A case study of the impact of increased computational resources on lattice gauge theory calculations: Constraints on Standard model parameters

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Abstract

In this document we discuss the impact of lattice calculations on fits of the parameters of the CKM matrix. We show the present status and estimate the resources needed to substantially reduce the lattice errors. For each quantity, we give both a summary and a detailed discussion of the basis of our estimates.

I. INTRODUCTION

The present status of CKM determinations is summarized in Fig. 1 [1]. The quantities refered to in the figure are the ratio of $b \to u$ and $b \to c$ weak coupling constants ($|V_{ub}/V_{cb}|$), the $K^0 - \overline{K^0}$ mixing strength ε_K, the $B_d - \overline{B_d}$ mixing amplitude ΔM_d, and the $B_s - \overline{B_s}$ mixing amplitude ΔM_s. The quantity "$\sin 2\beta$" is determined from measurements of CP-violation in the decay $B \to J/\psi K_s$. The quantities $\bar{\rho}$ and $\bar{\eta}$ are the least well known elements of CKM matrix. Each of the measured quantities constrains $\bar{\rho}$ and $\bar{\eta}$ as shown, with the error band resulting from a combination of experimental and theoretical errors. At present, all determinations are consistent, with the allowed region shown by the ovals in the figure.

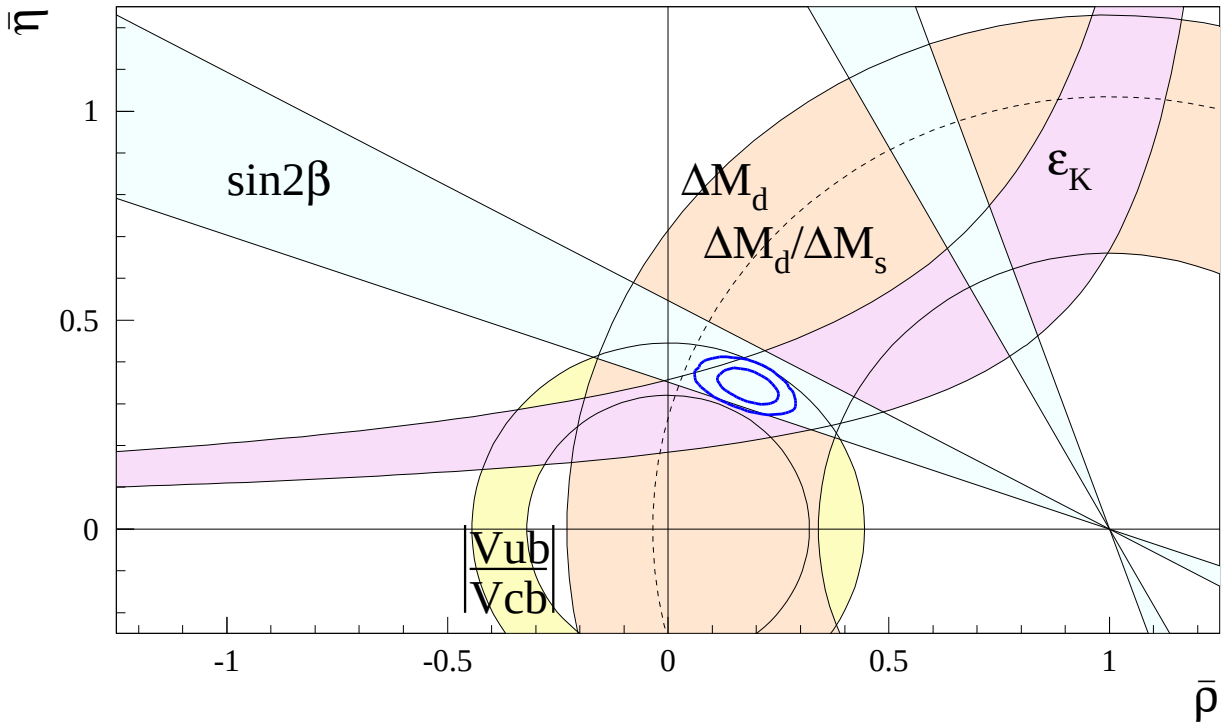

FIG. 1: Allowed regions for $\bar{\rho}$ and $\bar{\eta}$. The contours at 68% and 95% probability are shown. The full lines correspond to the 95% probability constraints given by the measurements of $|V_{ub}|\,/\,|V_{cb}|$, ε_K, ΔM_d and $\sin 2\beta$. The dotted curve bounds the region selected at 95% by the lower limit on ΔM_s.

It is clear from this figure (as has been the case for some time now) that a substantial reduction in the error bar in the constraint from ΔM_d has the greatest potential to improve the stringency of the test of the Standard Model (SM). Next in potential is the ε_K band. In both these quantities

the lattice error is dominant, as discussed in more detail below. Once ΔM_s is measured, a similar comment may apply there.

Another way of seeing the importance of the lattice calculations is shown in Fig. 2 [1]. This displays the allowed ranges of B_K and $f_{B_d}\sqrt{B_{B_d}}$ when these quantities are taken as outputs (which is possible because the CKM fits are now overconstrained). Present lattice results are consistent with these ranges, but have errors about as wide as the output bands, and thus add little to present constraints on CKM parameters. Reducing lattice errors to the few percent level will severely test the SM.

