What is a particle?

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Quantum Fields

- The universe as we know it is fundamentally described by a theory of fields which interact with each other "quantum mechanically"
- These fields take values over all of spacetime, and can be scalars, or other "representations of the Lorentz group"
- Several authors provide good discussions of why we use quantum fields, in particular Srednicki and Landau (volume 4)
- No matter what angle you attack the problem from, fairly general arguments always seem to lead back to a local QFT
- So let's assume we have a quantum field, and see what we can do with it...

Real Scalar Fields

- We can start with the simplest object possible, a scalar field
- We can give the field some dynamics by giving it an action, and let's choose:

$$S = \int d^4x \left[-\frac{1}{2}\partial^\mu \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 \right] \quad (1)$$

- Why do we choose this action?
- Action reproduces the Klein-Gordon equation
- Action can also be thought of as a "lowest order expansion" of something more fundamental

Diagonalizing in Plane Waves

- How do we make this theory "Quantum?"
- Let's try the usual trick of canonical quantization, by promoting observables to operators, and see where this gets us
- In this case, the observable is the field at any given spacetime point
- However, we first want to make a "change of basis" in order to diagonalize the system more readily
- Tackling the strictly "classical theory" (before canonical quantization), we know we can decompose any solution of the Klein-Gordon equation into the form

$$\varphi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} + i\omega t}\right] \tag{2}$$

- The expression in the denominator of the volume element is used to maintain Lorentz invariance of the measure

A First Hint of Particles

- Notice that the frequency and wave-vector are necessarily related by the equation

$$\omega = \left(\mathbf{k}^2 + m^2\right)^{1/2} \tag{3}$$

- This looks promising for the energy of a relativistic particle!
- If we use the reality condition, and do a little more rearranging, we find

$$\varphi\left(\mathbf{x}, t\right) = \int \frac{d^3 k}{\left(2\pi\right)^3 2\omega} \left[a\left(\mathbf{k}\right) e^{ikx} + a^*\left(\mathbf{k}\right) e^{-ikx}\right]. \tag{4}$$

- The restricted number of degrees of freedom will correspond to the resulting particles being their own antiparticles
- With appropriate interaction terms, a theory of this form might describe, for example, the (physical) Higgs boson

The Hamiltonian

- The last thing we need to do before going to canonical quantization is to find the Hamiltonian in terms of the Fourier coefficients
- In single particle mechanics, we find the canonical momentum by the prescription

$$p = \partial L / \partial \dot{q} \tag{5}$$

- We can generalize this to the case of our field theory, by writing

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \tag{6}$$

- The Hamiltonian (density) is then written

$$\mathcal{H} = \Pi \dot{\varphi} - \mathcal{L}. \tag{7}$$

The Hamiltonian

- In our case, the canonical momentum becomes the time derivative of the field, and so we ultimately find that we can write

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \tag{8}$$

- The full Hamiltonian is the space integral of this Hamiltonian density
- If we write the field in terms of its Fourier expansion, and perform some tedious computations, we ultimately find

$$H = \frac{1}{2}\int \widetilde{dk}\ \omega\left[a^*(\mathbf{k})\,a(\mathbf{k}) + a(\mathbf{k})\,a^*(\mathbf{k})\right] \tag{9}$$

- So far everything is still "classical"

Quantization At Last

- Now that we've identified a Hamiltonian, it's time to quantize
- In regular quantum mechanics, canonical quantization amounts to declaring the "canonical commutation relations"

$$[p_i, p_j] = [q_i, q_j] = 0 \tag{10}$$

$$[p_i, q_j] = i\delta_{ij} \tag{11}$$

- For the case of a continuum field theory, we assume that we can generalize this

$$[\varphi(\mathbf{x}, t) \ , \ \varphi(\mathbf{y}, t)] = [\Pi(\mathbf{x}, t) \ , \ \Pi(\mathbf{y}, t)] = 0 \tag{12}$$

$$[\varphi(\mathbf{x}, t) \ , \ \Pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{13}$$

Quantization At Last

- If we are now interpreting our field as a quantum mechanical operator, then the same must apply to the Fourier coefficients
- In this sense, the Fourier decomposition is thought of as a linear combination of the field operators, a "change of basis"
- In particular, it is possible to reverse this decomposition, in order to find that

$$a(\mathbf{k}) = \int d^3x \; e^{-ikx} \left[i\dot{\varphi}(x) + \omega\varphi(x)\right] \tag{14}$$

