# Moyal Formulation of Witten's Star Product in the Fermionic Ghost Sector

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**Abstract**

In this paper, we recast the fermionic ghost sector of Witten's open bosonic string field theory in the language of noncommutative field theory. In particular, following the methods of hep-th/0202087, we find that in Siegel gauge Witten's star product roughly corresponds to a continuous tensor product of Clifford Algebras, and we formulate important operators of the theory in this language, notably the kinetic operator of vacuum string field theory and the BRST operator describing the vacuum of the unstable D-25 brane. We find that the BRST operator is singular in this formulation. We explore alternative operator/Moyal representations of the star product analogous to the split string description and the discrete Moyal basis developed extensively in recent work by Bars and Matsuo (hep-th/0204260). Finally, we discuss some interesting singularities in the formalism and how they may be regulated.

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# I. Introduction

Since Sen described his famous conjectures on tachyon condensation[1], there has been renewed interest in the framework of string field theory[2], both as a tool for studying tachyon condensation and as a possible nonperturbative and background independent formulation of string theory. To this day, most of our knowledge of the theory has come about through more or less brute force numerical analysis of the string field equations[3]—level truncation. Some analytic insight however has emerged through the study of vacuum string field theory (VSFT), a theory constructed so as to supposedly describe physics around the locally stable closed string vacuum, and which possesses a particularly simple kinetic operator, $\mathcal{Q} = c^+(\frac{\pi}{2})+c^-(\frac{\pi}{2})$ [4, 5]. However, it remains unclear whether VSFT is a nonsingular or even accurate description of string theory—around the closed string vacuum or otherwise. We need more powerful analytic techniques, particularly to study the full open string field theory, whose kinetic operator, the BRST operator, is much more complicated.

Other than the usual oscillator and conformal field theory methods[6, 7, 8], three alternative approaches have been recently proposed for studying string field theory in a possibly simpler framework: the split string formalism[9, 10, ?, 12], the discrete Moyal formalism advocated by Bars and Matsuo[13, 14], and the continuous Moyal formalism[15, 16, 17, 18]. All of these approaches have a common goal, which is to reformulate string field theory in the language of operator algebras and noncommutative geometry[19, 20]. Witten's star product is then simple, roughly corresponding to matrix multiplication or noncommutative Moyal star multiplication. Of course, these two descriptions are naively isomorphic. This represents a marked improvement over the usual oscillator approach, where the star product is notoriously complicated. However, the down side of these new approaches is that the BRST operator seems very difficult to formulate, and may in fact be ill-defined. Worse, in the split string formalism it does not even seem possible to accommodate operators which act nontrivially on the string midpoint, such as the pure ghost $\mathcal{Q}$ of VSFT.

Of course, a clear understanding of these issues requires an adequate formulation of the ghost sector of the theory. For the most part, the ghost sector has been avoided for the simple reason that it is unclear how Witten's product in the ghost sector should be described as either a matrix or Moyal product. In references [10, 14] the approach has been to bosonize the ghosts, in which case the star product is given by a simple half-string overlap (times a phase factor inserting ghost momentum at the midpoint) which makes an operator/Moyal description seem natural. However, in the bosonized language the BRST operator is extremely complicated, and it becomes a labor even to verify the axioms of string field theory for the extremely simple pure ghost kinetic operators studied in ref.[10]. Furthermore, no nontrivial solution to VSFT has been satisfactorily constructed using the bosonized ghosts. Therefore it would seem extremely advantageous to find an operator/Moyal formulation of string field theory using the ordinary fermionic ghosts. It is known, for instance, that star multiplication in the fermionic ghost sector roughly corresponds to a half string overlap in the

$b$ ghost and an *anti*overlap (overlap up to a sign) in the $c$ ghost. This suggests that it may be possible to write the fermionic star product as a matrix product but with an additional operator insertion to impose the requisite anti-overlap conditions on the $c$ ghost:

$$\Psi * \Phi \sim \hat{\Psi}\hat{I}^{-1}\hat{\Phi} \tag{1}$$

The half string operator $\hat{I}^{-1}$ should be the inverse of the identity string field[1]. However, if one pursues this avenue, one finds that the identity string field is roughly,

$$\hat{I} = N\bar{b}\int [d\lambda][d\mu]|\lambda,\mu\rangle\langle\lambda,-\mu| \tag{2}$$

where $N$ is a normalization and,

$$\bar{b} = b(\tfrac{\pi}{2}), \quad \lambda(\sigma) = -\frac{2}{\pi}\sigma\bar{b} + b(\sigma), \quad \mu(\sigma) = c(\sigma)$$

for $\sigma \in [0, \frac{\pi}{2}]$. The problem is that $\hat{I}$ is not invertable! The $\bar{b}$ multiplying the integral clearly has no inverse, and comes from necessary midpoint insertions[7, 21] giving the identity string field the correct ghost number. This is somewhat baffling since in the bosonized ghost language $\hat{I}$ is a perfectly invertible number, $M^{-1} = e^{-i3\bar{\phi}/2}$, where $\bar{\phi}$ is the midpoint value of the bosonized ghost. Nevertheless, it is clearly impossible to make sense of eq.(1) given an identity field which is not invertible, so it is very unclear how the fermionic star product may be realized as any sort of matrix or Moyal product.

However recent studies of the spectrum of eigenvalues of the Neumann coefficients[22] have made it possible to choose a (continuous) basis of oscillators which diagonalize the quadratic form appearing in zero momentum three string vertex[15]. The authors of [15] discovered that the diagonalized vertex exactly describes a continuous tensor product of Heisenberg algebras, and were able to make a direct connection with the split string and discrete Moyal formalisms developed earlier. The remarkable thing about their approach is that it can be extended straightforwardly to the fermionic ghost sector, since in Siegel gauge the three string ghost vertex may be diagonalized by more or less the same techniques. In this way we may hope to recover an operator/Moyal formulation of the fermionic ghost star product even though other attempts have failed[10, ?].

This is the subject of the current paper. We find that, in Siegel gauge, the so-called reduced star product[23] ($= b_0$ times the full star product) corresponds to a continuous tensor product, from $\kappa = 0$ to $\infty$, of a pair of Clifford Algebras $Cl_{1,1}$, the first with a metric scaling as $\coth\frac{\pi\kappa}{4}$ and the second with a metric scaling as $-\coth\frac{\pi\kappa}{4}$. Clifford Algebra, in a sense we will explain, can be thought of as defining non-anticommutative geometry on a Grassmann space. Therefore our results show that string field theory may indeed be formulated in the language of noncommutative field theory. We take advantage of this and

[1] Note that the identity string field serves as the identity of Witten's star product, *not* the matrix product of half string operators. This is why the insertion $\hat{I}^{-1}$ in eq.(1) is nontrivial

formulate both the kinetic operator of VSFT and the BRST operator in this new basis. The BRST operator in particular is divergent, and must somehow be regulated to get a consistent formulation. We use our results in the continuous Moyal formalism to construct split string and discrete Moyal representations of the reduced star product in Siegel gauge. We hope these results encourage new progress in formulating string field theory in a language where Witten's product is simple.

As a matter of notation, we will denote the Moyal product with a five pointed star, $\star$, Witten's product with a six pointed one $*$, and the reduced star product with $*_{b_0}$.

# II. Non-anticommutative geometry on Grassmann spaces

In this section we would like to develop the notion of non(anti)commutative geometry on a space with Grassmann valued coordinates[2]. Let us start with familiar notions from noncommutative geometry on ordinary flat space. Given any vector space $\Re^{2n}$ with a nondegenerate 2-form $\theta$, we can construct an associative, noncommutative algebra $\mathcal{A}_\theta$ defined as the quotient:

$$\mathcal{A}_\theta \equiv \frac{\mathbf{C}\otimes \mathrm{Tensor}(\Re^{2n})}{\{[\mathbf{u},\mathbf{v}]\sim i\theta(\mathbf{u},\mathbf{v}),\quad \mathbf{u},\mathbf{v}\in\Re^{2n}\}} \tag{3}$$

Where $\mathrm{Tensor}(\Re^{2n})$ denotes the tensor algebra over $\Re^{2n}$. Intuitively, $\mathcal{A}_\theta$ is the algebra generated by taking all products and complex linear combinations of $2n$ basis vectors $\mathbf{x}_i$ for $\Re^{2n}$ subject to the multiplication rule $[\mathbf{x}_j,\mathbf{x}_k]=i\theta_{jk}$. We should think of $\mathcal{A}_\theta$ as a noncommutative deformation of the algebra[3] of continuous complex functions on $\Re^{2n}$. In this sense $\mathcal{A}_\theta$ defines what we mean by *noncommutative geometry* on $\Re^{2n}$. Since $\theta$ is nondegenerate, we can always choose a basis so that $\theta_{ij}$ is block diagonal, and in this basis we have:

$$[\mathbf{x}_{2j-1},\mathbf{x}_{2j}]=i,\quad j=1,...,n, \tag{4}$$

with all other commutators vanishing. With simple renaming of variables, $\hat{\mathbf{x}}_i=\mathbf{x}_{2i-1}$ and $\hat{\mathbf{p}}_i=\mathbf{x}_{2i}$, the commutation relations read:

$$[\hat{\mathbf{x}}_j,\hat{\mathbf{p}}_k]=i\delta_{jk},\quad [\hat{\mathbf{x}}_j,\hat{\mathbf{x}}_k]=0,\quad [\hat{\mathbf{p}}_j,\hat{\mathbf{p}}_k]=0.$$

This, of course, looks just like the canonical commutation relations for the position and momentum operators describing the quantum mechanics of a point particle moving in $n$ spatial dimensions. To make this correspondence exact, however, we must define a Hilbert space so that the $\mathbf{x}_i$'s act as Hermitian operators. This defines a representation of the algebra $\mathcal{A}_\theta$, which, in turn, induces a natural *involution* on $\mathcal{A}_\theta$ corresponding to Hermitian conjugation.

[2]Related discussion can be found in [16, 24].

[3]The algebra of continuous functions is, naturally, the algebra defined with the addition and multiplication laws $(f\cdot g)(x)=f(x)g(x)$ and $(f+g)(x)=f(x)+g(x)$.

Our discussion suggests that we might define non-anticommutative geometry on a *Grassmann space* by following the above argument in reverse. In particular, start with the quantum mechanics of a particle with Grassmann-valued coordinates; then we have hermitian position and momentum operators $\hat{\xi}_i$, $\hat{\pi}_i$ $i = 1, ..., n$ satisfying canonical anticommutation relations:

$$\{\hat{\xi}_j, \hat{\pi}_k\} = \delta_{jk}, \quad \{\hat{\xi}_j, \hat{\xi}_k\} = 0, \quad \{\hat{\pi}_j, \hat{\pi}_k\} = 0.$$

Now simply rename $\hat{\xi}_i = \gamma_{2i-1}$ and $\hat{\pi}_i = \gamma_{2i}$, then we get:

$$\{\gamma_{2j-1}, \gamma_{2j}\} = 1,$$

with all other anticommutators vanishing. Through a (real) linear change of basis, $\gamma_i \to M_{ij}\gamma_j$ with $M = M^*$ this relation can be put in the general form:

$$\{\gamma_i, \gamma_j\} = 2g_{ij} \tag{5}$$

with

$$g = \tfrac{1}{2} M f M^T, \qquad f = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

We instantly recognize eq.(5) as the defining relation for a Clifford Algebra. Note in particular that we can diagonalize $f$ through a suitable choice of $M$:

$$M = \begin{pmatrix} 1 & -1 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 1 & -1 \\ & & & 1 & 1 \end{pmatrix}, \quad \Longrightarrow \quad g = \begin{pmatrix} -1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & -1 & 0 \\ & & & 0 & 1 \end{pmatrix}.$$

Therefore, eq.(5) defines a Clifford Algebra with respect to a metric of signature $(n, n)$. Putting this all together, we define an associative algebra:

$$\mathcal{B}_g = \mathbf{C} \otimes Cl_{n,n} = \frac{\mathbf{C} \otimes \mathrm{Tensor}(\Re^{2n})}{\{\mathbf{v}^2 \sim g(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \Re^{2n}\}}, \tag{6}$$

which is a non-anticommutative deformation of the (Grassmann) algebra of complex functions on a $2n$ dimensional Grassmann space. This algebra defines what we mean by non-anticommutative geometry on a Grassmann space.