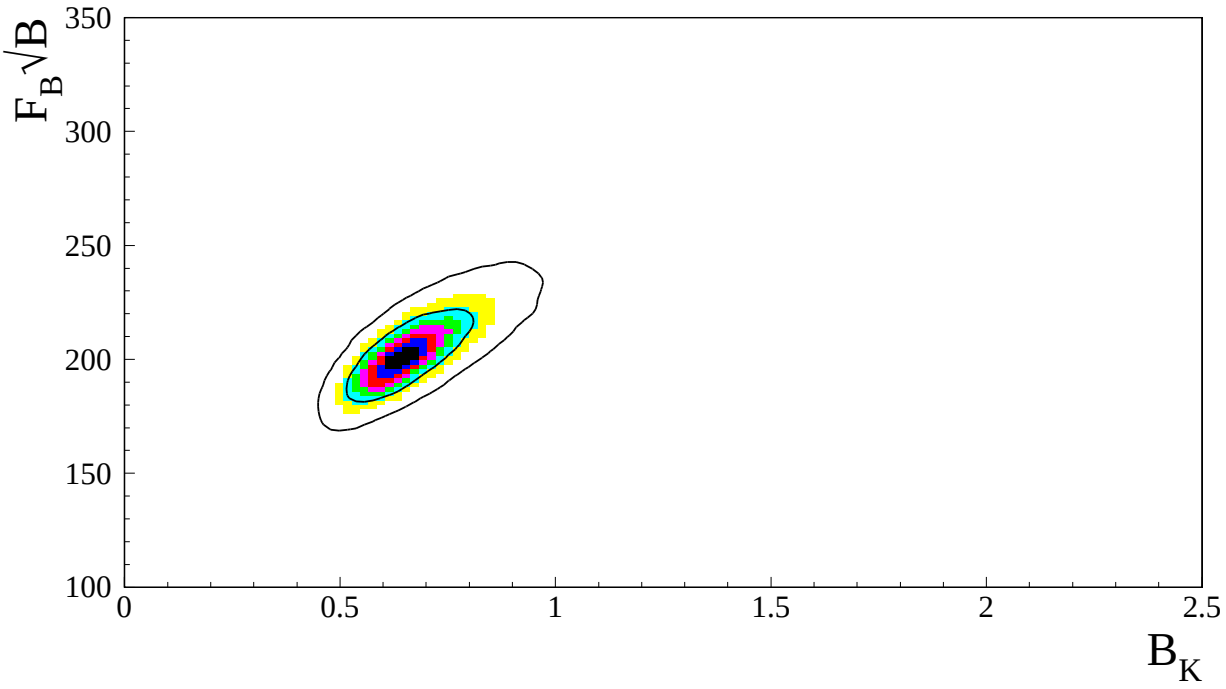

FIG. 2: The 68% and 95% contours in the $(f_{B_d}\sqrt{B_{B_d}},\ \widehat{B}_K)$ plane, using the $|V_{ub}|/|V_{cb}|$, Δm_s and sin2β constraints.

II. LATTICE CONTRIBUTIONS TO PRESENT ERRORS

Here we collect the standard formulae for the quantities requiring lattice calculations, and describe the current state of the errors.

A. ε_K

CP violation in $K_0 \leftrightarrow \bar{K}_0$ mixing is parameterized by ε_K. The standard expression for this is (see e.g. Ref. [2])

$$|\varepsilon_K| = C_\varepsilon A^2 \lambda^6 \bar{\eta} \left[\eta_2 S(x_t) A^2 \lambda^4 (1-\bar{\rho}) + \text{charm}-\text{contribs} \right] \widehat{B}_K \tag{1}$$

where

$$C_\varepsilon = \frac{G_F^2 f_K^2 m_K m_W^2}{6\sqrt{2}\pi^2 \Delta m_K}\,, \tag{2}$$

is a well measured combination of quantities, A, λ, $\bar{\eta}$, $\bar{\rho}$ are the usual Wolfenstein parameters in the CKM matrix, η_2 is a QCD Wilson coefficient, known to next-to-leading-order (NLO), and B_K is the kaon B-parameter. Only the dominant top quark loop contribution is shown.

This equation is to be used to constrain $\bar{\rho}$ and $\bar{\eta}$. Thus the relevant errors are [1, 2]

$$|\varepsilon_K| = [2.28 \pm 0.019] \times 10^{-3}\,, \tag{3}$$

$$A\lambda^2 = |V_{cb}| = [41.4 \pm 0.7 \pm 0.6] \times 10^{-4}\,, \tag{4}$$

$$\lambda = 0.2240 \pm 0.0036\,, \tag{5}$$

$$\eta_2 = 0.574 \pm 0.004 \tag{6}$$

where the first error is statistical, the second (if present) theoretical, except for η_2 where the error is theoretical. Combining all these errors in quadrature one finds that the present errors in all quantities in eq. (1) aside from $\widehat{B}_K$ add up to 9.5%. This should be compared to the current error usually quoted on the lattice result for $\widehat{B}_K$ used in CKM fits [3]:

$$\widehat{B}_K = 0.86 \pm 0.06 \pm 0.14\,. \tag{7}$$

Here the first error is a combination of statistical error and that from the use of only one-loop perturbation theory, while the second is an estimate of the systematic errors due to quenching and chiral extrapolation. Adding these in quadrature gives an 18% error (typically rounded up to 20% in the previous reviews).