- Note that time-independent Fourier coefficients become time-independent operators in the Heisenberg picture sense
- Using this expression for the Fourier coefficients, along with the field commutators and a notion of what it means to integrate an operator, we arrive at the commutation relations

$$\left[a(\mathbf{k}), a(\mathbf{p})\right] = \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{p})\right] = 0 \tag{15}$$

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] = (2\pi)^3\, 2\omega\delta^3(\mathbf{k} - \mathbf{p}) \tag{16}$$

Quantization At Last

- Upon interpreting the Fourier coefficients as operators, one writes the Hamiltonian as

$$H = \frac{1}{2} \int \widetilde{dk}\ \omega \left[a^{\dagger}(\mathbf{k})\, a(\mathbf{k}) + a(\mathbf{k})\, a^{\dagger}(\mathbf{k})\right] \tag{17}$$

- This can be rewritten as

$$H = \int \widetilde{dk}\ \omega \left[a^{\dagger}(\mathbf{k})\, a(\mathbf{k}) + \frac{1}{2}\left[a(\mathbf{k}), a^{\dagger}(\mathbf{k})\right]\right] \tag{18}$$

- Keeping mind of the normalization on the commutator, we can write this in terms of the usual "number operator" as

$$H = \int dk\ N(\mathbf{k})\, \omega + \int \widetilde{dk}\ \frac{\omega}{2}\left[a(\mathbf{k}), a^{\dagger}(\mathbf{k})\right] \tag{19}$$

Particles

- The first term of this Hamiltonian, along with the commutation relations, describe a theory with an infinite number of decoupled "harmonic oscillators," one for each momentum
- The energy momentum relation is exactly what we would expect for relativistic "particles"
- Any "physical" particle can be written as a linear combination of these states with a suitable envelope function

$$|\mathbf{k}_0\rangle = \int f(\mathbf{k}, \mathbf{k}_0) |\mathbf{k}\rangle = \int f(\mathbf{k}, \mathbf{k}_0) a^\dagger(\mathbf{k}) |0\rangle \tag{20}$$

- States like this carry the correct energy momentum relation, are quantized in integer "packets," and in every other way look like particles

The Vacuum Energy

- Unfortunately, the second term is not so nice looking
- Using the commutation relations, we find

$$\int \widetilde{dk}\ \frac{\omega}{2} \left[a\left(\mathbf{k}\right), a^\dagger\left(\mathbf{k}\right)\right] \propto \int d^3k\ \delta^3\left(0\right) \tag{21}$$

- Formally, this is infinity integrated over all of momentum space
- This term results from integrating over all of space time, and also from taking the continuum limit down to zero length
- Since it is a "constant" number, we will ignore it as an irrelevant shift in the total energy, but in other contexts it can be a much bigger problem

Other Options: Wave Functionals

- Before talking about interactions, let's discuss some other ways we could have approached the problem of quantizing a field
- Generalizing the notion of a wavefunction from regular quantum mechanics, we could have tried to find a probability distribution for different possible field configurations
- In particular, there would be field eigenstates

$$\varphi(\mathbf{x}, 0) |A\rangle = A(\mathbf{x}) |A\rangle \tag{22}$$

- For the real scalar field, one can show that the wave-*functional* for the ground state is

$$\langle A|0\rangle \propto \exp\left[-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega(\mathbf{k})\,\widetilde{A}(\mathbf{k})\,\widetilde{A}(-\mathbf{k})\right]. \tag{23}$$

- This could be used to calculate, for example, the energy of the ground state, by taking a functional integral over all possible field configurations
- This approach does not turn out to be very useful for particle physics applications, but we will run into functional methods again

Other Options: Promoting Time to an Operator

- Another approach is to maintain the idea of a position operator for an individual particle, but also promote the "time" coordinate to an operator as well, in the spirit of relativity
- These coordinates would be viewed on the worldline of the particle, and they would be parametrized by some affine parameter (possibly the proper time)
- This method runs into a number of complications, and is not as well-suited to describing particle creation and annihilation
- A related idea, however, is used in string theory, for describing the world-sheet of a string
- They key take-home point, however, is that QFT is a quantum theory, just like any other (with a few pesky issues that come from infinite-dynamical systems)

