an algorithm for deciding which collections of simplices are members of the class and which are not. (2) To recover the principle of equivalence in classical physics, the action and class must be such that manifolds dominate the sum over histories in the classical limit.

In § 3 we find the largest natural class of geometries which satisfy the above conditions in two-dimensional quantum gravity with the simplest extension of the action. Two-dimensional quantum gravity is metrically trivial. There are neither dynamical degrees of freedom nor field equations. Two-dimensional gravity is not, however, topologically trivial in the sense that the action depends on the topology of the geometry [17]. Topological issues are therefore clearly separated from metrical ones in two dimensions and easily investigated. With the simplest extension of the gravitational action we find that the largest natural class of two-dimensional geometries which are straightforwardly identifiable and which possess a reasonable classical limit are those built on pseudo-manifolds. Roughly, a two-dimensional pseudo-manifold is a collection of triangles which fails to be a manifold at some collection of vertices while being no more disconnected by these failures than a manifold is. Pseudo-manifolds thus are the unruly topologies of two-dimensional quantum gravity.

In higher and more interesting dimensions metric and topology are coupled and the problem of evaluating the classical limit is more complicated. Without reaching definite conclusions we discuss this problem in § 4 and exhibit possible candidates for unruly topologies in higher dimensions.

2. Simplicial approximation, simplicial complexes and simplicial manifolds

Sums over geometries may be given concrete meaning by taking limits of sums over simplicial approximations to them [6-11]. Simplicial geometries are built out of simplices joined together in a specific way together with a metric fixed by an assignment of lengths to their edges and a flat metric to their interiors. The action for gravitation becomes a function of the edge lengths and the way the simplices are joined together. For example, one could give a concrete meaning to the sum over geometries in (1.1) as follows. (1) Fix a total number of vertices n_0. (2) Approximate the sum over topologies by a sum over the different ways, K, of appropriately putting together 4-simplices. (3) Approximate the sum over metrics by a multiple integral over the squared edge lengths s_i. (4) Take the limit of these sums as n_0 becomes large. In short, express $\langle A\rangle$ as

$$\langle A\rangle = \lim_{n_0\to\infty} \frac{\Sigma_K \int_K \mathrm{d}\Sigma_1 A(s_i, K)\exp[-I(s_i, K)]}{\Sigma_K \int_K \mathrm{d}\Sigma_1 \exp[-I(s_i, K)]}. \tag{2.1}$$

A specification of the measure in the space of squared edge lengths, $d\Sigma_1$, and the contour of integration is needed to complete this prescription.

Simplicial complexes provide the widest reasonable framework in which to investigate simplicial geometries. A *simplicial complex* is a finite collection of simplices such that if a simplex of dimension k is in the collection then so are all of its faces of dimension less than k and whenever two simplices intersect they do so in a common face. (From now on we shall omit the qualification 'simplicial' from complexes, manifolds, pseudo-manifolds etc, it being understood that geometries are built out of simplices.) The maximum dimension of a simplex in the complex is the complex's dimension, n. It seems reasonable today to restrict attention to complexes which are of homogeneous dimension in the sense that any simplex of dimension $k<n$ is the face of some n-simplex. We thereby recover the notion of a uniform dimension of spacetime.

For computing expectation values in the state of minimum excitation for closed cosmologies, closed, four-dimensional geometries are appropriate. In other situations other classes of geometries may be of interest. For example, if spacetime is really ten- or eleven-dimensional, geometries of this dimension would be of interest. For a direct sum over histories evaluation of the minimum excitation wavefunction for closed cosmologies, four-geometries with a boundary would be of interest. To obtain concreteness, however, we shall focus on sums like (2.1) over closed geometries but in a general number of dimensions n. We are therefore interested in homogeneously n-dimensional complexes without boundary (closed), i.e. such that there are no $(n-1)$-simplices incident on an odd number of n-simplices†. Without any loss of generality we can further restrict attention to *connected complexes*—those for which there is a sequence of edges connecting any two vertices. For expectation values of quantities concerning a single universe the contributions of disconnected geometries cancel between the numerator and denominator of (2.1).