Thus, the lattice error is roughly twice that from other sources. Important milestones in the calculation of $\widehat{B}_K$ are, first, reducing the error to 10%, comparable to the error from other sources, and second, reducing the error to 5%, half of that from other sources. These require a reduction in the lattice error by factors of approximately 2 and 4, respectively.

Note that the size of the "other" error depends on what one is using the result for B_K to constrain. The discussion here is in the context of constraining $\bar{\rho}$ and $\bar{\eta}$—the standard CKM triangle. Alternatively, one could use B_K to determine the CKM element $Im(V_{td}^2) = A^2\lambda^6 2\bar{\eta}(1-\bar{\rho})$, in which case the "experimental error" comes only from that on ε_K, $A^2\lambda^4$ and η_2 and is considerably smaller. The latter presentation, with its correspondingly smaller experimental error, was used in previous SciDAC documents. Here we prefer to uniformly consider the constraints on $\bar{\rho}$ and $\bar{\eta}$.

B. ΔM_d

The mass difference in $B_d - \overline{B_d}$ mixing is given theoretically by

$$\Delta M_d = \frac{G_F^2 M_W^2 M_{B_d}}{6\pi^2}\eta_c S(x_t) A^2\lambda^6 \left[(1-\bar{\rho})^2 + \bar{\eta}^2\right] f_{B_d}^2 \widehat{B}_{B_d}\,. \tag{8}$$

Here there is a new perturbative Wilson coefficient, with value at NLO

$$\eta_c = 0.55 \pm 0.01\,. \tag{9}$$

The present measurement of the mixing gives

$$\Delta M_d = 0.503 \pm 0.006 \text{ps}^{-1}\,. \tag{10}$$

Adding all non-lattice errors in quadrature one finds 6%. By contrast the value for the lattice result used in recent CKM fits is [1]

$$f_{B_d}\sqrt{B_{B_d}} = 223 \pm 33 \pm 12 \text{ MeV}\,, \tag{11}$$

where the first error is statistical, the second mainly due to uncertainties from the chiral extrapolation. Adding errors in quadrature gives a total error of 15% in $f_{B_d}\sqrt{B_{B_d}}$, which should then be doubled because the square of this quantity appears in eq. (8). Thus the lattice error is about 5 times that from other sources.

To show that this lattice number is fairly robust, we also quote a recent result from the JLQCD collaboration using two flavors of dynamical improved Wilson quarks and non-relativistic QCD (NRQCD) for the *b*-quark [4]:

$$f_{B_d}\sqrt{B_{B_d}} = 215 \pm 11 \begin{matrix}+0\\-23\end{matrix} \pm 15 \text{ MeV}\,. \tag{12}$$

Here the first error is statistical, the second from chiral extrapolations, and the third from other systematics. This result also serves to emphasize the importance of working at smaller quark

masses to reduce the error from chiral extrapolations, and that the systematics associated with perturbative matching can be substantial.

Thus, if we want to lower the error so as to be comparable with other sources of error, we need, in this case, a reduction of about a factor of 5 (corresponding to an error of 3% in $f_{B_d}\sqrt{B_{B_d}}$); lowering to half of the other errors requires a factor of 10 reduction (corresponding to an error of 1.5% in $f_{B_d}\sqrt{B_{B_d}}$). On the other hand, *any* significant reduction, e.g. by a factor 2, would have a large, immediate impact on CKM fits.

C. ΔM_s

The mixing of B_s and $\bar{B_s}$ has not yet been measured, but is likely to be so in the next few years at the Tevatron. Forming a ratio to B_d mixing cancels many factors leaving:

$$\frac{\Delta M_d}{\Delta M_s} = \frac{M_{B_d}}{M_{B_s}}\lambda^2\left[(1-\bar{\rho})^2+\bar{\eta}^2\right]\frac{1}{\xi^2} \tag{13}$$

where the lattice quantity is

$$\xi = \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}}. \tag{14}$$

The present standard number for ξ from the lattice is [1]

$$\xi = 1.24 \pm 0.04 \pm 0.06 \tag{15}$$

where the second error is mainly from chiral extrapolations. Combining in quadrature gives a 6% error in ξ, and thus a 12% error in ξ^2. Even though there is no experimental number to compare to, a reasonable goal is to reduce this error by a factor of 4, i.e. down to 3% in ξ^2. or 1.5% error in ξ.

The recent JLQCD result for ξ is [4]

$$\xi = 1.14 \pm 0.03 \,{}^{+0.13}_{-0.02} \pm 0.02 \,{}^{+0.07}_{-0}\,, \tag{16}$$

with the first error statistical, the second from chiral extrapolation, the third from other systematics, and the last from the uncertainty in the strange quark mass. This is reasonably consistent with the standard number quoted above, and shows again how errors due to chiral extrapolations are dominant (and that perturbative errors largely cancel in this ratio).

D. V_{ub}/V_{cb}

At present the lattice does not contribute to the constraint from this quantity, but it is likely to in the future, first in the determination of V_{ub}, and perhaps later in that of V_{cb}. We thus discuss the situation for both of these quantities.