An important point: Since we derived $\mathcal{B}_g$ from the quantum mechanics of a fermionic point particle, the algebra inherits a natural involution corresponding to Hermitian conjugation. This allows us to distinguish between the algebras $\mathbf{C} \otimes Cl_{n,n}$ and $\mathbf{C} \otimes Cl_{k,2n-k}$, which after all only differ by absorbing a factor of $i$ in the choice of the basis. However, this type of redefinition is not allowed, since a new basis obtained this way would not be hermitian. This is why we required $M = M^*$ in eq.(5). This, however, raises the question of whether we should generalize our definition of non-anticommutative geometry by setting $\mathcal{B}_g = \mathbf{C} \otimes Cl_{p,q}$ for arbitrary metric signature $(p,q)$. The problem is that these more general algebras will not have a simple representation in terms of the quantum mechanics of a fermionic point particle. For the case of ordinary noncommutative geometry, the correspondence to the quantum mechanics of a point particle is automatic, regardless of our choice of $\theta$. This is not true for the fermionic case regardless of our choice of $g$. For the purposes of this paper, the narrower definition eq.(6) of $\mathcal{B}_g$ is the most appropriate.

Often in noncommutative geometry we represent the algebra $\mathcal{A}_\theta$ as an algebra of continuous functions on $\Re^{2n}$ which are multiplied together with the noncommutative and nonlocal Moyal star product. We can similarly find a representation of $\mathcal{B}_g$ in terms of a nonlocal product between functions on a Grassmann space (whose coordinates we write as $\phi_i, i = 1...2n$),

$$f \star g(\phi) = \exp\left[g_{ij}\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi'_j}\right] f(\phi')g(\phi)\Big|_{\phi=\phi'} . \tag{7}$$

The relationship between this analytic representation of $\mathcal{B}_g$ and the algebraic representation described in the last paragraph is given precisely as follows. Given a function $f(\psi)$ on our Grassmann space, we can construct an anti-symmetrically ordered operator $O_f$ as follows,

$$O_f \equiv \int d^{2n}\phi \tilde{f}(\phi)e^{-\phi^i\gamma_i}, \quad \tilde{f}(\phi) = \int d^{2n}\psi f(\psi)e^{\phi^i\psi_i},$$

where the operators $\gamma_i$ satisfy eq.(5). Then we can write:

$$O_f O_g = O_{f\star g}.$$

The operator and Moyal representations of the algebra are isomorphic.

Let's focus on the most elementary case of a two dimensional Grassmann space with coordinates $x$ and $y$ and metric $g_{ij} = g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij}$. For short we will denote $X = (x,y)$. For the purposes of this paper we will find it extremely useful to represent non-anticommutative multiplication in terms of a fermionic integral kernel,

$$f \star g(X^3) = \int \prod_{B=1,2} dx^B dy^B K(X^1, X^2, X^3) f(X^1) g(X^2), \tag{8}$$

From eq.(7) we know that,

$$f\star g(X^3) = \int dx^1 dy^1 dx^2 dy^2 (y^2-y^3)(x^2-x^3)(y^1-y^3)(x^1-x^3)e^{g(\partial_x^2\partial_y^1+\partial_y^2\partial_x^1)} f(X^1)g(X^2).$$

Integrating by parts, we can take the derivatives off of $f$ and $g$ leading to an expression for the kernel:

$$K(X^1, X^2, X^3) \quad = \quad e^{g(\partial_x^2\partial_y^1+\partial_y^2\partial_x^1)}(y^2-y^3)(x^2-x^3)(y^1-y^3)(x^1-x^3)$$

$$
\begin{aligned}
&= (y^2-y^3)(x^2-x^3)(y^1-y^3)(x^1-x^3) - g(y^2-y^3)(x^1-x^3) \\
&\quad -g(x^2-x^3)(y^1-y^3)+g^2 \\
&= g^2 \exp\left[g^{-1}(x^1y^2+x^2y^3+x^3y^1-x^2y^1-x^3y^2-x^1y^3)\right]
\end{aligned}
$$

More succinctly,

$$
K(X^1,X^2,X^3) = g^2 e^{x^A K^{AB} y^B}, \qquad K^{AB} = g^{-1}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{AB}, \tag{9}
$$

with an implied summation over the indices $A, B = 1,2,3$. The kernel is invariant under cyclic permutations of $1,2,3$. An important property of this kernel is,

$$
(\partial^1+\partial^2+\partial^3)K(X^1,X^2,X^3) = 0
$$

where $\partial$ represents a derivative with respect to either $x$ or $y$. This statement is equivalent to the nontrivial fact that $\partial$ acts as a derivation of the $\star$ algebra,

$$
\partial(f\star g) = (\partial f)\star g + (-1)^{G(f)} f\star \partial g
$$

where $G(f)$ is the Grassmann rank of $f$ (assuming it has a definite rank). This allows us to represent the action of the derivative on the algebra as a star (anti)commutator, just as in ordinary noncommutative field theory:

$$
\partial_x f(x,y) = \frac{1}{2g}\left(y\star f(x,y) - (-1)^{G(f)} f(x,y)\star y\right) \tag{10}
$$

with a similar relation for $\partial_y$. To see that these two operations are equivalent, all we have to do is check that they act identically on $1$, $x$, and $y$ and it follows that they will act identically on $f(x,y)$ from linearity and the above derived product rule.

# III. Diagonalizing the fermionic ghost vertex

We will now simplify the three-string vertex in the fermionic ghost sector, along similar lines as ref.[15], by choosing a basis of operators which diagonalize the quadratic form appearing in the vertex. In this paper we use the same conventions for the fermionic ghosts as [25, 7]. The three string vertex in the fermionic ghost sector was constructed in ref.[7, 21]:

$$
|V_3\rangle = \exp\left[-b_0^A \tilde{N}^{AB}_{0m}\sqrt{m}c_m^{B+} - b_m^{A+}\frac{1}{\sqrt{m}}\tilde{N}^{AB}_{mn}\sqrt{n}c_n^{B+}\right]|+\rangle_1|+\rangle_2|+\rangle_3. \tag{11}
$$

Here, $A, B$ are Hilbert space indices ranging from one to three, and $m, n$ are level number indices ranging from 1 to $\infty$, repeated indices summed. Given two string fields $|\phi\rangle$, $|\psi\rangle$ we calculate[4] their star product by evaluating,

$$
|\phi*\psi\rangle = \langle\phi|\langle\psi||V_3\rangle \tag{12}
$$

[4]For simplicity we will assume our string fields are real, so eq.(12) can be taken as the definition of the star product. For fields which are not real, the left hand side of eq.(12) should be $\langle\phi*\psi||V_2\rangle$ where $|V_2\rangle$ is the two string vertex.

We will find it useful to restrict ourselves to evaluating the star product in Siegel gauge. String fields will then be of the form $|\psi\rangle = |\hat{\psi}\rangle \otimes |-\rangle$ where $|\hat{\psi}\rangle$ is in the Fock space generated by the level $n > 1$ $bc$-oscillators. Multiplying eq.(12) by $b_0$, we obtain a formula for the so-called "reduced"[23] star product in Siegel gauge:

$$|\phi *_{b_0} \psi\rangle \equiv b_0|\phi * \psi\rangle = |-\rangle \otimes \langle\hat{\phi}|\langle\hat{\psi}||\hat{V}_3\rangle, \tag{13}$$

where,

$$|\hat{V}_3\rangle = \exp\left[-b_m^{A+}\frac{1}{\sqrt{m}}\tilde{N}^{AB}_{mn}\sqrt{n}c_n^{B+}\right]|0\rangle_1|0\rangle_2|0\rangle_3. \tag{14}$$

with $|0\rangle$ being the vacuum annihilated by $b_n, c_n$ for $n \geq 1$. As explained in ref.[23], if we know the reduced star product of two string fields in Siegel gauge, at least formally we can calculate their full star product using the relation:

$$|\phi * \psi\rangle = \mathcal{Q}|\phi *_{b_0} \psi\rangle \tag{15}$$

where $\mathcal{Q} = c^+(\frac{\pi}{2}) + c^-(\frac{\pi}{2})$ is the canonical choice of pure ghost kinetic operator in vacuum string field theory, as discussed in [27, 5, 26]. Equation (15) follows from the observation of ref.[23] that the star product of two fields in Siegel gauge is formally $\mathcal{Q}$ exact.

To better understand the structure of the star product for the fermionic ghosts, we can therefore gauge fix and focus on the vertex for the reduced star product, $|\hat{V}_3\rangle$. Following [15], we will simplify this vertex by choosing a particular basis of oscillators in which the Neumann coefficients $\tilde{N}^{AB}$ are diagonal. To do this, however, we must understand the spectrum of eigenvectors and eigenvalues of $\tilde{N}^{AB}$. Fortunately, this is not too difficult, since the $\tilde{N}^{AB}$ are closely related to the zero momentum Neumann coefficients $N^{AB}$ appearing in the matter sector vertex[6] whose eigenvectors and eigenvalues have been explicitly calculated in [22].

It is convenient to define matrices $M^{AB} \equiv CN^{AB}$ ($C_{mn} = (-1)^m\delta_{mn}$) which satisfy the important relations[27]:

$$\begin{gathered}[M^{AB}, M^{A'B'}] = 0, \quad (M^{AB})^+ = M^{AB}, \quad M^{AB} = M^{A+1,B+1} \\ M + M^{12} + M^{21} = 1, \quad M^{12}M^{21} = M(M-1).\end{gathered} \tag{16}$$

These relations imply that there are three independent Neumann matrices, $M^{12}$, $M^{21}$, and $M^{11} = M$ which can all be simultaneously diagonalized. Similar relations hold for the ghost matrices $\tilde{M}^{AB} \equiv C\tilde{N}^{AB}$. In fact, we can write the ghost matrices $\tilde{M}^{AB}$ explicitly in terms of the matrices $M^{AB}$ as follows[5]:

$$\tilde{M}^{12} = \frac{1 + M - M^{21}}{1 + 2M}, \quad \tilde{M}^{21} = \frac{1 + M - M^{12}}{1 + 2M} \quad \tilde{M} = -\frac{M}{1 + 2M}. \tag{17}$$

These nonlinear relations are different from the usual formula defining $\tilde{M}^{AB}$ in terms of six string Neumann coefficients[7], but they can be implied from

[5] Our expressions for the Neumann matrices differ from those in ref.[17].

the expressions in ref.[7] or derived independently from the Moyal formalism developed extensively in ref.[14] (in particular, from equations 5.35 and 5.47 of that reference). It is an easy check to verify that the ghost matrices in eq.(17) satisfy eq.(16). Clearly the eigenvectors of $M^{AB}$ will also be eigenvectors of $\tilde{M}^{AB}$.

So let us recall the basic results of [22, 26]. The matrices $M^{AB}$ have a continuous spectrum of eigenvectors $v_n(\kappa)$ for $-\infty < \kappa < \infty$:

$$M^{AB}_{mn} v_n(\kappa) = \mu^{AB}(\kappa) v_m(\kappa), \tag{18}$$

with,

$$\begin{aligned} \mu(\kappa) &= -\frac{1}{1+2\cosh\frac{\pi\kappa}{2}}, \\ \mu^{12}(\kappa) &= \frac{1+\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}, \\ \mu^{21}(\kappa) &= \frac{1+\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}. \end{aligned} \tag{19}$$

The special functions $v_n(\kappa)$ can be derived from the generating functional:

$$\sum_{n=1}^{\infty} \frac{v_n(\kappa)}{\sqrt{n}} z^n = \frac{1-e^{-\kappa \tan^{-1} z}}{\kappa\sqrt{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}} \tag{20}$$

The normalization factor in the denominator is an odd function of $\kappa$. The functions $v_n(\kappa)$ enjoy the following properties:

$$\begin{aligned} (-1)^{n+1} v_n(\kappa) &= v_n(-\kappa), \\ v_n(\kappa)v_n(\kappa') = \delta(\kappa-\kappa'), \qquad & \int_{-\infty}^{\infty} d\kappa\, v_n(\kappa) v_n(\kappa) = \delta_{mn}. \end{aligned} \tag{21}$$

Given equations (17,18,19) we deduce,

$$\tilde{M}^{AB}_{mn} v_n(\kappa) = \tilde{\mu}^{AB} v_m(\kappa), \tag{22}$$

with,

$$\begin{aligned} \tilde{\mu}(\kappa) &= \frac{1}{2\cosh\frac{\pi\kappa}{2}-1}, \\ \tilde{\mu}^{12}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}-1}{2\cosh\frac{\pi\kappa}{2}-1}, \\ \tilde{\mu}^{21}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}-1}{2\cosh\frac{\pi\kappa}{2}-1}. \end{aligned} \tag{23}$$

Apparently, the eigenvalues of $\tilde{M}$ lie in the range $(0, 1]$ and are doubly degenerate for $\kappa \neq 0$ since $v_n(\kappa)$ and $v_n(-\kappa)$ are linearly independent but have the same eigenvalue: $\tilde{\mu}(\kappa) = \tilde{\mu}(-\kappa)$. The $\kappa = 0$ eigenvalue $\tilde{\mu}(0) = 1$ is the exception, having only one twist odd eigenvector $v_{2n-1}(0)$.