Scattering

- We now want to discuss something which is more relevant to actual particle physics: scattering processes
- Because the theory we just studied was exactly solvable in terms of a set of plane-wave energy eigenstates, the theory does not describe any sort of "interactions"
- Particle wave-packets will simply pass through each other, traveling off to infinity
- In the free theory, we can define a physical one-particle creation operator according to

$$a_1^\dagger \equiv \int d^3k \; f_1(\mathbf{k}) \, a^\dagger(\mathbf{k}) \tag{24}$$

- The envelope function can be taken to be, for example, a Gaussian wave-packet centered around the momentum $\mathbf{k}_1$

Scattering

- For a scattering experiment in general, the object we want to compute is the matrix element

$$\mathcal{M} = \langle f|i\rangle \tag{25}$$

- This quantity is then used to compute physically measurable objects like differential cross sections, using standard methods which are not unique to QFT (Fermi's Golden Rule, etc.)
- An appropriate initial state for a "scattering" experiment involving two particles in the free theory might read

$$|i\rangle = a_1^\dagger a_2^\dagger|0\rangle \tag{26}$$

- A final state can be constructed similarly. Because this theory is trivial, the final state will again consist of two particles

Interacting Theories

- Now, what would we do in an interacting theory?
- We need to modify the Lagrangian
- A possible extension is

$$S = \int d^4x \, \left[-\frac{1}{2}\partial^\mu \partial_\mu \varphi - \frac{1}{2} m^2 + \frac{\lambda}{4!}\varphi^4 \right] \tag{27}$$

- While we could have added a cubic term, adding just a cubic term alone would lead to a Hamiltonian unbounded below, which has obvious problems
- The quartic term without any cubic contributions also has nicer symmetry properties
- One might consider this to be a higher order expansion in whatever the "true" theory is
- However, deeper considerations involving the "renormalization group" show that for large scale physics, well above the true microscopic theory, we will never have to deal with any terms higher than fourth order

Asymptotic States

- This new quartic theory can no longer be diagonalized in the usual way, plane-wave states do not constitute energy eigenstates, and if we write our field in the usual Fourier decomposition, the Fourier modes will now depend on time
- However, we generally tend to associate plane-wave states with "particles," and suspect there might be some way to describe particles "coming in from infinity" in this way
- In regular QM, asymptotically incoming and outgoing states are treated as plane-waves, which then interact when the particles are "close enough"
- If we assume that this still works in the interacting quantum field theory, we would suspect that we can write

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)\, a_2^\dagger(t)\,|0\rangle \tag{28}$$

- Even in the interacting theory, we define the Fourier coefficients in terms of the fields using the same expression
- Outgoing states are constructed similarly, with the time going to positive infinity

Validity of the Asymptotic States and Field Renormalization

- Before deriving some consequences of this identification, let's see if we can decide to what extent this construction is physically reasonable
- We assume there is a physical ground state of the theory, with zero energy and momentum
- The first excited state contains a particle with mass m, with arbitrary momentum
- The second excited state can have two particles, which in general can have an energy larger than $2m$, due to the particles having relative momentum
- We are explicitly assuming that we have a mass gap in our theory, which is valid here, but will be a source of problems in other theories

Validity of the Asymptotic States and Field Renormalization

- Let's now consider the action of the field operator on the physical vacuum.
- The relativistic equivalent of the Heisenberg equation is

$$\varphi(x) = \exp\left(-iP^{\mu}x_{\mu}\right)\varphi(0)\exp\left(iP^{\mu}x_{\mu}\right) \quad (29)$$

- Because we are assuming that the physical ground state is an eigenstate with zero energy and momentum, we find that

$$\langle 0|\varphi(x)|0\rangle = \langle 0|\varphi(0)|0\rangle \quad (30)$$

Validity of the Asymptotic States and Field Renormalization

- We want this expectation value to be zero, or else the state created by $a_1^\dagger(\pm\infty)$ will contain some overlap with physical single particle states
- If this value is not zero, we will simply shift the definition of the field by this constant, and make the appropriate change in the original Lagrangian

$$\varphi(x) \to \varphi(x) + \langle 0|\varphi(0)|0\rangle \tag{31}$$

- This is just a change in the name of the field, which does not change the physics

Validity of the Asymptotic States and Field Renormalization

- We also consider a physical one particle eigenstate with a given momentum, and find

$$\langle p|\varphi(x)|0\rangle = e^{-ipx}\langle p|\varphi(0)|0\rangle \tag{32}$$