Of the two, V_{ub} is less well determined. The best measurement, in terms of experimental errors, now comes from the exclusive $b \to u$ decays $B \to \rho\ell\nu$ or $B \to \pi\ell\nu$. According to Ref. [1] the present result is

$$|V_{ub}| = [33.0 \pm 2.4 \pm 4.6] \times 10^{-4}\,, \tag{17}$$

with the first error experimental, the second from theory (mainly from comparing models). The theoretical input is the form-factor of the left-handed current; this can be calculated directly on the lattice for large q^2 using B mesons at rest, and for smaller q^2 with moving B mesons.

The present lattice errors in the quenched approximation are around 15% [5]. These will be soon superseded by a FNAL/MILC/HPQCD calculation using existing MILC configurations with dynamical staggered quarks. With the present experimental error being about 7%, the aim should be to reduce lattice errors first to this level, and ultimately to about 3%. This may well be easier than achieving 1.5% errors in $f_{B_d}\sqrt{B_{B_d}}$ and ξ.

For V_{cb} the situation is, from the lattice perspective, more complicated. The best measurement is from measurements of the inclusive semi-leptonic decay rate:

$$|V_{cb}| = [41.4 \pm 0.7 \pm 0.6] \times 10^{-3}\,, \tag{18}$$

with the first error is experimental and the second theoretical. Present lattice methods do not contribute to the theoretical error. The total combined error is 2.2%. On the other hand, the lattice does contribute to the measurement from the exclusive $B \to D^*$ semileptonic decays, which presently give [1]

$$|V_{cb}| = [42.1 \pm 1.1 \pm 1.9] \times 10^{-3}\,. \tag{19}$$

The experimental error is thus 2.6%. The theoretical error enters through the uncertainty in the end-point $B \to D^*$ form factor $\mathcal{F}(1)$, and is presently estimated as 4.4% in Ref. [1], with the lattice calculation being a major contributor in this estimate.

Given this situation, an important objective is to reduce the lattice error by about a factor of three, to about 1.5%, so that it is comparable with the theoretical and experimental errors in the inclusive measurement.

III. RESOURCES NEEDED TO ACHIEVE THE DESIRED LATTICE ERRORS

In this section we estimate the computational resources required to substantially reduce the errors in the lattice matrix elements discussed above. These are based on extrapolations from what we know from present simulations, and assume no advances in algorithms.

The key issues are reducing the light quark masses so that a controlled chiral extrapolation can be done, and reducing the lattice spacing so that discretization errors are controlled. In the near term (resources at the 10 Teraflops scale) calculations will likely be mainly done using dynamical improved staggered fermions, which are significantly computationally cheaper than the alternatives of improved Wilson or Domain-Wall/Overlap fermions. The main disadvantages of staggered fermions are the practical problem that the continuum and chiral extrapolations are intertwined due to taste symmetry violations, and the theoretical uncertainty introduced by the need to take fractional powers of the fermion determinant. The strategy will be to calculate a sufficient number of experimentally measured quantities to high accuracy, so that one can judge the uncertainty from the results themselves, as well as from consistency checks based on the accuracy of staggered chiral perturbation theory in describing the data.

As resources increase, we will likely begin to shift to Domain-Wall or overlap dynamical fermions. This is likely to occur when resources reach the 10 Teraflops level. These fermions have exact or nearly exact chiral symmetry, so that continuum and chiral limits are decoupled, and one can directly simulate the desired continuum quark content. Improved Wilson fermions, including twisted mass QCD, are intermediate in computational difficulty and may well be used at an intermediate stage.

There will need to be a parallel effort at improving certain analytic calculations, in particular the matching of continuum and lattice operators, the determination of coefficients in improved fermion actions, and the calculation of the predicted mass and lattice spacing dependence using staggered chiral perturbation theory.

A. Timing estimates based on MILC simulations

In this section we give estimates of the computer time required to generate lattices using the "Asqtad" improved staggered fermion action developed by the MILC collaboration. To do this we use the timing information for existing runs, and extrapolate based on reasonably established

TABLE I: *Estimates of computer time needed to generate 120 independent lattices using improved staggered fermions. Results are given in Teraflop-years. See text for further discussion.*

m_l/m_s	a (fm)	Size	L (fm)	Tfl-yrs	m_π/m_ρ	Label
0.20	0.09	$28^3 \times 96$	2.5	0.09	0.39	MILC0
0.10	0.09	$40^3 \times 96$	3.6	1.5	0.30	MILC1
0.05	0.09	$56^3 \times 96$	5.0	23	0.22	
0.20	0.06	$42^3 \times 138$	2.5	1.5	0.39	MILC1
0.10	0.06	$60^3 \times 138$	3.6	25	0.30	MILC2
0.05	0.06	$84^3 \times 138$	5.0	390	0.22	
0.20	0.045	$56^3 \times 192$	2.5	12	0.39	
0.10	0.045	$80^3 \times 192$	3.6	190	0.30	
0.05	0.045	$112^3 \times 192$	5.0	2950	0.22	

scaling laws. These laws are conservative—in some cases the scaling is less rapid than what we assume. Furthermore, it is to be hoped that advances in algorithms will reduce the required computer time.