We will find it helpful to decompose the set of functions $v_n(\kappa)$ into odd/even components, $v_{2n}(\kappa)$ and $v_{2n-1}(\kappa)$. By symmetry, $v_{2n}(0) = 0$. The ghost matrices $\tilde{M}^{AB}$ act on $v_{2n}, v_{2n-1}$ as follows,

$$\begin{aligned}
\tilde{M}^{AB}_{2n,2m} v_{2m}(\kappa) &= \tfrac{1}{2}[\tilde{\mu}^{AB}(\kappa) + \tilde{\mu}^{BA}(\kappa)] v_{2n}(\kappa), \\
\tilde{M}^{AB}_{2n-1,2m} v_{2m}(\kappa) &= \tfrac{1}{2}[\tilde{\mu}^{AB}(\kappa) - \tilde{\mu}^{BA}(\kappa)] v_{2n-1}(\kappa), \\
\tilde{M}^{AB}_{2n-1,2m-1} v_{2m-1}(\kappa) &= \tfrac{1}{2}[\tilde{\mu}^{AB}(\kappa) + \tilde{\mu}^{BA}(\kappa)] v_{2n-1}(\kappa), \\
\tilde{M}^{AB}_{2n,2m-1} v_{2m-1}(\kappa) &= \tfrac{1}{2}[\tilde{\mu}^{AB}(\kappa) - \tilde{\mu}^{BA}(\kappa)] v_{2n}(\kappa),
\end{aligned} \tag{24}$$

Now eq.(21) takes the form,

$$v_{2m-1}(\kappa) v_{2m-1}(\kappa') = \tfrac{1}{2}\delta(\kappa - \kappa') \qquad \int_0^\infty d\kappa\ v_{2m-1}(\kappa) v_{2n-1}(\kappa) = \tfrac{1}{2}\delta_{2m-1,2n-1}$$

$$v_{2m}(\kappa) v_{2m}(\kappa') = \tfrac{1}{2}\delta(\kappa - \kappa') \qquad \int_0^\infty d\kappa\ v_{2m}(\kappa) v_{2n}(\kappa) = \tfrac{1}{2}\delta_{2m,2n} \tag{25}$$

We have now gathered all the technology we need to diagonalize the quadratic form appearing in the reduced three string vertex. We will write the vertex in terms of the following continuous collection of oscillators:

$$\beta_{\rm o}(\kappa) \equiv \sqrt{2}\sum_{n=1}^\infty \frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}} b_{2n-1} \qquad \beta_{\rm e}(\kappa) \equiv i\sqrt{2}\sum_{n=1}^\infty \frac{v_{2n}(\kappa)}{\sqrt{2n}} b_{2n},$$

$$\chi_{\rm o}(\kappa) \equiv \sqrt{2}\sum_{n=1}^\infty \sqrt{2n-1}\, v_{2n-1}(\kappa) c_{2n-1} \qquad \chi_{\rm e}(\kappa) \equiv i\sqrt{2}\sum_{n=1}^\infty \sqrt{2n}\, v_{2n}(\kappa) c_{2n}. \tag{26}$$

These satisfy a $bc$ oscillator algebra with a continuous mode label:

$$\{\beta_{\rm e}^+(\kappa), \chi_{\rm e}(\kappa')\} = \{\beta_{\rm o}^+(\kappa), \chi_{\rm o}(\kappa')\} = \delta(\kappa - \kappa'). \tag{27}$$

All other distinct anticommutators between the $\beta$s and $\chi$s vanish. To avoid cluttered notation, we will often write $\vec{\beta}(\kappa) = \begin{pmatrix} \beta_{\rm e}(\kappa) \\ \beta_{\rm o}(\kappa) \end{pmatrix}$, and similarly for $\vec{\chi}$, and sometimes suppress the "index" $\kappa$ whenever this won't lead to confusion. We can invert eq.(26) using eq.(25),

$$b^+_{2n-1} = \sqrt{2}\sqrt{2n-1}\int_0^\infty d\kappa\ v_{2n-1}\beta_{\rm o}^+ \qquad b^+_{2n} = i\sqrt{2}\sqrt{2n}\int_0^\infty d\kappa\ v_{2n}\beta_{\rm e}^+,$$

$$c^+_{2n-1} = \sqrt{2}\frac{1}{\sqrt{2n-1}}\int_0^\infty d\kappa\ v_{2n-1}\chi_{\rm o}^+ \qquad c^+_{2n} = i\sqrt{2}\frac{1}{\sqrt{2n}}\int_0^\infty d\kappa\ v_{2n}\chi_{\rm e}^+. \tag{28}$$

Now simply plug these formulas into the quadratic form,

$$
\begin{aligned}
& b_m^{A+}\tfrac{1}{\sqrt{m}}\tilde{N}^{AB}_{mn}\sqrt{n}c_n^{B+} \\
&= b_{2m}^{A+}\sqrt{\frac{2n}{2m}}\tilde{M}^{AB}_{2m,2n}c_{2n}^{B+} + b_{2m}^{A+}\sqrt{\frac{2n-1}{2m}}\tilde{M}^{AB}_{2m-1,2n}c_{2n-1}^{B+} \\
&\quad -b_{2m-1}^{A+}\sqrt{\frac{2n-1}{2m}}\tilde{M}^{AB}_{2m,2n-1}c_{2n}^{B+} - b_{2m-1}^{A+}\sqrt{\frac{2n-1}{2m-1}}\tilde{M}^{AB}_{2m-1,2n-1}c_{2n-1}^{B+} \\
&= \sum_{m,n=1}^{\infty}\Big[-2\int_0^\infty d^2\kappa\ \beta_e^{A+}(\kappa)\chi_e^{B+}(\kappa')v_{2m}(\kappa)\tilde{M}^{AB}_{2m,2n}v_{2n}(\kappa') \\
&\quad +2i\int_0^\infty d^2\kappa\ \beta_e^{A+}(\kappa)\chi_o^{B+}(\kappa')v_{2m}(\kappa)\tilde{M}^{AB}_{2m,2n-1}v_{2n-1}(\kappa') \\
&\quad -2i\int_0^\infty d^2\kappa\ \beta_o^{A+}(\kappa)\chi_e^{B+}(\kappa')v_{2m-1}(\kappa)\tilde{M}^{AB}_{2m-1,2n}v_{2n}(\kappa') \\
&\quad -2\int_0^\infty d^2\kappa\ \beta_o^{A+}(\kappa)\chi_o^{B+}(\kappa')v_{2m-1}(\kappa)\tilde{M}^{AB}_{2m-1,2n-1}v_{2n-1}(\kappa')\Big] \\
&= -\tfrac{1}{2}\int_0^\infty d\kappa\ \big[(\tilde{\mu}^{AB}+\tilde{\mu}^{BA})\vec{\beta}^{B+}\cdot\vec{\chi}^{B+} + (\tilde{\mu}^{AB}-\tilde{\mu}^{BA})\vec{\beta}^{A+}\cdot\sigma_y\cdot\vec{\chi}^{B+}\big].
\end{aligned}
$$

Where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Explicitly, then, the reduced three string vertex can be written in the following form:

$$
\begin{aligned}
|\hat{V}_3\rangle &= \exp\Big[\tfrac{1}{2}\int_0^\infty d\kappa\ \Big\{2\tilde{\mu}(\vec{\beta}^{1+}\cdot\vec{\chi}^{1+} + \text{cyclic.}) \\
&\quad +(\tilde{\mu}^{12}+\tilde{\mu}^{21})(\vec{\beta}^{1+}\cdot\vec{\chi}^{2+} + \vec{\beta}^{2+}\cdot\vec{\chi}^{1+} + \text{cyclic.}) \\
&\quad +(\tilde{\mu}^{12}-\tilde{\mu}^{21})(\vec{\beta}^{1+}\cdot\sigma_y\cdot\vec{\chi}^{2+} - \vec{\beta}^{2+}\cdot\sigma_y\cdot\vec{\chi}^{1+} + \text{cyclic})\Big\}\Big] \\
&\quad \times|0\rangle_1|0\rangle_2|0\rangle_3
\end{aligned}
\tag{29}
$$

So, we have succeeded diagonalizing the quadratic form in the vertex. This form makes it clear that the reduced star algebra in the fermionic ghost sector is a continuous tensor product[6], from $\kappa = 0$ to $\infty$, of mutually commuting algebras. Our task is now to understand the nature of these algebras.

# IV. Moyal structure in the ghost vertex

On general grounds we expect the star product in the fermionic ghost sector to be closely related to a Moyal product. Somehow we need to translate the Fock space representation of the star product, eq.(29), into an integral kernel

[6]When we talk about taking a "tensor product" of two algebras in this paper, we mean it in a special sense. The naive multiplication rule for the tensor product of two algebras is $(a\otimes c)(b\otimes d) = ab\otimes cd$ for $a$, $b$ in the first algebra and $c$, $d$ in the second. This is not the natural definition here, since we're dealing with Grassmann quantities. Rather, we should define $(a\otimes c)(b\otimes d) = (-1)^{G(b)+G(c)}ab\otimes cd$, where $G(b)$ and $G(c)$ are the Grassmann rank of $b$ and $c$, respectively. With this definition, note in particular that $Cl_{m,n}\otimes Cl_{p,q} = Cl_{m+p,n+q}$.

representation so we can explicitly compare this to the Moyal product, eq.(9). The crucial step is to construct "position" and "momentum" operators out of the $bc$ oscillator algebra, and use the eigenvectors of these operators to construct wavefunctions out of the Hilbert space states. Performing a contraction between two states and the three vertex should then correspond, in a simple way, to a Moyal product between the associated wavefunctions. So, given a $bc$ oscillator algebra we define the hermitian operators,

$$\begin{aligned} \hat{x} &= \frac{i}{\sqrt{2}}(b - b^+) \qquad & \hat{y} &= \frac{1}{\sqrt{2}}(c + c^+) \\ \hat{p} &= \frac{i}{\sqrt{2}}(c - c^+) \qquad & \hat{q} &= \frac{1}{\sqrt{2}}(b + b^+) \end{aligned} \tag{30}$$

It is simple to see that these operators are hermitian and satisfy $\{\hat{x}, \hat{p}\} = \{\hat{y}, \hat{q}\} = 1$, with all other anticommutators vanishing. As the notation suggests, we will think of $\hat{x}$, $\hat{y}$ as coordinates and $\hat{p}$, $\hat{q}$ as momenta. This is largely a matter of convention, but eq.(30) was chosen so that Fourier modes of the ghost fields $b(\sigma)$ and $c(\sigma)$ (defined in the next section) would be coordinates. We can write the eigenstates of $\hat{x}$, $\hat{y}$,

$$|x, y\rangle = \exp\big[-b^+c^+ + i\sqrt{2}xc^+ + \sqrt{2}b^+y - ixy\big]\,|0\rangle. \tag{31}$$

The inner product of two such states is,

$$\langle x, y|x', y'\rangle = -2i(x - x')(y - y') = \delta(x' - x)\delta(y' - y).$$

The factor of $i$ is present to ensure that the inner product is real. The eigenstates are also complete,

$$\int dX\ |x, y\rangle\langle x, y| = 1, \qquad dX = -\tfrac{i}{2}\ dxdy. \tag{32}$$

Therefore, we may express the state $|\psi\rangle$ as a wavefunction $\psi(x, y) = \langle x, y|\psi\rangle$. We put a factor of $\frac{1}{2}$ in the measure rather than $\frac{1}{\sqrt{2}}$ in the normalization of $|x, y\rangle$ so that we wouldn't have to keep track of divergent normalizations when taking infinite tensor products of such states, as we will need to do in the next section. Given a 3-vertex $|V\rangle$, allowing us to define a product between states, $|\psi * \phi\rangle = \langle\psi|\langle\phi|V\rangle$, it follows that we can write $\psi * \phi(X) = \int dX^1 dX^2\ K(X^1, X^2, X)\psi(X^1)\phi(X^2)$ with $K(X^1, X^2, X) = \langle X|\langle X^1|\langle X^2||V\rangle$, denoting $X = (x, y)$.

Taking a hint from eq.(27), let us consider the tensor product of two $bc$ oscillator algebras, which we will write for illustrative purposes $(b_{\rm e}, c_{\rm e})$ and $(b_{\rm o}, c_{\rm o})$. A kernel of particular interest is,

$$K(X^1, X^2, X^3) = 16Ng^4 \exp\left[-i\ \vec{x}^A \cdot W^{AB} \cdot \vec{y}^B\right],$$

with,

$$W^{AB} = g^{-1} \begin{pmatrix} 0 & -\sigma_y & \sigma_y \\ \sigma_y & 0 & -\sigma_y \\ -\sigma_y & \sigma_y & 0 \end{pmatrix}^{AB}, \tag{33}$$

The index $A$ ranges from 1 to 3 and $\vec{x} = \begin{pmatrix} x_{\rm e} \\ x_{\rm o} \end{pmatrix}$ and likewise for $\vec{y}$. We can also rewrite the quadratic form as,

$$-i\ \vec{x}^A \cdot W^{AB} \cdot \vec{y}^B = -x_{\rm o}^A K^{AB} y_{\rm e}^B + x_{\rm e}^A K^{AB} y_{\rm o}^B$$

where $K^{AB}$ is the matrix in eq.(9) defining the kernel for the Moyal product. Therefore, eq.(33) is just a kernel for the tensor product of two Clifford Algebras, up to some normalization[7], with $(x_{\rm e}, y_{\rm o})$ and $(x_{\rm o}, y_{\rm e})$ forming canonical Moyal pairs.