- The expectation value on the right depends in some Lorentz invariant way on the four-momentum, but the only available option is the (physical) mass-squared, leading to a constant
- We want this to have the value it has in free theory, which is unity, in order for our asymptotic operators to properly create single particle states
- If this is not the case, we will rescale the value of the field so that it is the case

Validity of the Asymptotic States and Field Renormalization

- Lastly, we want to check the overlap with a general multiparticle state

$$\langle p, n|\varphi(x)|0\rangle = e^{-ipx}\langle p, n|\varphi(0)|0\rangle \tag{33}$$

- We would like this to be zero, but this is actually a little restrictive
- Instead, we consider physical multiparticle wavepackets

$$|\psi\rangle = \sum_n \int d^3p\ \psi_n(\mathbf{p})|p, n\rangle \tag{34}$$

- By making use of the existence of the mass gap, and by using some other physical arguments, one can show that

$$\lim_{t\to\pm\infty}\langle\psi|a_1^\dagger(t)|0\rangle = 0 \tag{35}$$

Validity of the Asymptotic States and Field Renormalization

- The previous discussion shows that the "creation operators" we would like to work with do create single particle states in the physical sense, at asymptotic times
- These states will serve as suitable initial conditions for scattering experiments
- This argument needs to be reconsidered when the physics becomes more complicated, for example, when we have massless particles in our theory
- In general, the price we pay is needing to do some field redefinition, so that our "new" Lagrangian might look like

$$\mathcal{L} = -\frac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 - \frac{\lambda}{4!} Z_\lambda \varphi^4 + Y\varphi \qquad (36)$$

- We will adjust these parameters in the course of studying the theory in such a way so that the creation operators work the way we want them to
- Their value is determined by requiring, for example, m to be the actual physical mass of the particle
- We'll have more to say about "renormalization" in the second talk

The LSZ Formula

- Now that we've decided what states we want to work with, we need to compute the matrix element
- I will omit the details of the derivation, and simply state that with sufficient use of the field equations, and one or two clever insights, this matrix element can be rearranged into the *Lehmann-Symanzik-Zimmermann reduction formula*

$$\begin{aligned}\langle f|i\rangle = &\int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} \\ &\times e^{ik_1x_1}\left(-\partial_1^2+m^2\right)e^{ik_2x_2}\left(-\partial_2^2+m^2\right) \\ &\times e^{-ik_{1'}x_{1'}}\left(-\partial_{1'}^2+m^2\right)e^{-ik_{2'}x_{2'}}\left(-\partial_{2'}^2+m^2\right) \\ &\times\langle 0|T\left\{\varphi\left(x_1\right)\varphi\left(x_2\right)\varphi\left(x_{1'}\right)\varphi\left(x_{2'}\right)\right\}|0\rangle\end{aligned} \tag{37}$$

- With this formula, the computation of the matrix element becomes nothing other than computing some correlation functions in the physical ground state
- This result generalizes to an arbitrary number of incoming and outgoing particles, with a factor of $i^{n+n'}$

Correlation Functions with Sources

- The question we now face is how to compute correlation functions in the ground state
- Unlike in regular quantum mechanics, the "best" way is to use functional integral methods
- While I can't give a full explanation of how to use path integrals in field theory, I can state some plausible results, and go into more detail for those interested
- The punchline is that I want to study the following path integral, in the presence of a "source"

$$Z(J) = \langle 0|0\rangle_J = \int D\varphi \; e^{i\int d^4x \;\; \mathcal{L}+J\varphi} \tag{38}$$

- With this, vacuum expectation values can be given by

$$\langle 0|\,T\left\{\varphi(x_1)\ldots\right\}|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}Z(J)\,|_{J=0} = \int D\varphi \;\, \varphi(x_1)\ldots e^{i\int d^4x \;\; \mathcal{L}} \tag{39}$$

- We can pretend that the functional derivative is like a continuum version of the partial derivative, and that it works in all of the ways we naturally want it to

A Side Note on the Functional Derivative

- The functional derivative generalizes the idea of a partial derivative
- We can imagine defining our theory on a lattice

$$\varphi(x_i) \equiv \varphi_i \tag{40}$$

- The path integral then becomes an integral over finitely many field variables

$$Z(J) = \langle 0|0\rangle_J = \int \prod_i d\varphi_i \; e^{i\sum_i \;\mathcal{L}+J_i\varphi_i} \tag{41}$$