The following table summarizes the estimates of the costs of reducing both the quark masses and lattice spacing, while holding $m_\pi L = 4.2$ and the number of uncorrelated lattices fixed. The simulations at $m_l/m_s = 0.2$ (where m_l is the common mass taken for the up and down quark, and m_s is the strange quark mass, which is kept at or near the physical value) and $a = 0.09\,\mathrm{fm}$ have already been completed by the MILC collaboration, and we quote the time required for 3000 molecular dynamics time units, which is about 120 independent configuratons. In scaling to smaller light-quark masses, we assume that resources needed are proportional to $m_l^{-2.5}$ at fixed lattice spacing and volume.[1] Scaling to smaller lattice spacing at fixed physical volume we assume that resources vary as a^{-7}.[2] Finally, the resources are assumed to grow proportional to the four-volume when increasing the size of lattices at fixed a and quark masses.

[1] One power of m_l^{-1} from the inversion of the Dirac operator, another from the reduction in the step size, and a factor of $m_\pi^{-1} \propto m_l^{-0.5}$ from the increase in the trajectory length needed to generate an independent configuration.

[2] Four powers of a^{-1} from the change in the number of lattice points, and one power each from the inversion of the Dirac operator, the decrease in step size, and the increase in trajectory length.

The increments in the table (by factors of two in quark masses and a^2) are chosen to represent significant changes which would lead to significant reductions in errors, as discussed in more detail below.

We also quote values for m_π/m_ρ. This ratio is useful since its square governs the chiral expansion: successive terms are suppressed by m_π^2/m_ρ^2, up to factors of $O(1)$.

We note in passing that it may not be necessary to use lattices as large as those in the table at small pion masses. Even if $m_\pi L \lesssim 1$, one can still predict the form of correlation functions using chiral perturbation theory as long as $L \gg 1/\Lambda_{\rm QCD}$. This is the so-called epsilon regime. On the other hand, staggered chiral perturbation theory has not been worked out in this regime, so it is conservative to avoid it. Furthermore, experience suggests that one does not lose statistical power by working on larger volumes—it is as though one has more samples.

We use the following abbreviations to refer to possible future choices of simulations:

- **MILC0**: This is the present set of lattices—a suite including $a = 0.13\,$fm and $a = 0.09\,$fm with masses running down to $m_l/m_s = 0.2$, i.e. running down to the parameters in the first row of the table. Approximately 0.6 Tflop-years of CPU time have been used to generate these lattices and calculate physical quantites on them.

- **MILC1**: In this set of simulations one advances by "one step" in the table separately to smaller m_l and a, i.e. one undertakes simulations with parameters listed in the second and fourth rows of the table. One will need, in addition, to have runs at larger quark masses at $a = 0.06$, and to calculate the large number of valence quark propagators needed for extracting physical quantities. If we assume an overhead of a factor of 2 from these additional calculations, this step requires ~ 6 Tflop-years. It is thus at the level accessible to the QCDOC and planned large clusters.

- **MILC2**: Here one advances a further step, probably to smaller quark mass at the smaller lattice spacing (see the discussion of f_{B_d} and related quantities below). Whatever the details, the table shows that this step requires of order 50-60 Tflop-years, and thus the next generation of machines after that presently being considered.

- **DWF1**: As noted above, at some stage extensive simulations with dynamical domain-wall fermions (or other similar discretizations with exact or nearly exact chiral symmetry) will become feasible. We include a rough estimate of computational requirements for this step,

using ongoing simulations by the RBC collaboration as a benchmark. The need for a fifth dimension increases the cost by a factor of at least 12 compared to Wilson fermions, and thus by at least 12-24 compared to improved staggered fermions. (The range indicates the uncertainty in the relative number of conjugate gradient iterations needed for domain-wall and improved staggered fermions. Note that, in a calculation requiring about 500 conjugate gradient iterations, improved staggered and Wilson simulations have comparable cost, the factor of four gain with unimproved staggered fermions being counterbalanced by the factor of two cost increase due to the Naik term and a second factor of two from the calculation of the fermion force with smeared gauge links.) Thus a simulation with domain-wall fermions at $m_l/m_s = 0.1$ and $a = 0.09$ fm will require at least $20-40$ Teraflop-years. A thorough study of mass and lattice spacing dependence is thus likely to require resources of order 100 Teraflop-years, i.e of a similar scale to the MILC2 sample.

It is reasonable to assume that the chiral and continuum extrapolations will be better controlled than with staggered fermions, since they are almost independent. Thus it is likely that such a simulation will yield results of comparable precision to those from the MILC2 sample, even though the lattice spacing is larger. Furthermore, it will remove the uncertainty associated with taking roots of the fermion determinant. We also note that non-perturbative methods for normalizing operators can be implemented much more easily with domain-wall fermions than with staggered fermions.

We stress that the scaling of CPU time with domain-wall fermions is less well studied than with other discretizations, and the algorithms that are used are less developed, so the uncertainty in this estimate (in both directions) is greater than those for improved staggered fermions.

B. B_K

1. *Summary*

The present estimated lattice error on B_K is 20%, compared to 10% from other sources. Using the existing MILC0 data set, and one-loop perturbative matching (which has not yet been carried out for the improved gluon action used as part of Asqtad, although it will be straightforward to do so) the lattice error should be reduced to 12% (and made much more reliable). The MILC1

data set, together with staggered chiral perturbation theory (again, which doesn't yet exist for this quantity), should lower this error to 10% with one-loop perturbative operator matching. To achieve a 5% error will require two-loop or non-perturbative operator matching. Once this is available, further reduction in errors will be possible with the MILC2 or DWF1 ensembles.