As described above, to define a sum over geometries through simplicial approximation one can first sum over geometries defined on complexes with a fixed number of vertices n_0 and then consider the limit as n_0 becomes large. Indeed, to the extent that this limit represents a sum over continuum geometries, one suspects that this is the only reasonable procedure. This is because in sums like (2.1) one expects both numerator and denominator to diverge in the limit of large n_0 because there will be many simplicial geometries which approximate a given continuum one. The sums diverge like the curvature scale to the power nn_0 and this divergence corresponds to recovering the diffeomorphism group in the continuum limit (see, e.g., [9, 11]). The divergence should cancel between the numerator and denominator to yield a finite result for physical quantities with a continuum limit. We consider sums over complexes with a fixed number of vertices n_0 because in any approximation which involved several different n_0 the largest n_0 would eventually dominate anyway. Such a sum is finite since there are a finite number of complexes which can be constructed with n_0 vertices.

To define a sum over geometries one must at least be able to identify the complexes which contribute to the sum and define the gravitational action on them. Let us begin by briefly reviewing the situation for the class of complexes which are manifolds. A closed n-manifold is a complex such that there is a neighbourhood of any point which is topologically equivalent (homeomorphic) to an open ball in $\mathbb{R}^n$. This is a strong condition and one which has little to do with the combinatoric properties of a complex.

† We thus follow the classic topological definition of the boundary of a complex. For more precision and discussion of this and the other definitions we have introduced informally, the reader is referred to [18].

There is no difficulty with defining the action for general relativity on a geometry defined on manifolds. The continuum Euclidean action for a closed geometry on a manifold M is

$$g_n l^{n-2} I = -\int_M \mathrm{d}^n x \sqrt{g}(R - 2\Lambda), \tag{2.2}$$

where Λ is a cosmological constant, $l = (16\pi G)^{1/2}$ is the Planck length, g_n is a dimensionless coupling constant, and we use units where $\hbar = c = 1$. On a simplicial manifold (2.2) becomes exactly [19]

$$g_n l^{n-2} I = -2 \sum_{\sigma \in \Sigma_{n-2}} V_{n-2}(\sigma)\theta(\sigma) + 2\Lambda \sum_{\tau \in \Sigma_n} V_n(\tau). \tag{2.3}$$

The first sum is over the $(n-2)$-dimensional simplices Σ_{n-2} on which the curvature is concentrated and the second over the n-simplices. V_k is the volume of a k-simplex and $\theta(\sigma)$ is the deficit angle of an $(n-2)$-simplex σ which measures the curvature there. Specifically

$$\theta(\sigma) = 2\pi - \sum_{\tau \in \Sigma_n} \theta(\sigma, \tau). \tag{2.4}$$

The sum here is over the n-simplices which are incident on the given $(n-2)$-simplex and $\theta(\sigma, \tau)$ is the 'dihedral angle' in the n-simplex τ between the two $(n-1)$-simplices which contain σ. By standard flat space geometrical formulae every constituent quantity in the action may be expressed as a function of the edge lengths of the simplices.

By introducing the notion of the link of a simplex we can formulate the condition for a complex to be a manifold. The *link* of a simplex σ is the complex consisting of those simplices which (1) are faces of simplices which have σ as a face but which (2) do not themselves intersect σ. See figure 1 for an example. A necessary and sufficient condition for a complex to be an n-manifold is that the link of every k-simplex be homeomorphic to a sphere of dimension $n-1-k$ [20].

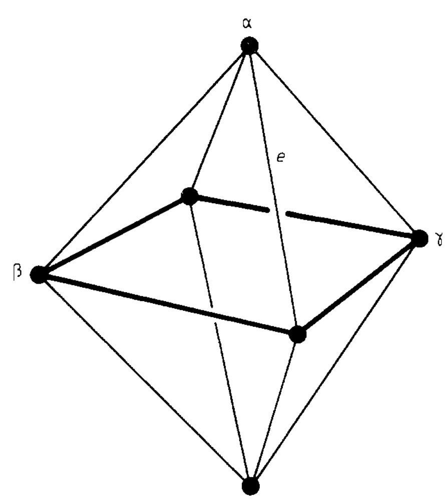

Figure 1. The surface of an octahedron is a simple example of a two-manifold. The link of every edge consists of two points and is thus a 0-sphere (the boundary of a 1-ball). The link of every vertex consists of four connected edges which are topologically a 1-sphere (a circle). For example, the link of edge e consists of the vertices β and γ. The link of vertex α is the heavily drawn quadrilateral.