Let us derive the 3-vertex corresponding to the kernel eq.(33). Assume it is of the form,

$$|V\rangle = \exp[\vec{b}^{A+} \cdot V^{AB} \cdot \vec{c}^{B+})]|0\rangle_1|0\rangle_2|0\rangle_3$$

Then we have,

$$\begin{aligned} K(X^1, X^2, X^3) &= \langle 0|^3 \exp[\vec{b}^A \cdot \vec{c}^A + i\sqrt{2}\vec{x}^A \cdot \vec{c}^A - \sqrt{2}\vec{b}^A \cdot \vec{y}^A - i\vec{x}^A \cdot \vec{y}^A] \\ &\qquad \times \exp[\vec{b}^{A+} \cdot V^{AB} \cdot \vec{c}^{B+})]|0\rangle^3, \\ &= \det(1+V)\exp\left[-i\vec{x}^A \cdot \left(\frac{1-V}{1+V}\right)^{AB} \cdot \vec{y}^B\right]. \end{aligned}$$

This implies,

$$N = \frac{\det(1+V)}{16g^4} = \left(\frac{2}{g^2+3}\right)^2, \qquad V^{AB} = \left(\frac{1-W}{1+W}\right)^{AB}. \tag{34}$$

Plugging in eq.(33) and working through some matrix algebra we find:

$$V^{AB} = \frac{1}{g^2+3}\begin{pmatrix} g^2-1 & 2+2g\sigma_y & 2-2g\sigma_y \\ 2-2g\sigma_y & g^2-1 & 2+2g\sigma_y \\ 2+2g\sigma_y & 2-2g\sigma_y & g^2-1 \end{pmatrix}^{AB} \tag{35}$$

The vertex is then becomes,

$$\begin{aligned} |V\rangle &= \exp\left[\frac{g^2-1}{g^2+3}(\vec{b}^1 \cdot \vec{c}^1 + \text{cyclic.}) + \frac{2}{g^2+3}(\vec{b}^1 \cdot \vec{c}^2 + \vec{b}^2 \cdot \vec{c}^1 + \text{cyclic})\right. \\ &\qquad \left. + \frac{2g}{g^2+3}(\vec{b}^1 \cdot \sigma_y \cdot \vec{c}^2 - \vec{b}^2 \cdot \sigma_y \cdot \vec{c}^1 + \text{cyclic})\right]|0\rangle_1|0\rangle_2|0\rangle_3 \end{aligned} \tag{36}$$

A quick comparison with eq.(29) shows that this vertex is in the same form as the diagonalized three string vertex. However, if the two vertices are really the same, we must be able to find a metric $g(\kappa)$ satisfying,

$$\tilde{\mu}(\kappa) = \frac{g(\kappa)^2-1}{g(\kappa)^2+3}, \qquad \tfrac{1}{2}(\tilde{\mu}^{12}(\kappa) + \tilde{\mu}^{21}(\kappa)) = \frac{2}{g(\kappa)^2+3},$$

[7] According to eq.(9) the conventional normalization would be $N = 1$. The factor of 16 in front of the above kernel is necessary to cancel four factors of $\frac{1}{2}$ which go into our definition of the measure eq.(32).

$$\tfrac{1}{2}(\tilde{\mu}^{12}(\kappa)-\tilde{\mu}^{21}(\kappa))=\frac{2g(\kappa)}{g(\kappa)^2+3}.$$

Remarkably, a solution exists:

$$g(\kappa)=\coth\tfrac{\pi\kappa}{4}. \tag{37}$$

Therefore, the reduced star product in the fermionic ghost sector is given by a continuous tensor product—from $\kappa>0$ to $\infty$—of a *pair* of Clifford Algebras $\mathbf{C}\otimes Cl_{1,1}$, the first having a metric $+g(\kappa)$ and the second $-g(\kappa)$. Next section we will explicitly construct a path integral representation for the reduced star product using a continuous product of kernels of the form eq.(33).

It is interesting to compare eq.(37) to the analogous result in the matter sector, where in ref.[15] they found a noncommutativity parameter $\theta(\kappa)$ scaling as $\tanh(\frac{\pi\kappa}{4})$. The fact that $\theta(\kappa)$ vanishes at $\kappa=0$ is significant, indicating the presence of a commutative coordinate. Here we have no purely anticommutative coordinate, but instead an infinitely nonanticommutative algebra at $\kappa=0$. This seems most peculiar, and its relevance is not immediately obvious.

We should mention that our definition of the wavefunction $\psi(x,y)=\langle x,y|\psi\rangle$ is crucial for establishing the above result. If instead we had defined $\psi$ to depend on the eigenvalues of $\hat{p}$ or $\hat{q}$, we would find that the vertex describing the Moyal product of these functions would not be of the form eq.(29). This is hardly a surprise, since the reduced star product cannot be a Moyal product in position and momentum space simultaneously. Still, one wonders why the Moyal structure emerges in "position" space, especially considering that our definition of position versus momentum seemed somewhat arbitrary. The reason comes down to our choice of multiplicative phase in the definition of $\beta,\ \chi$ in eq.(26). If we redefined the phase so that $\beta\to -i\beta$ and $\chi\to i\chi$ then the three string vertex would actually correspond to fermionic Moyal star multiplication in "momentum space." Clearly there is no physical content in our choice of conventions.

# V. Path integral kernel representation of the reduced star product

In the last section we argued that the diagonalized vertex eq.(29) actually represents a continuous tensor product of Clifford Algebras, $\mathbf{C}\otimes Cl_{2,2}$, up to some normalization (except at $\kappa=0$). In this section we would like to translate the reduced star product from an oscillator vertex into a path integral kernel, analogous to eq.(9). Let us begin by defining position and momentum modes as in eq.(30),

$$\begin{aligned}\hat{x}_n &= \frac{i}{\sqrt{2}}(b_n-b_n^+) \qquad & \hat{y}_n &= \frac{1}{\sqrt{2}}(c_n+c_n^+)\\ \hat{p}_n &= \frac{i}{\sqrt{2}}(c_n-c_n^+) \qquad & \hat{q}_n &= \frac{1}{\sqrt{2}}(b_n+b_n^+)\end{aligned} \tag{38}$$

and the associated eigenstate,

$$|x_n, y_n\rangle = \exp\big[-b_n^+ c_n^+ + i\sqrt{2}x_n c_n^+ + \sqrt{2}b_n^+ y_n - ix_n y_n\big]|0\rangle. \tag{39}$$

($|0\rangle$ again is the vacuum annihilated by $b_n,\ c_n,\ n > 0$.) The operators $x_n$ and $y_n$ are the Fourier modes of the ghost fields,

$$\begin{aligned} b(\sigma) &= i\sqrt{2}\sum_{n=1}^{\infty} \hat{x}_n \sin n\sigma \\ c(\sigma) &= c_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{y}_n \cos n\sigma, \end{aligned} \tag{40}$$

where,

$$c^{\pm}(\sigma) = c(\sigma) \pm i\pi_b(\sigma) \qquad b_{\pm\pm}(\sigma) = \pi_c(\sigma) \mp ib(\sigma)$$

The string field in the fermionic ghost sector is often written as a functional of $b(\sigma)$ and $c(\sigma)$,

$$\Psi[b(\sigma), c(\sigma)] = \Psi[c_0, x_n, y_n] = \langle c_0| \otimes \langle x_n, y_n||\Psi\rangle \tag{41}$$

where $|c_0\rangle = \exp[\hat{b}_0 c_0]|+\rangle$ is the eigenstate of $\hat{c}_0$. Remember that we are working in Siegel gauge, which means,

$$b_0|\Psi\rangle = 0 \quad \Rightarrow \quad \frac{\partial}{\partial c_0}\Psi[c_0, x_n, y_n] = 0 \quad \Rightarrow \quad \Psi = \Psi[x_n, y_n].$$

As implied by our discussion in the last section, the Moyal structure of the reduced star product is clearly manifest when the string field is expressed as a functional of the eigenvalues of the operators,

$$\begin{aligned} \hat{x}_{\rm e}(\kappa) &= \tfrac{i}{\sqrt{2}}(\beta_{\rm e}(\kappa) - \beta_{\rm e}^+(\kappa)) = -\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}v_{2n}(\kappa)\hat{q}_{2n} \\ \hat{x}_{\rm o}(\kappa) &= \tfrac{i}{\sqrt{2}}(\beta_{\rm o}(\kappa) - \beta_{\rm o}^+(\kappa)) = \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}v_{2n-1}(\kappa)\hat{x}_{2n-1} \\ \hat{y}_{\rm e}(\kappa) &= \tfrac{1}{\sqrt{2}}(\chi_{\rm e}(\kappa) + \chi_{\rm e}^+(\kappa)) = \sqrt{2}\sum_{n=1}^{\infty}\sqrt{2n}v_{2n}(\kappa)\hat{p}_{2n} \\ \hat{y}_{\rm o}(\kappa) &= \tfrac{1}{\sqrt{2}}(\chi_{\rm o}(\kappa) + \chi_{\rm o}^+(\kappa)) = \sqrt{2}\sum_{n=1}^{\infty}\sqrt{2n-1}v_{2n-1}(\kappa)\hat{y}_{2n-1} \end{aligned} \tag{42}$$

The string field as it is usually written in eq.(41) is not a functional of the eigenvalues of eq.(42). We need to perform a Fourier transform on the even coordinates:

$$\tilde{\Psi}[x_{2n-1}, y_{2n-1}, p_{2n}, q_{2n}] = \int \left(\prod_{m=1}^{\infty} \tfrac{1}{i}dx_{2m}dy_{2m}e^{p_{2n}x_{2n}+q_{2n}y_{2n}}\right)\Psi[x_n, y_n]. \tag{43}$$

Now define,

$$\Psi^M[\vec{x}(\kappa), \vec{y}(\kappa)] = \tilde{\Psi}[x_{2n-1}, y_{2n-1}, p_{2n}, q_{2n}]$$

This is the form of the string field we require.

From eq.(39) it follows that the ground state of the ghost Fock space is the functional,

$$\begin{aligned} \Psi_{|0\rangle}[x_n, y_n] &= e^{-ix_n y_n} \\ \tilde{\Psi}_{|0\rangle}[x_{2n-1}, y_{2n-1}, p_{2n}, q_{2n}] &= \exp\left[-i(x_{2n-1}y_{2n-1} + p_{2n}q_{2n})\right] \end{aligned} \tag{44}$$

The states $|x_n, y_n\rangle$ are normalized, just as in eq.(32), so that they furnish a resolution of the identity with respect to a measure given as an infinite product of $-\frac{i}{2}dx_n dy_n$. Consistently, we have,

$$\begin{aligned} \langle 0|0\rangle = 1 &= \int \left( \prod_{n=1}^{\infty} \tfrac{1}{2i} dx_n dy_n \right) e^{-2ix_n y_n} \\ &= \int \prod_{n=1}^{\infty} \tfrac{1}{2i} dx_{2n-1} dy_{2n-1} \tfrac{1}{2i} dp_{2n} dq_{2n} e^{-2ix_{2n-1}y_{2n-1} - 2ip_{2n}q_{2n}}. \end{aligned}$$

From eq.(42) we see that the ground state can also be expressed as a functional of $\vec{x}(\kappa), \vec{y}(\kappa)$ (again we will suppress the $\kappa$ index when this is unambiguous),

$$\Psi^M_{|0\rangle}[\vec{x}(\kappa), \vec{y}(\kappa)] = \exp\left[-i\int_0^\infty d\kappa (x_o y_o - x_e y_e)\right]. \tag{45}$$

Let us define a path integral measure over $\vec{x}$ and $\vec{y}$, so that this ground state functional has unit norm:

$$\int [dxdy] \exp\left(-2i\int_0^\infty d\kappa [xy + j_x y + x j_y]\right) = \exp\left(2i\int_0^\infty d\kappa\ j_x j_y\right), \tag{46}$$

from which we can see that,

$$\int [d\vec{x} d\vec{y}] \left| \Psi^M_{|0\rangle}[\vec{x}, \vec{y}] \right|^2 = 1$$

We will also need to consider the path integral,

$$\int [dxdy] \exp\left(-2i\int_0^\infty d\kappa\ Gxy\right) = \exp\left(\delta(0)\int_0^\infty d\kappa\ \ln[G]\right) \tag{47}$$

The infinite $\delta(0)$ is the same as the divergent factor $\frac{\ln L}{2\pi}$, with $L$ is the "level regulator," discussed in [22, 26, 15]. We can see intuitively that this divergent factor is necessary, since rescaling a particular $x(\kappa')$ by $G$ should scale the whole path integral by $G$. This is consistent with setting $G(\kappa) = 1 + (G-1)\delta(\kappa - \kappa')/\delta(0)$, and plugging into eq.(47).