2. Details of estimates

In all previous calculations of B_K statistical errors with about 100 independent lattices have been at the 1% level. The use of improved staggered fermions has been found to slightly increase the errors [7] so we assume 2% statistical errors in the following. The precise choice is not important as other errors dominate.

The first step is to calculate B_K using the present MILC0 ensemble and operators containing some form of smeared links (e.g. HYP smearing). This will result in a substantial reduction in the largest source of error, namely quenching. To estimate this we compare to the standard quenched JLQCD calculation [6]. The first issue is the size of discretization errors. The JLQCD calculation uses unimproved staggered fermions, and finds it necessary to use a minimum lattice spacing of $0.05\,\mathrm{fm}$ in order to carry out a continuum extrapolation accurate to 1%. The use of improved staggered fermions (in both the action and operators) allows one to work at a larger minimum lattice spacing for the same continuum extrapolation errors. We take the appropriate minimum value to be $a \approx 0.09$ fm.[3] Finally, we note that the MILC0 ensemble has only two lattice spacings, rather than the five used in the JLQCD extrapolation. Thus, to be conservative, we assume a 5% continuum extrapolation error. This is, nevertheless, a substantial reduction from the 14% quenching error included in the present error budget.

Chiral extrapolation is a sub-leading issue with B_K. As long as one uses a kaon with the physical mass, the dependence on the mass difference between the quark and antiquark in the "kaon" only enters at next-to-leading order in chiral perturbation theory. Assuming chiral logarithms dominate, Ref. [8] estimated that using a degenerate quark and antiquark leads to a 5% error. This assumes

[3] This is based on two arguments. First, the discretization errors are increased by a factor $(9/5)^2$ (from the change in lattice spacing) and decreased by a factor of roughly α_s (due to improvement), and these approximately cancel. Second, recent quenched results for B_K with improved staggered fermions at $a = 0.1\,\mathrm{fm}$ are consistent with the continuum extrapolated results with uinmproved staggered fermions, while lying significantly below the unimproved results at $a = 0.1$ fm [7].

that certain low-energy constants in chiral perturbation theory are not unusually large, which must be checked. The MILC0 ensemble will allow some checking, but we retain the 5% error estimate. The difference is that this error will be based on data rather than assumption.

The largest error is from operator matching. In the present quenched calculation, the error from using one-loop perturbative matching is about 5%. This estimate is obtained by comparing the results from two different discretizations of the operator, and is consistent with assuming a two loop term of size $1 \times \alpha_s^2$. This error will increases in the Asqtad ensemble, because the smallest lattice spacing is larger. The appropriate value of the coupling becomes $\alpha_s = 0.3$, implying a two-loop error of 9%.

Taking this value and adding the three errors plus statistics in quadrature gives a total error of 12% from the MILC0 ensemble. We do not include a scale error since B_K is dimensionless, although there is a small uncertainty since B_K has a logarithmic dependence on scale. A 12% error is close to the first goal of 10%.

The MILC1 ensemble will reduce the errors both in the continuum and chiral extrapolations. Indeed, at this stage one should do a combined continuum-chiral extrapolation using staggered chiral perturbation theory. The dominant logarithm in B_K has a known coefficient (as for f_π and f_K, but not f_{B_d}), which helps, although this is counterbalanced by the fact that the logarithm has a larger coefficient, so the corrections are larger than for f_π and f_K. We use a combined chiral-continuum error of 2.5% (the value proposed for f_{B_d} on the MILC0 ensemble). Combining this with 2% statistical errors and 9% perturbative errors leads to a total error of 10%.

Clearly the key to further reduction in errors is to use two-loop or non-perturbative matching. The latter is, in principle, feasible although quite involved using staggered fermions. (A test case with bilinears has been worked out in Ref. [9].) Since a non-perturbative determination will involve statistical and systematic errors of its own, it might be quite hard to lower the error below the 3% obtained with a two-loop analytic calculation. Assuming a 3% matching error, the MILC1 ensemble would yield a total error of about 5%.

Further approaches towards the chiral and continuum errors would presumably lower the error towards the limiting 3% from matching.

C. $f_{B_d}\sqrt{B_{B_d}}$ and ξ

1. Summary

Present errors in $f_{B_d}\sqrt{B_{B_d}}$ are estimated to be 15%, compared to the "other" error of 3%. In the near future, an analysis using the MILC0 ensemble and FNAL heavy quarks is expected to reduce the errors to 8–13%, with the range resulting from the uncertainty in the size of perturbative matching errors. The MILC1 ensemble will reduce extrapolation errors leading to an expected total error of 7–12%, the largest component of which is the error due to use of one-loop perturbation theory. We estimate that the inclusion of two-lop perturbation theory would reduce this error to 4–5%. To reduce the errors to 3–4% will likely require the MILC2 or DWF1 ensembles, together with the two-loop matching calculation.

For ξ, present errors are 6% (and there is as yet no measurement to compare with). In the near future, using MILC0 lattices, this will likely be reduced to 4% (assuming staggered chiral perturbation theory calculations). With the MILC1 ensemble, the expectation is a reduction to 3% errors, and a further reduction to 1.5–2% should follow with the MILC2 ensemble and two-loop perturbative matching. As noted above, we expect the DWF1 ensemble to have similar errors.