Complexes with n_0 vertices can be enumerated by listing all collections of n-simplices which can be constructed from n_0 vertices. There are a finite number. The lower-dimensional simplices of these complexes are the faces of their n-simplices. One could weed this list of complexes by enforcing necessary conditions, but, in the end, to check which complexes are manifolds one will have to check that the link of each simplex is topologically a sphere of correct dimension. In particular one must check that the link of each vertex is an $(n-1)$-sphere.

The problem of deciding whether a given manifold is homeomorphic to a k-sphere is a more specific one than the problem of deciding when *any* two given manifolds are homeomorphic. The non-existence, in a given dimension, of a universal algorithm for the latter problem, therefore, does not preclude finding an algorithm for the former. The current mathematical situation on the existence of an algorithm to determine when a given manifold is homeomorphic to a k-sphere appears to be the following. For $k=1$ and $k=2$ there exist algorithms. For $k=3$, Haken [21] has announced the proof of an algorithm, but it is not likely to be easily implementable. For $k \geqslant 4$ it is highly unlikely that an algorithm exists [12] and there are preliminary indications [22] that a proof of this can be supplied for $k \geqslant 5$. Thus, for dimensions $n>5$, and possibly in some useful sense in dimension $n=4$, it appears likely that there does not exist a universal algorithm for deciding when a given complex is a manifold. The inability in a general dimension of deciding whether a given complex is a manifold does not make a definition of a sum over manifolds impossible [11], but it does make such definitions much less straightforward.

Complexes which are not manifolds fail to be so because the link of some simplex is not a sphere of appropriate dimension. Some two-dimensional examples are shown in figures 2 and 3. Complexes which are not manifolds are thus singular at a discrete

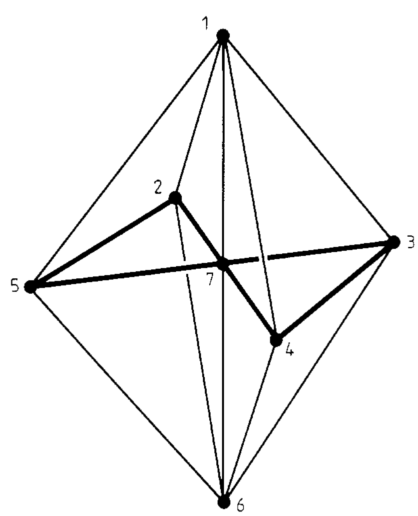

Figure 2. This closed, two-dimensional complex is not a manifold. There are neighbourhoods of points in the edges $(1,7)$ and $(6,7)$ which are not homeomorphic to a disc but rather to a region in the intersection of two planes. Four triangles intersect each of the edges $(1,7)$ and $(6,7)$. The complex thus branches on these edges. Their links consist of four vertices each and are not the 0-sphere. For example, the link of edge $(1,7)$ consists of the vertices 2, 3, 4 and 5. The links of vertices 1, 6 and 7 are not 1-spheres. For example, the link of vertex 1 is the heavily drawn set of edges. It is not topologically a circle nor a set of disconnected circles.