From eq.(39) we can derive the eigenstate of the operators $\hat{x}_{\rm e}$, $\hat{x}_{\rm o}$, $\hat{y}_{\rm e}$, and $\hat{y}_{\rm o}$ through taking Fourier transforms and the appropriate linear combinations:

$$|\vec{x},\vec{y}\rangle = \exp\left[\int_0^\infty \left(-\vec{\beta}^+\cdot\vec{\chi}^+ + i\sqrt{2}\vec{x}\cdot\vec{\chi}^+ + \sqrt{2}\vec{\beta}^+\cdot\vec{y} - i\vec{x}\cdot\vec{y}\right)\right]|0\rangle. \tag{48}$$

We can then calculate the wavefunctional by performing the contraction,

$$\Psi^M[\vec{x},\vec{y}] = \langle c_0|\otimes\langle\vec{x},\vec{y}||\Psi\rangle$$

This is consistent with eq.(45). Therefore, with the choice of measure eq.(46), we have a resolution of the identity,

$$\int[d\vec{x}\ d\vec{y}]\ |\vec{x},\vec{y}\rangle\langle\vec{x},\vec{y}| = 1 \tag{49}$$

Armed with this we can easily translate the three string vertex eq.(29) into the kernel of the path integral representing the reduced star product ($\vec{X}=(\vec{x},\vec{y})$):

$$\Phi^M *_{b_0} \Psi^M[\vec{X}^3] = \int\prod_{A=1,2}[d\vec{x}^A\ d\vec{y}^A]\Phi^{M*}[\vec{X}^1]\Psi^{M*}[\vec{X}^2]K[\vec{X}^1,\vec{X}^2,\vec{X}^3] \tag{50}$$

where

$$K[\vec{X}^1,\vec{X}^2,\vec{X}^3] = \langle\vec{X}^1|\langle\vec{X}^2|\langle\vec{X}^3||\hat{V}_3\rangle$$

Performing the contraction, we get,

$$K[\vec{X}^1,\vec{X}^2,\vec{X}^3] = \tfrac{1}{8}\exp\left[-i\int_0^\infty d\kappa\vec{x}^A\cdot W^{AB}\cdot\vec{y}^B - 2\delta(0)\int_0^\infty d\kappa\ \ln\left(\frac{3+g^2}{8g^2}\right)\right] \tag{51}$$

where $W^{AB}(\kappa)$ the same matrix as in eq.(33) with $g=g(\kappa)=\coth\frac{\pi\kappa}{4}$. This is exactly as anticipated in the previous section. Note that the normalization factor relating the reduced star product to a canonically normalized fermionic Moyal star is,

$$\mathcal{N} = \tfrac{1}{8}\exp\left[-2\delta(0)\int_0^\infty d\kappa\ \ln\tfrac{1}{2}\left(3+g^2\right)\right].$$

$\mathcal{N}$ is the analogy of the constant $N$ calculated in eq.(34). In ref.[15] it was suggested that $\mathcal{N}$ might cancel a corresponding divergent normalization in the matter sector kernel. However, explicitly calculating the product of the two factors,

$$\mathcal{N}C' = 8\exp\left[\delta(0)\left\{D\int_0^\infty d\kappa\ \ln\tfrac{1}{8}\left(12+\theta^2\right) - 2\int_0^\infty d\kappa\ \ln\tfrac{1}{2}\left(3+g^2\right)\right\}\right],$$

where $\theta(\kappa)=2\tanh\frac{\pi\kappa}{4}$ is the bosonic noncommutativity parameter[15], we see that the product is infinite, at least for $D\neq 2$, though the ghost contribution does temper the divergence as expected. This implies that the reduced star algebra (with the matter part) does not really define an algebra of bounded operators; the reduced star product of two string fields of finite norm will generally yield a state of infinite norm. Of course, the fact that the reduced star algebra is unbounded does not necessarily imply that the *full* star algebra is unbounded[8]; it might be possible that the full star algebra is bounded in $D = 26$. This is an important question which we leave for future inquiry.

Let us summarize our results as follows: the reduced star product can be written,

$$\Phi^M *_{b_0} \Psi^M[\vec{x}(\kappa), \vec{y}(\kappa)] = \mathcal{N}\Phi^M \star \Psi^M[\vec{x}(\kappa), \vec{y}(\kappa)],$$

with $\star$ the canonically normalized Moyal product satisfying,

$$\{x_{\rm e}(\kappa), y_{\rm o}(\kappa')\}_\star = -\{x_{\rm o}(\kappa), y_{\rm e}(\kappa')\}_\star = 2\coth\tfrac{\pi\kappa}{4}\delta(\kappa-\kappa') \tag{52}$$

All other anticommutators between $\vec{x}(\kappa)$ and $\vec{y}(\kappa)$ vanish. This completes the Moyal formulation of the reduced star product in the fermionic ghost sector.

# VI. Kinetic and ghost number operators

In this section we will formulate some important operators in string field theory using the continuous oscillator basis introduced in eq.(26). Sofar we have established that the reduced star algebra can be formulated as an infinite tensor product of Clifford Algebras, but eventually we are interested in the *full* star product, which as explained before is related to the reduced star product in Siegel gauge through the action of the operator,

$$\mathcal{Q} = c(\tfrac{\pi}{2}) = c_0 - (c_2 + c_2^+) + (c_4 + c_4^+) - ..., \tag{53}$$

which also happens to be the canonical choice of pure ghost kinetic operator in VSFT (up to a possibly infinite normalization)[27, 5, 23]. Clearly we would rather not translate back into the old oscillator basis whenever we need to calculate the full star product from the reduced one, so it is advantageous to formulate $\mathcal{Q}$ within the language developed here. Write,

$$\mathcal{Q} = c_0 - i\sqrt{2}\int_0^\infty d\kappa K(\kappa)(\chi_{\rm e}(\kappa) - \chi_{\rm e}^+(\kappa))$$

where,

$$K(\kappa) = \sum_{n=1}^\infty (-1)^n \frac{v_{2n}(\kappa)}{\sqrt{2n}} = \frac{2 - e^{-\kappa\tan^{-1} i} - e^{-\kappa\tan^{-1}(-i)}}{2\sqrt{2\kappa\sinh\frac{\pi\kappa}{2}}}$$

We took the liberty of evaluating the sum using eq.(20). Unfortunately this expression is undefined, since the inverse tangent has simple poles at $\pm i$. We can regulate it by replacing $\tan^{-1} i$ by $\tan^{-1} ia$ and taking the limit $a \to 1$:

$$K(\kappa) = \lim_{a\to 1}\frac{2 - e^{-i\kappa\tanh^{-1} a} - e^{i\kappa\tanh^{-1} a}}{2\sqrt{2\kappa\sinh\frac{\pi\kappa}{2}}} = \frac{2}{\sqrt{2\kappa\sinh\frac{\pi\kappa}{2}}}\lim_{w\to\infty}\sin^2(w\kappa)$$

[8] Thanks to L. Rastelli for pointing this out to me.

but this doesn't help since the above expression oscillates with infinite frequency and does not converge in the limit. However, we should really think of $K(\kappa)$ as an object which properly belongs under an integral. Consider, for instance,

$$\begin{aligned}\int_a^b d\kappa\ \kappa^{2n}\sin^2(w\kappa) &= \tfrac{1}{2}\int_a^b d\kappa\ \kappa^{2n} - \tfrac{1}{2}\left(-\tfrac{1}{4}\right)^n \frac{\partial^{2n}}{\partial w^{2n}}\int_a^b d\kappa\ \cos(2w\kappa)\\ &= \tfrac{1}{2}\int_a^b d\kappa\ \kappa^{2n} - \tfrac{1}{2}\left(-\tfrac{1}{4}\right)^n \frac{\partial^{2n}}{\partial w^{2n}}\frac{\cos(2wb)-\cos(2wa)}{2w}.\end{aligned}$$

Notice that the second term on the right hand side, for all $n \geq 0$, dies off for large $w$ at least as $1/w$. A similar argument follows for odd powers of $\kappa$. Therefore,

$$\lim_{w\to\infty}\int_a^b d\kappa\ f(\kappa)\sin^2(w\kappa) = \tfrac{1}{2}\int_a^b d\kappa\ f(\kappa)$$

for any function $f$ with a local Taylor expansion. Taking this into account, we can write the following expression for $\mathcal{Q}$:

$$\mathcal{Q} = c_0 - \sqrt{2}\int_0^\infty \frac{d\kappa\ \hat{p}_{\rm e}(\kappa)}{\sqrt{\kappa\sinh\frac{\pi\kappa}{2}}} \tag{54}$$

where $\hat{p}_{\rm e} = \frac{i}{\sqrt{2}}(\chi_{\rm e} - \chi_{\rm e}^+)$ [9]. In the spirit of noncommutative field theory, one might like to express the action of $\mathcal{Q}$ on a state as a star-(anti)commutator with the associated wavefunctional. We can do this easily by writing the state as a functional of $\vec{x}(\kappa), \vec{y}(\kappa)$, in which case we represent the action of $\hat{p}^{\rm e}(\kappa)$ as the functional derivative $\delta/\delta x_{\rm e}(\kappa)$. Define the funtional/operator,

$$\mathcal{Q}^M = -\tfrac{1}{2}\int_0^\infty d\kappa\ \left(\frac{\sinh\frac{\pi\kappa}{4}}{\kappa\cosh^3\frac{\pi\kappa}{4}}\right)^{\frac{1}{2}} y_{\rm o}(\kappa). \tag{55}$$

This allows us to write, invoking eq.(10) and eq.(52),

$$\mathcal{Q}(\Psi^M) = c_0\Psi^M + \mathcal{Q}^M \star \Psi^M + (-1)^{G(\Psi)}\Psi^M \star \mathcal{Q}^M. \tag{56}$$

where $\Psi^M = \Psi^M[\vec{x}(\kappa), \vec{y}(\kappa)]$ and $G(\Psi)$ denotes the ghost number[10] of $\Psi^M$. Similar looking expressions for the action of pure ghost kinetic operators appear in the split string formalism[10]. The continuous oscillator approach however has a marked advantage over the split string formalism, since in the latter it seems impossible to formulate $\mathcal{Q}$ in a well-defined way. This is both because $\mathcal{Q}$ acts only at one point on the string, and because this point happens to be the sting midpoint. In particular, $\mathcal{Q}$ creates a kink in the bosonized ghost

[9] The apparent pole in the integrand near $\kappa = 0$ is illusory since $\hat{p}^{\rm e}(\kappa)$ vanishes linearly near $\kappa = 0$.

[10] Note that the ghost number of the string field is 1 plus its Grassmann rank, so eq.(56) is consistent with eq.(52). This follows since the ground state functional eq.(44), corresponding to the $|-\rangle$ vacuum, is Grassmann even but has ghost number one.

field at $\sigma = \frac{\pi}{2}$, while $\mathcal{Q}^2$ creates two kinks; so $\mathcal{Q}$ is apparently not nilpotent. Furthermore, a configuration with a kink at the midpoint cannot be encoded with ordinary split string variables $\bar{\phi}$, $l(\sigma)$, $r(\sigma)$ with $l(\frac{\pi}{2}) = r(\frac{\pi}{2}) = 0$. Though one may attempt to deal with these problems by regulating and adding more degrees of freedom at the midpoint, the structure of the split string formalism, which necessarily treats the midpoint in a somewhat naive fashion, begins to fall apart. This is in large part the reason why no satisfactory formulation of VSFT has been constructed using bosonized ghosts.