- Average values are computed by taking partials

$$\langle 0|\varphi_j|0\rangle_J = \frac{\partial}{\partial J_j} \int \prod_i d\varphi_i \; e^{i\sum_i \;\mathcal{L}+J_i\varphi_i} = \int \prod_i d\varphi_i \; \varphi_j \; e^{i\sum_i \;\mathcal{L}+J_i\varphi_i} \tag{42}$$

A Side Note on the Functional Derivative

- The regular partial derivative obeys the rule

$$\frac{\partial}{\partial J_i} J_j = \delta_{ij} \tag{43}$$

- The continuum generalization of this is taken to be

$$\frac{\delta}{\delta J(x)} J(x') = \delta(x - x') \tag{44}$$

- From this, we find, for example

$$\begin{aligned} \frac{\delta}{\delta J(x)} \int d^4y\ F(y) J(y) &= \frac{\delta}{\delta J(x)} \int d^4y\ F(y) \frac{\delta}{\delta J(x)} J(y) \\ &= \int d^4y\ F(y) \delta(x - y) = F(x) \end{aligned} \tag{45}$$

- With suitable generalization, everything about regular integrals and derivatives can be carried over to functional ones (chain rule, product rule, integration by parts, etc.)

The Partition Function

- To get some physical insight into what's going on here, recall the formula for the partition function in Statistical Mechanics

$$\mathcal{Z} = \sum_s e^{-[E(s)-\mu_i N_i(s)]/T} \tag{46}$$

- We know that we can use this to compute physically interesting quantities

$$\langle N_i \rangle = \sum_s N_i\,(s)\, e^{-[E(s)-\mu_i N_i(s)]/T} = T\frac{\partial}{\partial \mu_i}\mathcal{Z} \tag{47}$$

- This is all actually very physical - we "poke" the system, and then watch what happens to it
- The idea is the same in field theory, with a few key differences, including continuum fields and quantum effects

The Partition Function in the Free Theory

- For a free theory, without any renormalizing terms, the path integral is "easy" to do, using methods similar to those that would be done for a quadratic theory in regular quantum mechanics
- The result we find is

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\ d^4x'\ J(x)\,\Delta(x-x')\,J(x')\right] \tag{48}$$

- Here we have introduced the *Feynman propagator*

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\ \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{49}$$

- Don't be alarmed by the iϵ - it is just a trick for making sure we select out the physical ground state

The Correlation Function in the Free Theory

- Using this result in our formula for the correlation function gives us some physical insight into what the "propagator" means
- We know that we should be able to write

$$\langle 0| T\varphi(x_1)\varphi(x_2)|0\rangle = -\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)} Z_0(J)|_{J=0} \tag{50}$$

- The functional derivative obeys the usual chain rule, and so we get

$$\langle 0| T\varphi(x_1)\varphi(x_2)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left[\int d^4x'\ \Delta(x_2 - x') J(x')\right] Z_0(J)|_{J=0} \tag{51}$$

- Each functional derivative "chews off" a source
- Ultimately, we find

$$\langle 0| T\varphi(x_1)\varphi(x_2)|0\rangle = \frac{1}{i}\Delta(x_2 - x_1) \tag{52}$$

- The correlation function is the probability amplitude for a particle at one place and time to propagate to another place and time

Computing Integrals Perturbatively

- In the interacting theory, we can't do this path integral exactly, and so we need to resort to perturbative means
- Consider the regular integral

$$\int_{-\infty}^{+\infty} e^{-x^2-gx^4}\, dx \tag{53}$$

- If we wanted to compute this in a power series in g, we could taylor expand the quartic part of the exponential

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-gx^4}\, dx \to \int_{-\infty}^{+\infty} e^{-x^2} \left(1 - gx^4 + \frac{1}{2}g^2x^8 + \ldots\right) dx \tag{54}$$

- Because we have exact results for integrating polynomials against Gaussian functions, this allows us to compute this integral perturbatively in powers of g
- Strictly speaking, this expansion has zero radius of convergence! It is an "asymptotic expansion"