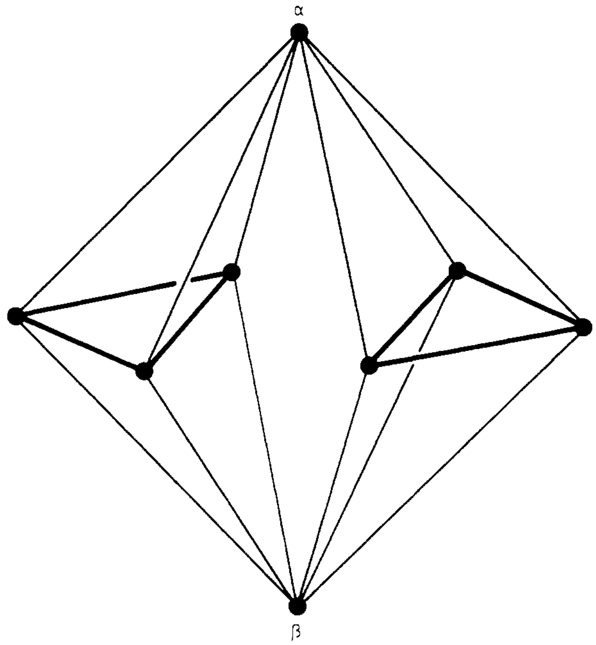

Figure 3. This two-dimensional complex is not a manifold. It is non-branching so that the links of edges are 0-spheres. The link of some vertices, however, does not consist of a single circle but of two. For example, the link of vertex α consists of the heavily drawn edges. The complex is not strongly connected and can be regarded as two almost disconnected spheres joined at the vertices α and β.

number of simplices of dimension less than n. An assignment of edge lengths to such complexes still defines a metric in the sense that the distance along any curve in the complex is then determined. A general complex with an assignment of edge lengths may thus be regarded as a geometry.

In order to investigate whether complexes which are not manifolds can reasonably contribute to the geometric sum over histories a gravitational action must be defined for them. The Regge action (equation (2.3)) extends naturally to all complexes of homogeneous dimension. This is because every $(n-2)$-simplex is contained in *some* number of n-simplices. Equation (2.4) can thus be used to define the deficit angle and equation (2.3) to define the action. One would expect a similar extension to hold for the approximate curvature squared actions [10]. Other extensions are certainly possible and in particular the Regge action can always be augmented by topological invariants of the complex. Such extensions must be consistent with the additive property of the action required by the composition law of quantum mechanics (see, e.g., [8]).

The conditions for a reasonable class of simplicial geometries with which to define the gravitational sum over histories may now be restated more precisely as follows. We seek a class of closed complexes of homogeneous dimension n together with an extension of the gravitational action to them such that

(1) There is an algorithm for deciding whether a given complex is a member of the class or not.

(2) A manifold dominates the sum over histories in the classical limit.

Finding a class of geometries and an extension of the action are coupled problems.

They could be attacked by starting with an extension of the action and attempting to identify a class of geometries satisfying conditions (1) and (2). Alternatively one could start with a class of geometries and try and find an action such that (1) and (2) hold. We shall illustrate both lines of investigation in the subsequent sections.

3. Two dimensions

The curvature part of the Regge action for two-dimensional quantum gravity is independent of the edge lengths of the closed complex on which it is evaluated. This can be seen directly from the defining relations (2.3) and (2.4). Write

$$g_2 I_E = -2 \sum_{\sigma\in\Sigma_0} \theta(\sigma). \tag{3.1}$$

Substitute (2.4) into (3.1), interchange the summations over vertices and triangles, and note that the sum over the interior angles of a triangle is π. There results

$$I_E = -(4\pi/g_2)(n_0 - n_2/2), \tag{3.2}$$

where n_0 is the total number of vertices and n_2 is the total number of triangles of the complex. The curvature part of the action thus depends only on how the triangles are put together to form the complex, not on their edge lengths.

Since there are no metric degrees of freedom in two-dimensional quantum gravity it would be reasonable to consider I_E as the total gravitational action for different topological configurations. Alternatively, one could define the total action by adding to I_E a cosmological term of the form constant $\times$ (total area of the complex). One could then carry out the integral over the edge lengths and be left with a sum over topologies. In either event, we shall assume that the interesting expectation values in two-dimensional Einstein gravity are to be computed in the form

$$\langle A\rangle = \lim_{n_0\to\infty} \frac{\Sigma_K\, \nu(K) A(K) \exp(-I_E[K])}{\Sigma_K\, \nu(K) \exp(-I_E[K])} \tag{3.3}$$

where the sum is over some class of closed, connected, two-dimensional complexes and $\nu(K)$ is an appropriate weight. In this section we shall find a class of closed, connected, two-dimensional complexes which is larger than manifolds and which satisfies the two conditions set forth in § 2: the existence of an algorithm for identifying members of the class and the domination of the sum over histories by manifolds in the classical limit.