An interesting and nontrivial check on the consistency of our approach and the correctness of our expression for $\mathcal{Q}$ is to verify that $\{\mathcal{Q}, \pi_c(\sigma)\} = 0$ for $\sigma \neq \frac{\pi}{2}$. Note,

$$\begin{aligned}\pi_c(\sigma) &= \tfrac{1}{2}\big[b_{++}(\sigma) + b_{--}(\sigma)\big] = b_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{q}_n \cos n\sigma \\ &= b_0 + 2\int_0^\infty d\kappa\ K_{\rm o}(\sigma,\kappa)\hat{q}_{\rm o}(\kappa) + 2\int_0^\infty d\kappa\ K_{\rm e}(\sigma,\kappa)\hat{x}_{\rm e}(\kappa)\end{aligned}$$

where,

$$\begin{aligned}K_{\rm e}(\sigma,\kappa) &= \sum_{n=1}^{\infty}\sqrt{2n}\ v_{2n}(\kappa)\cos 2n\sigma \\ &= \frac{\kappa}{4\sqrt{2\kappa\sinh\frac{\pi\kappa}{2}}}\left[\frac{\phi(e^{\kappa\tan^{-1}\phi} - e^{-\kappa\tan^{-1}\phi})}{1+\phi^2} + \text{c.c.}\right]\end{aligned}$$

where $\phi = e^{i\sigma}$ and c.c. denotes the complex conjugate of the first term. This equation is obtained straightforwardly by using the generating function for the $v_n$s to evaluate the sum. A similar expression defines $K_{\rm o}(\sigma,\kappa)$ but we will not need it here. Calculate the anticommutator,

$$\{\mathcal{Q}, \pi_c(\sigma)\} = \{c_0, b_0\} - 2\sqrt{2}\int_0^\infty d\kappa d\kappa'\ \frac{K^{\rm e}(\sigma,\kappa')\{\hat{p}_{\rm e}(\kappa), \hat{x}_{\rm e}(\kappa')\}}{\sqrt{\kappa\sinh\frac{\pi\kappa}{4}}}$$

If this vanishes, we must have,

$$\int_0^\infty d\kappa\ \frac{K_{\rm e}(\sigma,\kappa)}{\sqrt{2\kappa\sinh\frac{\pi\kappa}{2}}} = \frac{1}{4} \tag{57}$$

plugging in $K_{\rm e}$ we are lead to consider the integral,

$$E(\sigma) = \frac{\phi}{4(1+\phi^2)}\int_{-\infty}^{\infty} d\kappa\ \frac{e^{\kappa\tan^{-1}\phi}}{2\sinh\frac{\pi\kappa}{2}}$$

This expression plus its complex conjugate is the quantity in eq.(57). Consider,

$$\tan^{-1}\phi = \tfrac{1}{2}\tan^{-1}\left[\frac{\cos\sigma}{1-\sin\sigma}\right] - \tfrac{1}{2}\tan^{-1}\left[\frac{\cos\sigma}{1+\sin\sigma}\right] - \frac{i}{4}\ln\left[\frac{(1-\sin\sigma)^2+\cos^2\sigma}{(1+\sin\sigma)^2+\cos^2\sigma}\right]$$

The real part of this is always between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ so $E(\sigma)$ for $\sigma \neq \frac{\pi}{2}$ is always rendered convergent by the $\sinh\frac{\pi\kappa}{2}$ in the denominator. The imaginary part is always positive imaginary so we close the contour in the upper half plane:

$$\begin{aligned}
E(\sigma) &= \frac{2\pi i\phi}{4(1+\phi^2)}\left[\tfrac{1}{2}\mathrm{Res}(0)+\sum_{n=1}^{\infty}\mathrm{Res}(2in)\right] \\
&= \frac{2\pi i\phi}{4(1+\phi^2)}\left[\frac{1}{2\pi}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{1+i\phi}{1-i\phi}\right)^n\right] \\
&= \frac{\phi^2}{4(1+\phi^2)}
\end{aligned}$$

Above we took the principle value of the integral, though actually the final result is the same regardless of our contour prescription. So we are left to check,

$$E(\sigma)+\bar{E}(\sigma)=\frac{\phi^2}{4(1+\phi^2)}+\frac{\bar{\phi}^2}{4(1+\bar{\phi}^2)}=\frac{\phi^2}{4(1+\phi^2)}+\frac{1}{4(1+\phi^2)}=\frac{1}{4}$$

just as required. This argument does not work (fortunately) at $\sigma=\frac{\pi}{2}$, since $E(\sigma)$ is divergent there. Obviously, we expect a delta function at $\frac{\pi}{2}$.

Another interesting operator we should consider is the ghost number operator,

$$\begin{aligned}
G &= \tfrac{3}{2}+\tfrac{1}{2}(c_0b_0-b_0c_0)+\sum_{n=1}^{\infty}(c_n^+b_n-b_n^+c_n) \\
&= 1+\int_0^{\infty}d\kappa(\vec{\chi}^+\cdot\vec{\beta}-\vec{\beta}^+\cdot\vec{\chi})
\end{aligned} \tag{58}$$

In the second expression we assumed Siegel gauge. One of the crucial properties of Witten's star product is that the ghost number of the product of two string fields is the sum of the ghost numbers of the fields individually; this is what makes the star product of fields similar to the wedge product of differential forms. Let us see how this property emerges in the present formalism. Note that the reduced star product must be additive in $G_0=G-1$:

$$\begin{aligned}
G_0(A*_{b_0}B) &= G(A*_{b_0}B)-1=G(A*B)-2=G(A)+G(B)-2 \\
&= G_0(A)+G_0(B)
\end{aligned}$$

which we call the reduced ghost number. Acting on a string field expressed as a functional of $\vec{x}(\kappa),\vec{y}(\kappa)$, the reduced ghost number operator is,

$$G_0=\int_0^{\infty}d\kappa\left(\vec{y}(\kappa)\cdot\frac{\delta}{\delta\vec{y}(\kappa)}-\vec{x}(\kappa)\cdot\frac{\delta}{\delta\vec{x}(\kappa)}\right)$$

It turns out that $G_0$ acts as a derivation of the $\star$ algebra. We can argue this as follows. The kernel derived in eq.(9) is easily seen to satisfy,

$$\sum_{A=1}^{3}\left(y^A\frac{\partial}{\partial y^A}-x^A\frac{\partial}{\partial x^A}\right)K(X^1,X^2,X^3)=0$$

This can be rewritten,

$$\left(y^3\frac{\partial}{\partial y^3}-x^3\frac{\partial}{\partial x^3}\right)K(X^1,X^2,X^3)=\left(\frac{\partial}{\partial y^1}y^1-\frac{\partial}{\partial x^1}x^1+\frac{\partial}{\partial y^2}y^2-\frac{\partial}{\partial x^2}x^2\right)K(X^1,X^2,X^3)$$

The "zero point" contributions we get from switching the order of the coordinates and derivatives on the left fortunately cancel out. Plugging this into eq.(8) and integrating by parts, we immediately see that $y\partial_y - x\partial_x$ acts as a derivation. This argument obviously carries over to the continuum case as well. Therefore, we may write,

$$\left(\vec{y}(\kappa)\cdot\frac{\delta}{\delta\vec{y}(\kappa)}-\vec{x}(\kappa)\cdot\frac{\delta}{\delta\vec{x}(\kappa)}\right)=\frac{1}{2\coth\frac{\pi\kappa}{4}}\Big[y_{\mathrm{o}}(\kappa)\star x_{\mathrm{e}}(\kappa)-y_{\mathrm{e}}(\kappa)\star x_{\mathrm{o}}(\kappa),\quad\Big]_\star .$$

Both the left and right hand side act identically on $1,\vec{x}(\kappa),\vec{y}(\kappa)$, and because both are derivations, as just argued, they must act identically on all $\Psi^M$. We can then represent the action of the reduced ghost number operator as a star commutator,

$$G_0(\Psi^M)=G_0^M\star\Psi^M-\Psi^M\star G_0^M \tag{59}$$

with,

$$G_0^M=\tfrac{1}{2}\int_0^\infty d\kappa\ \tanh\tfrac{\pi\kappa}{4}\ \Big(y_{\mathrm{o}}(\kappa)\star x_{\mathrm{e}}(\kappa)-y_{\mathrm{e}}(\kappa)\star x_{\mathrm{o}}(\kappa)\Big) \tag{60}$$

It is easy to see that the derivation property of $G_0$ implies that the reduced star product is additive in reduced ghost number, as expected.

We now turn to the BRST operator, $Q_B$, describing the background of the unstable D-25 brane. Unfortunately, when we write it out in the continuous oscillator basis, $Q_B$ suffers from serious divergences. The divergences are the same as those encountered in ref.[15] when they tried to formulate $L_0$ in a continuous oscillator basis. Their presence is perhaps the most significant shortcoming of the continuous Moyal formulation, and it is crucial to have an explicit construction of $Q_B$ and a concrete understanding of its divergences if this problem is to be overcome. We are not at this point able to explicitly regulate $Q_B$ and demonstrate that it functions properly as the regulator is taken away, but it is still instructive to see how $Q_B$ looks in the continuous oscillator basis. We write[25],

$$\begin{aligned}Q_B&=\sum_{n=-\infty}^{\infty}:c_n(L_{-n}+\tfrac{1}{2}L^{gh}_{-n}-\delta_{n0}):\\&=:c_0(L_0+\tfrac{1}{2}L_0^{gh}-1):+\int_{-\infty}^{\infty}d\kappa\ :\vec{\chi}(\kappa)\cdot\big(\vec{L}(-\kappa)+\tfrac{1}{2}\vec{L}^{gh}(-\kappa)\big):\end{aligned}\tag{61}$$

The above expression follows from the definition $\vec{\chi}(-\kappa)\equiv\vec{\chi}^+(\kappa)$ and,

$$\begin{aligned}L_{\mathrm{e}}(\kappa)&=\begin{cases}-i\sqrt{2}\sum_{n=1}^\infty\frac{1}{\sqrt{2n}}v_{2n}(\kappa)L_{-2n} & \kappa<0\\ i\sqrt{2}\sum_{n=1}^\infty\frac{1}{\sqrt{2n}}v_{2n}(\kappa)L_{2n} & \kappa>0\end{cases}\\L_{\mathrm{o}}(\kappa)&=\begin{cases}\sqrt{2}\sum_{n=1}^\infty\frac{1}{\sqrt{2n-1}}v_{2n-1}(\kappa)L_{-2n+1} & \kappa<0\\ \sqrt{2}\sum_{n=1}^\infty\frac{1}{\sqrt{2n-1}}v_{2n-1}(\kappa)L_{2n-1} & \kappa>0\end{cases}\end{aligned}\tag{62}$$

Similar expressions define $\vec{L}^{gh}(\kappa)$. Let us concentrate on expressing the ghost Virasoro generators in the continuous $bc$ oscillator basis. The matter Virasoro generators may also be reformulated with an appropriate choice of basis, perhaps one which diagonalizes the relevant matter sector three string vertex. Consider first $L_0^{gh}$,

$$\begin{aligned} L_0^{gh} &= \sum_{n=1}^{\infty} n(b_n^+ c_n - b_n c_n^+) \\ &= \int_0^{\infty} d\kappa_1 d\kappa_2 \, (\vec{\beta}(-\kappa_1) \cdot K_{L_0}(\kappa_1,\kappa_2) \cdot \vec{\chi}(\kappa_2) - \vec{\beta}(\kappa_1) \cdot K_{L_0}(\kappa_1,\kappa_2) \cdot \vec{\chi}(-\kappa_2)) \end{aligned}$$

where,

$$\begin{aligned} K_{L_0}^{\mathrm{ee}}(\kappa_1,\kappa_2) &= K(\kappa_1,\kappa_2) - K(\kappa_1,-\kappa_2) \\ K_{L_0}^{\mathrm{oo}}(\kappa_1,\kappa_2) &= K(\kappa_1,\kappa_2) + K(\kappa_1,-\kappa_2) \\ K_{L_0}^{\mathrm{oe}}(\kappa_1,\kappa_2) &= K_{L_0}^{\mathrm{eo}}(\kappa_1,\kappa_2) = 0 \end{aligned} \tag{63}$$

and,

$$K(\kappa_1,\kappa_2) = \sum_{n=1}^{\infty} n \; v_n(\kappa_1) v_n(\kappa_2) \tag{64}$$

Unfortunately, the above sum does not converge, as can be seen from the the asymptotic behavior of the $v_n$s[15]:

$$\begin{aligned} v_{2n}(\kappa) &\sim \frac{(-1)^n}{\sqrt{2n}} \frac{1}{\sqrt{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}} \Im\left[\frac{(4n-2)^{i\kappa/2}}{\Gamma(1+i\frac{\kappa}{2})}\right] \\ v_{2n-1}(\kappa) &\sim \frac{(-1)^{n-1}}{\sqrt{2n-1}} \frac{1}{\sqrt{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}} \Re\left[\frac{(4(n-1))^{i\kappa/2}}{\Gamma(1+i\frac{\kappa}{2})}\right] \end{aligned} \tag{65}$$

Consider the ghost Virasoro generators with a continuous mode label:

$$\begin{aligned} L_{\mathrm{e}}^{gh}(\kappa) &= i\sqrt{2} \sum_{m=1}^{\infty} \frac{v_{2m}(\kappa)}{\sqrt{2m}} \left( \sum_{n=-\infty}^{\infty} (2m-n) b_{2m+n} c_{-n} \right) \\ &= 2b_0 \chi_{\mathrm{e}}(\kappa) - c_0 \int_0^{\infty} d\kappa_1 \; K_{L_0}^{\mathrm{ee}}(\kappa_1,\kappa)\beta_{\mathrm{e}}(\kappa_1) + \int_{-\infty}^{\infty} d\kappa_1 d\kappa_2 \; \vec{\beta}(\kappa_1) \cdot K_{L_{\mathrm{e}}}(\kappa_1,\kappa_2,\kappa) \cdot \vec{\chi}(\kappa_2) \end{aligned} \tag{66}$$