2. Details of estimates

We begin by estimating what the errors will be for the calculation presently underway on the MILC0 ensemble ($m_\ell/m_s^{\rm phys} \geq 0.12$ at $a = 0.12$ fm; $m_\ell/m_s^{\rm phys} \geq 0.22$ at $a = 0.09$). This uses improved staggered light valence quarks and the "standard" Fermilab approach for heavy quarks, correct through $O(a)$ and $1/M$. To make these estimates we compare to the recent calculation of f_π and f_K on these lattices using improved staggered valence quarks [10]. This is an appropriate benchmark since it is the only lattice calculation to date which extrapolates to the physical point with all sources of error controlled by the simulation. The results are that, for f_π and f_K, the errors are about 2% for the scale, 1 to 1.5% for the chiral plus continuum extrapolations (total, together, using staggered chiral perturbation theory) and 0.8% for statistics. For the ratio f_K/f_π, the errors are 1.5% for the chiral and continuum extrapolation and 0.5% for statistics, while the scale error is negligible.

Turning to f_{B_d}, the scale errors will remain at 2%. Assuming staggered chiral perturbation theory for heavy-light mesons has been worked out, the chiral extrapolation plus light quark $O(a^2)$

error will probably be about 2.5% (larger than for f_π because $g_{DD^*\pi}$ is not known very accurately). We take 3% for the statistical errors: heavy-lights fluctuate more than light-lights, and in old MILC results f_{B_d} had about three times the statistical error of f_π. Two new sources of error enter. The first is the heavy quark discretization error. We expect this to be smaller with the Fermilab discretization than the truncation errors of NRQCD, since the continuum limit is approachable. Our estimate is 3%, about half of the NRQCD truncation errors quoted in Ref. [4]. The second new error is that due to perturbative matching—this is absent for f_π where one can use the partially conserved current. If Z_{qQ} is computed in perturbation theory at one loop, the residual error will be $\sim 10\%$, since $\alpha_S \sim 0.3$. This may be reduced using the Fermilab trick of calculating the ratio $Z_{qQ}/\sqrt{Z_{qq}Z_{QQ}}$ perturbatively, and $\sqrt{Z_{qq}Z_{QQ}}$ nonperturbatively, because the perturbative series has smaller coefficients. What is not known about this approach at present is the size of the statistical errors in the non-perturbative determination of $\sqrt{Z_{qq}Z_{QQ}}$. Because of this, we take the range 5–10% for the matching error.

For the ratio f_{B_s}/f_{B_d}, we again estimate 2.5% errors from chiral extrapolation plus light quark discretization errors, while we expect statistical and heavy quark discretization errors to be reduced to 1% each. Perturbative errors will almost completely cancel in the ratio.

For the B-parameter B_{B_d} we estimate that the chiral plus light quark discretization error will be 2% (smaller than that for f_{B_d} since the chiral extrapolation is less steep), the statistical error will be 5% (larger than for the decay constant since this is the matrix element of a four-fermion operator), the heavy-quark discretization error will be 2% (smaller than those for the decay constants since B_{B_d} is a ratio), and the perturbation error will be 5-10% (as for f_{B_d}).

For the ratio of B-parameters, B_{B_s}/B_{B_d}, we keep the same 2% estimate of the chiral plus light quark discretization error as for B_{B_d}, but we expect, based on prior experience, some cancellation in statistical and heavy-quark discretization errors, and we assume 4% and 1% respectively.

Adding errors in quadrature gives a 7–11% error for f_{B_d}, and 8–13% for $f_{B_d}\sqrt{B_{B_d}}$. For f_{B_s}/f_{B_d}, the above considerations give 3%, and 4% for ξ. These of course would be already a lot better than the "current" errors quoted in the previous section.

We stress that the errors will increase if staggered chiral perturbation theory for heavy-light particles has not been worked out. Then one would have to extrapolate to the continuum at fixed light quark mass, and fit to continuum forms. Judging from f_π and f_K, the light quark errors will roughly double (and it is also no longer so clear that the light and heavy quark errors can be largely disentangled). This would boost the f_{B_d} error by about 1% to 8–12%, the error in $f_{B_d}\sqrt{B_{B_d}}$

to 9.5–13.5%, the f_{B_s}/f_{B_d} error to 5.2% and the ξ error to 6%.

We next consider the reduction in errors with the MILC1 ensemble. We estimate that better continuum and chiral extrapolations will reduce light quark errors in decay constants to $\sim 1.5\%$ (1% for $B-$parameters), while better control of the calculations used for setting the scale may reduce the scale errors in decay constants to 1.5%. For the heavy quark, having a smaller lattice spacing may reduce the putative 3% to 2% in decay constants, and from 2% to 1% in $B-$parameters. With more sets to fit, statistical errors should also be improved, say to 2% in decay constants, 3% in $B-$parameters. This gives 6–10% for f_{B_d} (and 7–12% for $f_{B_d}\sqrt{B_{B_d}}$), where the assumed 5–10% 1-loop perturbative error is what is holding things up. The ratios are a lot better, though. Assuming again a light quark error of 1.5% for f_{B_s}/f_{B_d} and somewhat improved statistical and heavy quark errors of 0.8% each, gives a 2.2% total error. For ξ, we estimate that this will increase to 3%.