The classical limit of a sum over histories like (3.3) is obtained by allowing $\hbar$ to tend to zero keeping other dimensional quantities fixed. Equivalently, in the units we are using it is obtained by allowing g_2 to tend to zero. We shall assume that the classical limit of the sums in (3.4) is entirely determined by the topological part of the action, I_E. That is, we shall assume that when g_2 becomes small the dominant contribution to the sum over histories comes from those complexes with least I_E.

Let us begin with connected, closed, two-dimensional complexes which are manifolds and enlarge this class as naturally as possible until the largest class is found which meets our conditions for the classical limit. For a closed complex to be a manifold every edge must intersect exactly two triangles. Otherwise there would be neighbourhoods of points on the edge which were not homeomorphic to a disc in $\mathbb{R}^2$. Such complexes are called *non-branching*. For non-branching complexes the number of edges n_1 and the number of triangles n_2 are related by

$$3n_2 = 2n_1. \tag{3.4}$$

The Regge action for non-branching complexes (equation (3.2)) can be rewritten as

$$I_E = -4\pi\chi, \tag{3.5}$$

where χ is the *Euler number*

$$\chi = n_0 - n_1 + n_2. \tag{3.6}$$

For non-branching complexes the action is therefore a topological invariant in the sense that it is unchanged by subdivision†. The Euler–Poincaré formula expresses χ in terms of the Betti numbers b_k

$$\chi = b_0 - b_1 + b_2. \tag{3.7}$$

b_k is the number of linearly independent, non-homologous k-cycles of the complex. Connected complexes have $b_0 = 1$. For manifolds $b_2 = 1$ if the manifold is orientable and $b_2 = 0$ if it is not. Thus for manifolds

$$\chi = \begin{cases} 2 - b_1, & \text{orientable,} \\ 1 - b_1, & \text{non-orientable,} \end{cases} \tag{3.8}$$

and so, for all manifolds $\chi \leqslant 2$. The action I_E is thus bounded below by -4π and the manifold of least action is the sphere with $b_1 = 0$, $\chi = 2$.

Complexes which are not manifolds fail to be so on some edges or vertices. A complex will not be a manifold at an edge if the edge does not intersect exactly two triangles‡. Were such branching complexes allowed in the sum over histories however, there would be branching complexes with the same action as any manifold. To see this imagine joining two closed complexes together along an edge. The resulting complex will branch at the edge. Its action, as follows from (3.2), is $I_1 + I_2 + 8\pi/g_2$ where I_1 and I_2 are the actions of the two complexes which are joined. By joining many spheres $(I = -8\pi/g_2)$ to a given manifold in this way we obtain a branching complex with the same action as the original manifold, and a manifold would not give the dominant contribution to the classical limit. We conclude that complexes contributing to the sum over histories must be non-branching. Indeed it would be difficult to see how to proceed otherwise. Unless the complexes are non-branching the Regge action is not invariant under subdivision and thus not a topological invariant.

Complexes which fail to be manifolds only at vertices can be divided into two groups depending on whether they are strongly connected or not. A *strongly connected* two-dimensional complex is one for which every pair of triangles can be connected by a sequence of triangles beginning with the first member of the pair and ending with the second such that successive members of the sequence have a common edge. As the example in figure 3 shows, connected complexes which are not strongly connected can be divided into components which are connected only at vertices. The components are 'almost disconnected'. The link of a vertex at which two or more complexes are joined consists of two or more disconnected pieces and therefore cannot possibly be a sphere. Every connected manifold is therefore strongly connected.

† In the succeeding discussion we shall be quoting a number of elementary results in topology. A general reference for all of them is [18].

‡ This is equivalent to the condition on the link of an edge because a 0-sphere consists of two points.