Perturbative Partition Function

- We can extend these perturbation ideas to the path integral we want to compute, with a few extra subtleties
- Considering the case of our interacting quartic theory, without worrying about renormalizing factors right now, we write the partition function as usual, but with the two pieces in separate exponentials

$$Z(J) = \int \mathcal{D}\varphi \;\exp\left[i\int d^4x\, \frac{\lambda}{4!}\varphi^4\right] \exp\left[i\int d^4x\; -\frac{1}{2}\partial^\mu\partial_\mu\varphi - \frac{1}{2}m^2 + J\varphi\right] \tag{55}$$

- The interacting exponential can again by Taylor expanded

$$Z(J) = \int \mathcal{D}\varphi \;\sum_{V=0}^{\infty}\frac{1}{V!}\left[i\int d^4x\,\frac{\lambda}{4!}\varphi^4\right]^V \times \exp\left[i\int d^4x\; -\frac{1}{2}\partial^\mu\partial_\mu\varphi - \frac{1}{2}m^2 + J\varphi\right] \tag{56}$$

Perturbative Partition Function

- Notice that the previous expression takes the form of expectation values of powers of fields. We can write this in a very clever way

$$Z(J) = \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i\lambda}{4!} \int d^4x \, \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right]^V \times \int \mathcal{D}\varphi \, \exp \left[i \int d^4x \, -\frac{1}{2} \partial^\mu \partial_\mu \varphi - \frac{1}{2} m^2 + J\varphi \right] \tag{57}$$

- The functional derivatives act in the correct way to pull down the appropriate powers of the field
- The remaining path integral is just the path integral for the free theory, and so we find

$$Z(J) = \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i\lambda}{4!} \int d^4x \, \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right]^V Z_0(J) \tag{58}$$

Perturbative Partition Function

- Using the result for the free theory, and performing a dual Taylor expansion, this could be written as

$$Z(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i\lambda}{4!} \int d^4x \, \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right]^V \times \sum_{P=0}^{\infty} \frac{1}{P!} \left[\frac{i}{2} \int d^4y \; d^4z \; J(y) \, \Delta(y-z) \, J(z) \right]^P \tag{59}$$

- Ultimately, we want to take this perturbative result, and plug it back into the expression for the correlation function, and then plug this back into the LSZ formula
- This is pretty gross, and if you want to see all of the details, you should read Srednicki

The Feynman Rules

- However, if you do this power by power, you start to notice some patterns, and in particular, there's a very nice way to organize the perturbation theory, using Feynman diagrams, which have certain *Feynman rules*
- We draw pictures with lines that meet at vertices, where the number of internal lines is the P in the expansion, and the number of vertices is the V in the expansion
- Each external leg has an associated external momentum which comes from the initial states that go into the LSZ formula, and we associate an incoming or outgoing particle with it
- Each internal line also has an associated momentum

The Feynman Rules

- Each vertex has zero net four-momentum flowing into it (sort of like Kirchoff's rules)
- This momentum conservation follows from the fact that a vertex always ends up being associated with a delta function that is only satisfied in this case
- For a "tree diagram," this constrains all of the momentum
- For "loop diagrams," some of the internal momentum will be unfixed
- Each object in the picture gets a factor. These factors will be different for every theory, but in our particular theory, each vertex gets a factor of $i\lambda$, and each internal line with momentum k gets a propagator

$$\frac{-i}{k^2 + m^2 - i\epsilon} \tag{60}$$

- Notice that while momentum is conserved at each vertex, energy is not! This internal particle need not be on-shell. This is just a result of the math that comes out of the perturbation theory.

The Feynman Rules

- After we have written down the diagram, and associated each object with one of these factors, we integrate over all of the unfixed internal momentum
- After integrating over all of the unfixed momentum, we add all of the topologically distinct diagrams
- One also needs to worry about "symmetry factors" to get counting statistics correct
- The result gives you $i\mathcal{T}$, which is defined by

$$\langle f | i \rangle = (2\pi)^4 \, \delta^4 \left(k_{\text{in}} - k_{\text{out}} \right) i\mathcal{T} \tag{61}$$

- Loop integrals will tend to diverge, which is a result of several nontrivial issues involving the perturbation theory.
- Renormalization will fix this, but that is a subject which will be (briefly) discussed in the next lecture

Why Do We Need Field Theory For This?