And similarly,

$$\begin{aligned} L_{\mathrm{o}}^{gh}(\kappa) &= \sqrt{2} \sum_{m=1}^{\infty} \frac{v_{2m-1}(\kappa)}{\sqrt{2m-1}} \left( \sum_{n=-\infty}^{\infty} (2m-1-n) b_{2m-1+n} c_{-n} \right) \\ &= 2b_0 \chi_{\mathrm{o}}(\kappa) - c_0 \int_0^{\infty} d\kappa \; K_{L_0}^{\mathrm{oo}}(\kappa_1,\kappa)\beta_{\mathrm{o}}(\kappa_1) + \int_{-\infty}^{\infty} d\kappa_1 d\kappa_2 \; \vec{\beta}(\kappa_1) \cdot K_{L_{\mathrm{o}}}(\kappa_1,\kappa_2,\kappa) \cdot \vec{\chi}(\kappa_2) \end{aligned} \tag{67}$$

The expressions defining the kernels $K_L$ above are somewhat lengthy, so we will not reproduce them here. For reference and the reader's amusement, we have written them explicitly in the appendix. They both diverge in a similar fashion as eq.(64). One way to regulate these divergences is to replace all $v_m(\kappa)$ with $q^m v_m(\kappa)$ for $0 < q < 1$. However, this will almost certainly lead to heinously complicated expressions, especially for the kernels in the appendix which are cubic in the $v_n(\kappa)$s. Maybe the current representation of the fermionic ghost star product is not the one best suited for treatment of the BRST operator. In recent work[14] Bars and Matsuo treated the Virasoro operators using a closely related *discrete* Moyal formalism with fairly good results, though they also encounter subtleties having to do with the fact that the Virasoro algebra doesn't close in their regulated theory. Next section we will see how to obtain a representation of the star product similar to theirs but in the fermionic ghost sector. Hopefully it is possible to construct a tractable and well-defined treatment of $Q_B$ in some formalism where Witten's product is simple.

The divergences we're grappling with are puzzling. They must be telling us something about our formalism, though it's unclear what it is. Experience with split strings suggests that the midpoint might be playing a role. We can test this hypothesis, since by this wisdom we would expect the midpoint preserving reparameterization generators, $\mathcal{K}_n = L_n - (-1)^n L_{-n}$ to be nonsingular. From the current standpoint this would be remarkable coincidence, since the $\mathcal{K}_n$s are the sum of operators which diverge. Consider for example,

$$
\begin{aligned}
\mathcal{K}_1^{gh} &= 2b_0(c_1 - c_1^+) + (b_1 - b_1^+)c_0 + \sum_{n=2}^{\infty}(1+n)(b_{n-1}^+ c_n - b_{n-1}c_n^+) \\
&\quad + \sum_{n=2}^{\infty}(1-n)(b_{1+n}c_n^+ - b_{1+n}^+ c_n) \\
&= -4ib_0\int_0^\infty d\kappa\ v_1(\kappa)\hat{p}_{\mathrm{o}}(\kappa) - 2i\int_0^\infty d\kappa\ v_1(\kappa)\hat{x}_{\mathrm{o}}(\kappa)\ c_0 \\
&\quad -i\int_0^\infty d\kappa d\kappa' K_1(\kappa,\kappa')[\beta_{\mathrm{o}}^+(\kappa)\chi_{\mathrm{e}}(\kappa') + \beta_{\mathrm{o}}(\kappa)\chi_{\mathrm{e}}^+(\kappa')] \\
&\quad +i\int_0^\infty d\kappa d\kappa'\ K_2(\kappa,\kappa')[\beta_{\mathrm{e}}^+(\kappa)\chi_{\mathrm{o}}(\kappa') + \beta_{\mathrm{e}}(\kappa)\chi_{\mathrm{o}}^+(\kappa')]
\end{aligned}
$$

with

$$
\begin{aligned}
K_1(\kappa,\kappa') &= 2\sum_{n=1}^{\infty}\left[(2n+1)\sqrt{\frac{2n-1}{2n}}v_{2n-1}(\kappa)v_{2n}(\kappa') + (2n-1)\sqrt{\frac{2n+1}{2n}}v_{2n+1}(\kappa)v_{2n}(\kappa')\right] \\
K_2(\kappa,\kappa') &= 2\sum_{n=1}^{\infty}\left[(2n+2)\sqrt{\frac{2n}{2n+1}}v_{2n}(\kappa)v_{2n+1}(\kappa') + 2n\sqrt{\frac{2n+2}{2n+1}}v_{2n+2}(\kappa)v_{2n+1}(\kappa')\right]
\end{aligned}
$$

We see the same $nv_n^2$ terms that caused the divergences in the BRST operator, but here these contributions actually cancel. This is because from eq.(65) we see that $v_{2n-1}$ and $v_{2n+1}$ roughly have opposite sign for large $n$, so the first and second divergent terms cancel. Hence $\mathcal{K}_1^{gh}$ converges. If we had considered $\mathcal{K}_2^{gh}$ we'd have needed to compare $v_{2n-2}$ with $v_{2n+2}$, which have the same sign, but a cancellation still occurs in that case because we subtract the Virasoro generators instead of adding them. In this way we can see that all the $\mathcal{K}_n$'s (including the matter ones in the basis of ref.[15]) are finite. This suggests that the midpoint may be a part of our troubles with the BRST operator.

# VII. Split string and discrete Moyal approach to the fermionic ghosts.

As explained in the introduction, there are at this time essentially three proposed formalisms expressing string field theory in the language of operator algebras and noncommutative geometry: the split string formalism, the discrete Moyal formalism, and the continuous Moyal formalism. So far we've mostly developed fermionic ghosts in the continuous Moyal formalism since this approach is the one most directly related to the three string vertex. But clearly it is advantageous to have an understanding of fermionic ghosts in all three formalisms, since for a particular purpose any one might be preferable to the others.

We begin by considering the Moyal star anti-commutator of the odd position modes and even momentum modes of $b(\sigma)$ and $c(\sigma)$:

$$\{y_{2m-1}, q_{2n}\}_\star = 2G_{2m-1,2n}, \qquad \{x_{2m-1}, p_{2n}\}_\star = 2\frac{2m-1}{2n}G_{2m-1,2n} \tag{68}$$

where,

$$G_{2m-1,2n} = -2\sqrt{\frac{2n}{2m-1}}\int_0^\infty d\kappa\ \coth\tfrac{\pi\kappa}{4}v_{2m-1}(\kappa)v_{2n}(\kappa) \tag{69}$$

We will evaluate this integral by comparing it to an integral explicitly calculated in ref.[15]:

$$\begin{aligned} T_{2m-1,2n} &= -2\sqrt{\frac{2m-1}{2n}}\int_0^\infty d\kappa\ \tanh\tfrac{\pi\kappa}{4}v_{2m-1}(\kappa)v_{2n}(\kappa) \\ &= \frac{2(-1)^{m+n+1}}{\pi}\left(\frac{1}{2m-1+2n}+\frac{1}{2m-1-2n}\right) \end{aligned} \tag{70}$$

The trick is to consider the sum,

$$\begin{aligned} \sum_{m=1}^\infty G_{2m-1,2n}T_{2m-1,2n'} &= 4\int_0^\infty d\kappa d\kappa'\ \coth\tfrac{\pi\kappa}{4}\tanh\tfrac{\pi\kappa'}{4}v_{2n}(\kappa)v_{2n'}(\kappa')\sum_{m=1}^\infty v_{2n-1}(\kappa)v_{2n'-1}(\kappa') \\ &= 2\int_0^\infty d\kappa d\kappa'\ v_{2n}(\kappa)v_{2n'}(\kappa)\delta(\kappa-\kappa') = \delta_{2n,2n'} \end{aligned}$$

Likewise we can argue that $\sum_{n=1}^\infty G_{2m-1,2n}T_{2m'-1,2n} = \delta_{2m-1,2m'-1}$. Therefore $G$ must be the inverse of $T$. The inverse of $T$ is known, and is a matrix often called $R$ in the literature [13, 14, 15]:

$$G_{2m-1,2n} = R_{2m-1,2n} = \frac{4n(-1)^{n+m}}{\pi(2m-1)}\left(\frac{1}{2m-1+2n} - \frac{1}{2m-1-2n}\right). \quad (71)$$

A fermionic ghost version of the discrete Moyal formalism advocated by Bars and Matsuo may be obtained by considering a particular choice of basis,

$$y^D_{2n} \equiv \sum_{m=1}^{\infty} T_{2m-1,2n}\ y_{2m-1} \qquad x^D_{2n} \equiv \sum_{m=1}^{\infty} \frac{2n}{2m-1} T_{2m-1,2n}\ x_{2m-1} \quad (72)$$

In this basis the Moyal star anti-commutator is particularly simple:

$$\{y^D_{2m}, q_{2n}\}_\star = \{x^D_{2m}, p_{2n}\}_\star = 2\delta_{2m,2n} \quad (73)$$

We should mention that this choice of basis encounters difficulties having to do with the fact that both $T$ and $R$ possess zero modes and are in a sense not invertible. While it's true that $TR = RT = 1$, this is only sufficient to imply that $T$ and $R$ are invertible when multiplication is *associative*, which sometimes it isn't for such infinite dimensional matrices[11]. See [28, 14, 15] for further discussion of this basis and its particular advantages and disadvantages. If we express the string field as a functional,

$$\Psi^D[x^D_{2n}, p_{2n}, y^D_{2n}, q_{2n}] = \Psi[x_{2n-1}, y_{2n-1}, p_{2n}, q_{2n}] = \Psi^M[\vec{x}(\kappa), \vec{y}(\kappa)]$$

then in the discrete Moyal formalism one would calculate the reduced star product as,

$$\Psi^D *_{b_0} \Phi^D = \mathcal{N}\exp\left[\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial x^D_{2n}}\frac{\partial}{\partial p'_{2n}} + \frac{\partial}{\partial p_{2n}}\frac{\partial}{\partial x^{D'}_{2n}} + \frac{\partial}{\partial y^D_{2n}}\frac{\partial}{\partial q'_{2n}} + \frac{\partial}{\partial q_{2n}}\frac{\partial}{\partial y^{D'}_{2n}}\right)\right]\Psi^{D'}\Phi^D\Bigg|_{X=X'} \quad (74)$$

This equation would be the starting point for a discrete Moyal formulation of string field theory in the fermionic ghost sector.

Let us move on and see how to construct a split string representation of the reduced star product. Consider as a simple example a function of two fermionic variables, $\tilde{A}(x,p)$, and an associated function which is given by a Fourier transform on the $p$ variable:

$$\hat{A}(x+y, -x+y) \equiv A(x,y) = \int dp\ e^{-py}\tilde{A}(x,p),$$
$$\tilde{A}(x,p) = \int dy\ e^{py}\hat{A}(x+y,-x+y).$$

[11] We can count on associativity only if we restrict our space of operators, in particular to those which act on the Hilbert space of square-integrable sequences, not the larger Banach space of bounded sequences[15]. Unfortunately, a satisfactory formulation of string field theory seems to require the latter space.