The action for a complex which consists of components connected only at vertices may be written as a sum over the different components as follows

$$I_E = -\left(\frac{4\pi}{g_2}\right) \sum_i [\chi_i - \tfrac{1}{2} m_i], \tag{3.9}$$

where χ_i is the Euler number of the ith component and m_i is the number of attached vertices. The action can be made arbitrarily negative. For example, a closed chain of N spheres, each joined at two vertices as in figure 4, would have action $-(4\pi/g_2)N$. In order to have the sum over topologies converge, and in order that manifolds dominate the classical limit, this result suggests that the sum over geometries must be restricted to complexes which are strongly connected. For the computation of expectation values of sufficiently local quantities one expects that this restriction is not an essential one. For quantities confined to one component of a complex consisting of almost disconnected components, the contribution of non-strongly connected complexes could, with a suitable choice of weight, cancel between the numerator and denominator of an expression like (3.3).

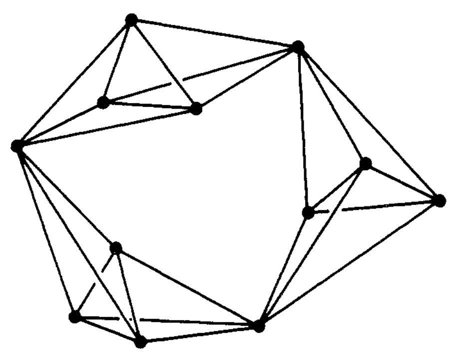

Figure 4. A chain of three almost disconnected spheres connected at two vertices of each sphere. The Euler number of this complex is 3 so that the action is less than that of a sphere.

Through the above two suggestive arguments we have isolated a class of complexes with which to potentially define a two-dimensional sum over topologies. These complexes are closed, compact, connected and are (1) homogeneously two-dimensional, (2) non-branching and (3) strongly connected. These properties are topologically invariant, i.e. they are unchanged by subdivision of the complex [18]. They are defining properties of a two-dimensional *pseudo-manifold*† [18].

Pseudo-manifolds are complexes which fail to be manifolds at a discrete number of vertices. The link of every vertex of a pseudo-manifold consists of some number of disconnected circles. Otherwise the complex would branch on some edge. The link of vertices where a pseudo-manifold fails to be a manifold is a number of disconnected circles greater than one. We may reasonably call these the pseudo-manifold's singular vertices. An example is shown in figure 5. As this example suggests, two-dimensional pseudo-manifolds may be thought of as manifolds in which points have been identified.

† Pseudo-manifolds have also been considered in quantum geometry by R Sorkin (private communication). See also [23].

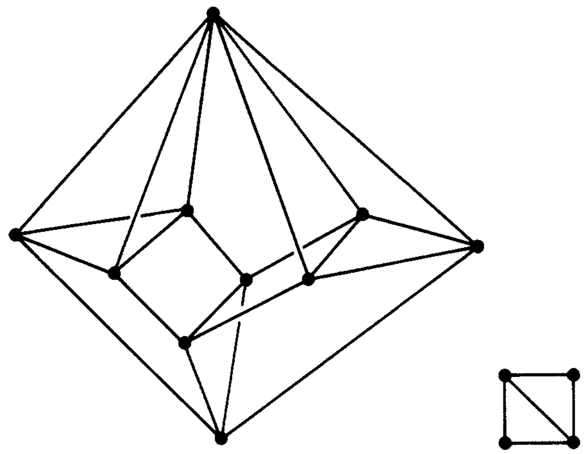

Figure 5. An example of a two-dimensional pseudo-manifold. This complex is non-branching and strongly connected. It fails to be a manifold at the topmost vertex. The complex is topologically equivalent to a sphere with two points identified. Its Euler number is 1 and so its action is greater than that of a sphere.

For pictorial clarity some of the edges triangulating quadrilaterals have been omitted in the figure. They should be imagined to be inserted as in the example at lower right.

The principle of equivalence is not satisfied at the singular vertices of a pseudo-manifold for spacetime cannot be locally flat there. We will now show that these more singular geometries meet the conditions for defining a sum over histories set forth in § 2.