- After looking at the final result, we may have guessed this, even without field theory
- Particles just "bounce off each other," and we sum over all paths
- However, the field theory has several nice features
- First, a wide variety of physics can be derived from a small number of parameters
- Second, there is a nice sense in which stipulating the most general possible theory of some sort of field naturally leads to the idea of a particle
- Third, the field theory is the only good way of systematically studying all of the underlying symmetries of the theory
- Fourth, there are certain non-perturbative results which can only come from studying the full field theory
- Lastly, we will find that when we try to generalize to broader contexts (like curved spacetimes), we need to modify the particle idea slightly

Generalization of the Rules to More Complex Theories

- Notice that the same type of "particle" shows up in the internal lines
- For more complicated theories with several types of fields that interact, a similar result holds
- Each free field has some sort of "propagator," which depends on the specific theory
- We again draw these diagrams, where the allowed vertices arise from the interaction terms in the Lagrangian
- Every internal line gets the appropriate propagator
- Every type of particle which can show up on the outside of the diagram also shows up on the inside
- With practice, the Feynman Rules can be "read off" from the Lagrangian

Real vs. Virtual Particles

- We often refer to the internal particles as virtual particles, and the external particles as real particles
- However, there is not actually a lot of physical difference between
- Any "real" particle is actually just an approximation, where we assume the last interaction it had was "infinitely far in the past"
- If we could "see" the particles on the inside of a diagram, then in some sense that would mean that they should be the "external lines" which go on to interact with a detector
- Unstable particles are often said to have an indefinite mass - this is a statement about internal particles not being on-shell, and the fact that unstable particles can not live asymptotically far into the future
- Keep in mind that the external lines are what correspond to the "physical states" that go into the LSZ formula

When perturbation Theory Fails: Loop Integrals and Large Coupling

- This procedure, unfortunately, does not always work
- The first indication of a problem is the divergence of loop integrals
- This issue can be resolved, but requires renormalization
- Another problem is when the coupling is not weak, and this is a much more serious problem, which still impedes progress in QCD to this day (although there are some non-perturbative results)
- Another disturbing result is Haag's theorem, which says the Hilbert spaces of the free and interacting theory are not the same, and so in some sense, perturbation theory is not valid
- A similar idea happens in regular quantum mechanics, with the finite square well
- For most practical applications, we can ignore Haag's theorem when doing calculations

When perturbation Theory Fails: Massless Particles

- Another instance in which perturbation theory fails is when we have massless particles
- Many of our physical arguments used when deriving the LSZ formula fail when there is no mass gap in the spectrum of states
- This needs to be treated with more sophisticated means, but all of the key ideas (involving diagrams, Feynman Rules, propagators, etc.) carry over
- In some sense, the perturbation theory causes problems due to the uncontrolled statistics of dividing a small amount of energy among an arbitrary number of massless particles
- In QED, in the classical limit, this shows up in the long-range nature of the Coulomb potential, which leads to an infinite total cross section

When perturbation Theory Fails: Bound States

- Another physical process which is difficult to access is the formation of bound states
- A bound state "lasts forever," and does not involve a scattering-like event
- In QED, there is a clever way to access bound states
- If we compute the lowest order scattering matrix element, take the relativistic limit, and compare it with the Born approximation from regular quantum mechanics, we can "read off' the resulting classical potential, and we indeed find the Coulomb potential
- This fails in QCD as a result of strong coupling
- In some sense, the coupling in QCD causes the particles in the composite to lose their individual identity

For the Future

- After this initial introduction to how to get some basic particle-like behaviour out of a quantum field, in the next lecture I'll talk about some deeper aspects of what a particle is
- In particular, I will discuss how particles can "hide" themselves in a theory, where they might not initially be seen - solitons, and "Higgs particles"
- I'll also talk about what "quasiparticles" are in condensed matter, and why they may not be as different from "real" particles as you think
- Some brief discussion of the RG will occur, and what it tells us about "fundamental physics"
- There will also be some discussion of how to generalize the idea of a particle when we no longer have the symmetries of Minkowski space (which is the case in the real world!), and why we can always make reference to the idea of a particle if we restrict to "local observers"
- Lastly, I'll say some words about how I have come to think about particles to this date, and offer whatever philosophical insights I can

Summary

- By stipulating that some sort of "field" exists, and then stipulating basic "quantum" ideas, we naturally find that particle-like things pop out
- With appropriate physical insight, we can see how to use these objects to do scattering experiments
- This procedure is inherently perturbative, and internal particles arise as terms in a perturbation series
- This perturbation theory is not always successful, and in those cases we must make reference to more sophisticated means
- However, the basic underlying ideas of the technique are always the same