Clearly, the key to further improvement in f_{B_d} is the reduction of the perturbative errors, using two loop (automated) perturbation theory. This reduces the 5–10% to a 1–3% perturbative error, to be compared to the sum of other lattice errors which is 3.5%. The result would be a total error of 3.5–4.5% in f_{B_d}, and 4–5% in $f_{B_d}\sqrt{B_{B_d}}$.

To go further, the MILC2 ensemble looks at present like the best next step: we expect that moving to somewhat lighter quarks at the smallest lattice spacing (where the taste violations and heavy quark errors are smallest) will pay better than pushing to very small mass on the coarser lattices ($a = .12$ or 0.09 fm). (Making the quark mass smaller doesn't help too much when the splittings are as large as the Goldstone mass.) We estimate that for decay constants this will reduce light quark, scale and statistical errors to 1% each, and heavy quark errors to 1.5% (heavy quark errors improve more slowly than light quark because of $O(a)$ vs. $O(a^2)$—of course the $O(a^2)$ Fermilab version would help further). For $B-$parameters, we estimate 0.6% light quark errors, 1.5% for statistical errors, and 0.8% for heavy quark discretization errors. Assuming a two-loop macthing calculation, this would give total errors of 2.5–3.8% in f_{B_d} calculation, and 2.7–4.2% on $f_{B_d}\sqrt{B_{B_d}}$. For the ratios, perturbation theory is not important. Assuming 1% light quark errors and 0.5% heavy quark and statistical errors, gives 1.2% for f_{B_s}/f_{B_d} and 1.5–2% for ξ.

D. $b \to u$ semileptonic form factors

1. Summary

We concentrate on the form factor f_+ in $B \to \pi\ell\nu$ decays at small pion recoil momenta. Present lattice errors are estimated to be about 15%, compared to experimental errors of 7%. We find that using the existing MILC0 ensemble will reduce the errors to 10–13%, where the range is due to the uncertainty in the size of perturbative errors on the MILC0 data set. These will improve to 7.5–12% on the MILC1 data set (or 5.5-6.5% with 2-loop perturbation theory), and 4–5% errors on the MILC2 ensemble (assuming 2-loop perturbation theory).

2. Details of estimates

Many errors in heavy-light semileptonic decays are very similar to those in the heavy-light leptonic decay constants discussed in the previous section. We therefore make an estimate of errors for $B \to \pi\ell\nu$ form factors by doing a comparison with those for f_{B_d} (expressing them as the ratio of percentage error in the form factor f_+ to that in f_{B_d}). Our estimates are based on our quenched studies of both quantities, using the same set of lattices and gauge configurations. The computational effort associated with the $B \to \pi$ study was higher, because additional quark propagators were needed to compute the three-point functions. Since we have results only for small pion recoil momenta, $p_\pi \leq 1$ GeV, the following statements are for this case only.

We find the following:

- Statistical errors increase by a factor of $1.3 - 1.5$ in comparison with those for f_{B_d}, due to the finite pion momentum.
- Chiral extrapolation errors are harder to estimate. It is not clear if we can assume that there will be a staggered chiral perturbation theory analysis available for this case. We assume a factor of 2 increase for this error.
- For the semileptonic decays, we saw lattice spacing dependence, which was similar in size to the statistical errors, while it was smaller than the statistical errors in the f_{B_d} case. Hence we estimate a factor of 2 increase for this error.

- Errors due to heavy quark discretization and (one-loop) perturbative matching are about the same in both cases.

- The error due to the uncertainty in a (scale) is smaller for form factors than for f_{B_d}, since they are dimensionless. This error is reduced by a factor of 2.

Combining these estimates with those of the previous section for f_{B_d} yields 10–13% errors on the MILC0 data set (assuming 1-loop perturbation theory), 7.5–12% errors on the MILC1 data set with 1-loop perturbation theory, 5.5–6.5% on the MILC1 data set with 2-loop perturbation theory, and 4–5% errors on the MILC2 data set (assuming 2-loop perturbation theory).

E. Contributing to the determination of V_{cb}

The aim here, as discussed above, is ultimately to reduce the errors to the 1.5% level. The present situation on the lattice is that the methodology for a very precise calculation has been developed, but there are as yet only quenched calculations for which it is difficult to reliably estimate the total error.

Our estimates are based on the Fermilab calculations of the endpoint form factors, $\mathcal{F}(1)$, for $B \to D\ell\nu$ and $B \to D^*\ell\nu$, using the ratio method. With the ratio method, the errors scale with $\mathcal{F}(1) - 1$, and are in general much smaller than in the f_{B_d} case. As a result, the statistical, perturbative, and lattice spacing and scale errors are expected to be multiplied by a factor of 0.3.

The error due the chiral extrapolation is similarly suppressed, but due to the likely lack of knowledge of staggered chiral perturbation theory, we estimate a factor of 0.7. Combining these estimates with those for f_{B_d} yields 3–4% errors on the MILC0 data set (assuming 1-loop perturbation theory), 2.3–3.4% errors on the MILC1 data set with 1-loop perturbation theory, 1.8–2.0% on the MILC1 data set with 2-loop perturbation theory, and 1–1.4% errors on the MILC2 data set (assuming 2-loop perturbation theory).

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