There is a straightforward algorithm for identifying pseudo-manifolds from among all connected, closed, two-dimensional complexes. To list all closed, connected, two-dimensional pseudo-manifolds with n_0 vertices one might proceed as follows. List all collections of triangles which can be constructed from n_0 vertices. There are a finite number. Consider these as complexes with the edges and vertices being those of the triangles. Homogeneity of dimension is therefore already satisfied. Discard all for which an edge is not a face of exactly two triangles. The remaining complexes will be closed and non-branching. Since there are a finite number of edges it takes a finite number of operations to do this. It remains to discard those complexes which are not connected and strongly connected. Since strongly connected complexes are connected we can do this in one step. We first note that if there is a sequence of triangles connecting two vertices then there is a connecting sequence in which each triangle occurs only once. Suppose there was a connecting sequence in which a triangle σ occurred twice. By omitting all triangles after the first occurrence of σ through its second one would obtain a connecting sequence in which σ occurred only once. While there are an infinite number of connecting sequences between two triangles, there are only a finite number of sequences in which a constituent triangle occurs at most once. These sequences can therefore be listed and it can be checked whether a complex is strongly connected in a finite number of steps. If complexes which are not strongly connected are discarded there remain complexes which are connected, homogeneously two-dimensional, closed, non-branching and strongly connected. That is, there remain the closed, connected, two-dimensional pseudo-manifolds.

The two-dimensional pseudo-manifold of least action is a manifold—the sphere. Thus a manifold dominates the sum over pseudo-manifolds in the classical limit. We can demonstrate this result as follows†. Since pseudo-manifolds are non-branching complexes their Regge action is determined by their Euler number through (3.5). Let P be a closed, connected two-dimensional pseudo-manifold with k singular vertices. The link of each of these vertices is some number of circles greater than one. If we delete from P the singular vertices together with the edges connecting them to their links we are left with a complex $\tilde{P}$ which is a manifold. This manifold has a boundary consisting of the circles which were the links of singular vertices. The number of these boundary components, b, is greater than $2k$. $\tilde{P}$ is connected because P is strongly connected. The Euler number of P is related to that of $\tilde{P}$ by

$$\chi(P)=\chi(\tilde{P})+k, \tag{3.10}$$

as follows directly from (3.6) and the above construction. Imagine joining hemispheres to $\tilde{P}$ at each of its boundaries. There results a closed connected manifold $\tilde{P}^{c}$. The Euler number of $\tilde{P}$ can be related to that of $\tilde{P}^{c}$ by noting that the Euler number of a hemisphere is 1 and that (from (3.6)) Euler numbers of manifolds joined across circles add. One has

$$\chi(\tilde{P})=\chi(\tilde{P}^{c})-b. \tag{3.11}$$

Closed, connected, two-dimensional manifolds have Euler numbers which are less than 2, the value for a sphere (see, e.g. (3.8)). Inserting this inequality in (3.11), the result in (3.10), and noting again that $b \geqslant 2k$ one finds

$$\chi(P)\leqslant 2-k. \tag{3.12}$$

The pseudo-manifold of largest Euler number and least action therefore has $k=0$. It is thus a manifold—the sphere.

The above result can be seen more constructively if one knows that each pseudo-manifold is a manifold with some vertices identified. Under this identification the number of vertices decreases while the number of edges and triangles remains unchanged. The Euler number (equation (3.6)) decreases under this identification. For every pseudo-manifold there is thus a manifold with larger Euler number and smaller action.

In two dimensions pseudo-manifolds meet the two conditions for defining a sum over topologies set forth in § 2. They are algorithmically identifiable from among all complexes and the principle of equivalence is recovered in the classical limit. Eliminating any of their defining conditions will allow complexes which violate these conditions. In this sense pseudo-manifolds are the largest class which can be defined naturally, although one can produce larger classes by *ad hoc* restrictions (e.g. considering almost disconnected complexes, each component of which is connected to others at four or more vertices).

Two-dimensional quantum gravity is interesting only in so far as it is suggestive of results in higher dimensions. Indeed, in the two-dimensional case it *is* possible to meet the two conditions just with manifolds. Two-dimensional manifolds are identifiable and the condition of the classical limit is trivially satisfied. The existence of a larger class, however, already raises the question of how the sum over topologies should be carried out in two dimensions. In higher dimensions such larger classes may be essential to obtain a computable procedure.

† The author learned this demonstration from Professor M Scharlemann. It is reproduced here for completeness and because the author does not know of another reference.

4. More than two dimensions

Defining a physically reasonable sum over topologies in higher dimensions is a more challenging problem than in two dimensions for two reasons. First, in higher dimensions there are a wider variety of topological invariants and therefore a wider variety of candidates for the gravitational action even if one is certain of its local nature. Second, an examination of the classical limit involves not only finding the topology but also the metric of least action. This, in turn, involves the measure, contour and action on the space of edge lengths. In this section we illustrate the issues which arise in higher dimensions in two ways. We shall show that, with the straightforward extension of the Regge action to all complexes described in § 2, pseudo-manifolds do *not* meet the two conditions for reasonable unruly topologies in higher dimensions. Then we shall exhibit possible restrictions on the class and illustrate modifications of the action by which the conditions might be met.

Pseudo-manifolds are straightforwardly generalised to arbitrary dimensions [18]. Replace 'triangle' and 'edge' in their defining properties in two dimensions by 'n-simplex' and '$(n-1)$-simplex' in n dimensions. Pseudo-manifolds may be constructed from manifolds by identifying simplices of dimension less than $n-1$, although for $n \geqslant 4$ not every pseudo-manifold can be realised in this way. This construction can be used to show that for $n>2$, with the straightforward extension of the Regge action to complexes discussed in § 2, there are geometries on pseudo-manifolds with the same action as any geometry on a manifold. Consider identifying two sufficiently separated vertices on an n-dimensional manifold and ask how the Regge action changes when the edge lengths are kept the same. The action (equation (2.3)) involves the volumes of the n- and $(n-2)$-simplices and the deficit angles defined by (2.4). The volumes are unchanged because edge lengths are unchanged. The deficit angles are unchanged because the interior angles of any n-simplex are unchanged and because the n-simplices intersecting a given $(n-2)$-simplex are unchanged by the identification of vertices. The Regge action is thus neutral to an identification of vertices. Manifolds therefore cannot be the only geometries of least action and manifolds will not dominate the classical limit. A suitable sum over histories in higher dimensions must either be defined with a different action or a different class of geometries or both.

To illustrate the possibilities of a different action consider the virtues in three dimensions of adding a negative multiple of the Euler number to the Regge action for closed geometries. χ is zero for manifolds [18]. Identification of vertices of a manifold decreases χ and thus increases the total action since the Regge action is unchanged. This addition of $-\chi$ therefore removes the neutral stability to identification described above. The nature of the minimum and the existence of analogous modifications in higher dimensions are interesting questions.

One can find a variety of subclasses of pseudo-manifolds in higher dimensions which are algorithmically identifiable from among all complexes (condition (1) of § 2) and for which the Regge action is not neutral to identification of vertices. The simplest class are pseudo-manifolds whose simplices all have connected links. Identification of vertices produces disconnected links and would thus be ruled out. A very restrictive alternative would be the class of homology manifolds—complexes for which the link of every simplex has all the homology groups of a sphere without necessarily being homeomorphic to a sphere. Since the homology groups of a complex are computable from its incidence matrices (with some considerable work!) [18], there exists an algorithm for deciding whether or not a complex is a homology manifold. Since the lowest homology group determines the connectedness of a complex, disconnected links would not be allowed and the objects obtained by the identification of vertices of manifolds would not be homology manifolds. In fact, in two and three dimensions homology manifolds are manifolds.

The above two examples serve to illustrate the variety of possible approaches to defining a sum over topologies in higher dimensions. The search is both for a physically motivated action and class of unruly topologies. If, however, an algorithmically identifiable class can be found and an action for them such that manifolds dominate the classical limit, then by relaxing the principle of equivalence in the quantum sum over histories, one will be able to implement in a computable way a quantum sum over topologies.

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