It turns out that Moyal star multiplication of functions of $x, p$ is equivalent to calculating,

$$\hat{A} \star \hat{B}(l, r) = -2 \int dw \hat{A}(l, -w) \hat{B}(w, r) \tag{75}$$

We can see this directly as follows:

$$\begin{aligned}
\tilde{A} \star \tilde{B}(x,p) &= e^{\frac{\partial}{\partial x}\frac{\partial}{\partial p'} + \frac{\partial}{\partial p}\frac{\partial}{\partial x'}} \int dy' dy \; e^{p'y'} A(x'+y', -x'+y') e^{py} B(x+y, -x+y) \Big|_{X=X'} \\
&= \int dy' dy \; e^{p'y'} e^{y\frac{\partial}{\partial x'}} A(x'+y', -x'+y') e^{-y'\frac{\partial}{\partial x}} e^{py} B(x+y, -x+y) \Big|_{X=X'} \\
&= \int dy' dy \; e^{p(y+y')} A(x+y+y', -x-y+y') B(x-y'+y, -x+y'+y) \\
&= -2 \int dz dw \; e^{pz} A(x+z, -w) B(w, -x+z)
\end{aligned}$$

The last line follows from the definitions $z = y + y'$ and $w = -x + y' - y$ and is easily seen to be equivalent to eq.(75)[12]. We can repeat the above argument and find a similar representation of Moyal-star multiplication of string fields. Define,

$$\hat{\Psi}[l_{2n}, m_{2n}, r_{2n}, s_{2n}] = \Psi[x_n, y_n] = \int \left( \prod_{m=1}^{\infty} \tfrac{1}{i} dp_{2m} dq_{2m} e^{-p_{2m}x_{2m} - q_{2m}y_{2m}} \right) \Psi^D[x^D_{2n}, p_{2n}, y^D_{2n}, q_{2n}]$$

$$\Psi^D[x^D_{2n}, p_{2n}, y^D_{2n}, q_{2n}] = \int \left( \prod_{m=1}^{\infty} \tfrac{1}{i} dx_{2m} dy_{2m} e^{p_{2m}x_{2m} + q_{2m}y_{2m}} \right) \hat{\Psi}[l_{2n}, m_{2n}, r_{2n}, s_{2n}]$$

where,

$$\begin{aligned}
l_{2m} &= \tfrac{1}{2m}(x_{2m} + x^D_{2m}), \qquad & m_{2m} &= y_{2m} + y^D_{2m}, \\
r_{2m} &= \tfrac{1}{2m}(x_{2m} - x^D_{2m}), \qquad & s_{2m} &= y_{2m} - y^D_{2m}.
\end{aligned} \tag{76}$$

Then we can calculate the reduced star product as,

$$\hat{\Psi} *_{b_0} \hat{\Phi}[l_{2n}, m_{2n}, r_{2n}, s_{2n}] = \mathcal{N} \int \prod_{n=1}^{\infty} \tfrac{8n}{i} du_{2n} dw_{2n} \; \hat{\Psi}[l_{2n}, m_{2n}, -u_{2n}, -w_{2n}] \hat{\Phi}[u_{2n}, w_{2n}, r_{2n}, s_{2n}] \tag{77}$$

This equation bears a very close resemblance to the split string formalism. Note that "left" and "right" modes are identified only up to a sign. Let us see how this formula may be interpreted more geometrically in terms of overlaps of $c(\sigma)$ and $b(\sigma)$. From eq.(76) and eq.(72) we have the correspondence,

$$m_{2n} = y_{2n} + \sum_{m=1}^{\infty} T_{2m-1,2n} y_{2m}, \quad s_{2n} = y_{2n} - \sum_{m=1}^{\infty} T_{2m-1,2n} y_{2m-1} \tag{78}$$

[12] A detailed point: above we assumed we could commute $dy'$ through $\hat{A}$, but in the following discussion we will actually not need this assumption since we will always have pairs $dxdy$ for our measure, which is Grassmann even.

This equation is easily seen to define left and right half string Fourier modes for $c(\sigma)$ with Neumann boundary conditions at the midpoint:

$$\begin{aligned} m(\sigma) &= m_0 + \sqrt{2}\sum_{n=1}^{\infty} m_{2n} \cos 2n\sigma = c(\sigma) \\ s(\sigma) &= s_0 + \sqrt{2}\sum_{n=1}^{\infty} s_{2n} \cos 2n\sigma = c(\pi - \sigma) \qquad \sigma \in [0, \pi/2] \end{aligned}$$

See ref.[28] for discussion of split strings with Neumann boundary conditions at the midpoint. Of course, now we see that eq.(77) imposes precisely the correct half string anti-overlap conditions on $c(\sigma)$. Note that our string fields do not depend on the half string zero modes $m_0$ and $s_0$ since we have taken Siegel gauge. Now let's consider the $x$ modes,

$$\begin{aligned} l_{2n} &= \frac{1}{2n} x_{2n} + \sum_{m=1}^{\infty} T_{2m-1,2n} \frac{1}{2m-1} x_{2m-1}, \\ r_{2n} &= \frac{1}{2n} x_{2n} - \sum_{m=1}^{\infty} T_{2m-1,2n} \frac{1}{2m-1} x_{2m-1} \end{aligned}$$

Just like eq.(78) this looks like half string Fourier modes for a field like $c(\sigma)$ but with Fourier modes $x_n/n$:

$$\tilde{b}(\sigma) \equiv \sqrt{2}\sum_{n=1}^{\infty} \frac{x_n}{n} \cos n\sigma \tag{79}$$

In particular, this field satisfies

$$\frac{\partial}{\partial\sigma}\tilde{b}(\sigma) = ib(\sigma)$$

So we have half string fields,

$$\begin{aligned} l(\sigma) &= l_0 + \sqrt{2}\sum_{n=1}^{\infty} l_{2n} \cos 2n\sigma = \tilde{b}(\sigma) \\ r(\sigma) &= r_0 + \sqrt{2}\sum_{n=1}^{\infty} r_{2n} \cos 2n\sigma = \tilde{b}(\pi - \sigma) \qquad \sigma \in [0, \pi/2] \end{aligned}$$

Our string fields do not depend on the half string zero modes $l_0$ and $r_0$. We can easily see that eq.(77) imposes half string anti-overlap conditions on $\tilde{b}(\sigma)$. This is exactly correct, since setting anti-overlap conditions on $\tilde{b}(\sigma)$ is equivalent to setting overlap conditions on $b(\sigma)$. Note that the overlap conditions for $\sigma \neq \frac{\pi}{2}$ should hold *both* for the full and reduced star product, since they only differ by the action of $\mathcal{Q}$, which commutes with these constraints. Therefore we see the half string overlap conditions, which were originally used to construct the fermionic ghost vertex in ref.[7], explicitly arise out of our construction. This is a good check of the consistency of our results[13].

The last element we need to define Witten's product in the discrete Moyal and split string formalisms is a legitimate formulation of $\mathcal{Q}$. This is easy to come by. Invoking eq.(73) we can write the action of $\mathcal{Q}$ on $\Psi^D[x_{2n}^D, p_{2n}, y_{2n}^D, q_{2n}]$ as,

$$\mathcal{Q}(\Psi^D) = c_0\Psi^D + \mathcal{Q}^D \star \Psi^D + (-1)^{G(\Psi)}\Psi^D \star \mathcal{Q}^D \tag{80}$$

where

$$\mathcal{Q}^D = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n y_{2n}^D$$

Translating into the split string formalism, we find similarly

$$\mathcal{Q}(\hat{\Psi}) = c_0\hat{\Psi} + \hat{\mathcal{Q}} \star \hat{\Psi} + (-1)^{G(\Psi)}\hat{\Psi} \star \hat{\mathcal{Q}} \tag{81}$$

with

$$\hat{\mathcal{Q}} = \frac{1}{\sqrt{2}}\prod_{n=1}^{\infty}\frac{1}{8ni}(l_{2n}+r_{2n})(m_{2n}+s_{2n})\sum_{n=1}^{\infty}(-1)^n m_{2n}$$

Actually, we can plug this directly into eq.(81) to find,

$$\mathcal{Q}(\hat{\Psi}) = \left(c_0 + \tfrac{1}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n(m_{2n}+s_{2n})\right)\hat{\Psi} = c(\tfrac{\pi}{2})\hat{\Psi}$$

This of course makes perfect sense. However, it is interesting to note that our approach apparently fixes the way that $c(\frac{\pi}{2})$ must be expressed in split string variables if it is to be identified with $\mathcal{Q}$. A priori, there is no unique way to write $c(\frac{\pi}{2})$ in terms of split string variables.

# IX. Conclusion

To summarize, we have shown that the reduced star algebra of fermionic ghosts in Siegel gauge may be formulated as a continuous tensor product of Clifford Algebras, up to a nontrivial normalization. We have also developed split string and discrete Moyal representations of the reduced star product so that the fermionic ghost sector may be studied from other viewpoints. We have formulated the kinetic operator of VSFT and the ghost number operator in the language of noncommutative field theory, with good results, but the BRST operator describing the vacuum of the unstable D-25 brane is divergent and must somehow be regulated to get a consistent formulation.

[13]Note: There is nothing deep about the fact that we found Neumann boundary conditions at the midpoint in our construction. We would have found Dirichlet boundary conditions if we had used an odd mode discrete Moyal basis $y_{2m-1}$, $q_{2m-1}^D = -\sum_{n=1}^{\infty} T_{2m-1,2n}q_{2n}$ and similarly for $x, p$. Furthermore, the field $b'(\sigma) = i\partial_\sigma b(\sigma)$ would have emerged in the split string description rather than $\tilde{b}(\sigma)$

This last point is perhaps the most crucial problem currently facing any attempt to formulate string field theory in the language of noncommutative geometry. After all, the only real use of these methods is to simplify Witten's open string field theory enough to possibly permit analytic solution to the field equations. Vacuum string field theory, of course, seems to work very naturally in the Moyal formulation, but to be honest VSFT is already under comparative analytic control. Indeed, it is the accumulation of all of this analytic understanding that makes the results of this paper possible. Of course, we hope that our framework might shed some light into further investigations of VSFT, particularly it's more difficult and controversial aspects, such as normalization of the action and the study of fluctuations about it's various classical solutions. Still, an adequate treatment of $Q_B$ is extremely important.

Another important open question is how one should formulate the star product outside of Siegel gauge as a Moyal or operator product of some sort. Siegel gauge may be useful for specific calculations, but it would be extremely interesting to understand the full structure of string field theory's gauge symmetry within the operator/Moyal language. Furthermore, gauge invariance may offer a method for generating solutions to the field equations, as suggested for instance in ref.[11].

Another issue is the normalization of the algebra, and whether Witten's star product including ghost contributions truly defines an algebra of bounded operators, as the CFT results seem to suggest.

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# Appendix

For reference we will write out the kernels in equations (66) and (67) explicitly:

$$\begin{aligned}
K^{\rm ee}_{L^e}(k_1,k_2,k) &= -4i\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\frac{(m+n)\sqrt{m-n}}{\sqrt{mn}}v_{2m}(k)v_{2(m-n)}(k_1)v_{2n}(k_2), \quad \text{for } k_1,k_2>0\\
&= 4i\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{(m+n)\sqrt{n-m}}{\sqrt{mn}}v_{2m}(k)v_{2(n-m)}(k_1)v_{2n}(k_2), \quad \text{for } k_1<0,\ k_2>0\\
&= 4i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m-n)\sqrt{n+m}}{\sqrt{mn}}v_{2m}(k)v_{2(n+m)}(k_1)v_{2n}(k_2), \quad \text{for } k_1>0,\ k_2<0\\
&= 0 \quad \text{for } k_1,\ k_2<0
\end{aligned}$$

$$K^{\rm oo}_{L^e}(k_1,k_2,k) = i2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{(2m+2n-1)\sqrt{2m-2n+1}}{\sqrt{2m(2n-1)}}v_{2m}(k)v_{2m-2n+1}(k_1)v_{2n-1}(k_2) \quad \text{for } k_1,\ k_2>0$$

$$
\begin{aligned}
&= i2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{(2m+2n-1)\sqrt{2n-2m-1}}{\sqrt{2m(2n-1)}}v_{2m}(k)v_{2n-2m-1}(k_1)v_{2n-1}(k_2) \\
&\qquad \text{for } k_1<0,\ k_2>0 \\
&= i2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(2m-2n+1)\sqrt{2n+2m-1}}{\sqrt{2m(2n-1)}}v_{2m}(k)v_{2n+2m-1}(k_1)v_{2n-1}(k_2) \\
&\qquad \text{for } k_1>0,\ k_2<0 \\
&= 0 \quad \text{for } k_1,\ k_2<0
\end{aligned}
$$

$$
K^{\rm oe}_{L^{\rm e}}(k_1,k_2,k) = K^{\rm eo}_{L^{\rm e}}(k_1,k_2,k)=0 \tag{82}
$$

$$
\begin{aligned}
K^{\rm eo}_{L^{\rm o}}(k_1,k_2,k) &= -8\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\frac{(m+n-1)\sqrt{m-n}}{\sqrt{(2m-1)(2n-1)}}v_{2m-1}(k)v_{2(m-n)}(k_1)v_{2n-1}(k_2) \\
&\qquad \text{for } k_1,\ k_2>0 \\
&= 8\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{(m+n-1)\sqrt{n-m}}{\sqrt{(2m-1)(2n-1)}}v_{2m-1}(k)v_{2(n-m)}(k_1)v_{2n-1}(k_2) \\
&\qquad \text{for } k_1<0,\ k_2>0 \\
&= 8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m-n)\sqrt{m+n-1}}{\sqrt{(2m-1)(2n-1)}}v_{2m-1}(k)v_{2(m+n-1)}(k_1)v_{2n-1}(k_2) \\
&\qquad \text{for } k_1>0,\ k_2<0 \\
&= 0 \quad \text{for } k_1,\ k_2<0
\end{aligned}
$$

$$
\begin{aligned}
K^{\rm oe}_{L^{\rm o}}(k_1,k_2,k) &= 2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\frac{(2m+2n-1)\sqrt{2m-2n-1}}{\sqrt{(2m-1)2n}}v_{2m-1}(k)v_{2m-2n-1}(k_1)v_{2n}(k_2) \\
&\qquad \text{for } k_1,\ k_2>0 \\
&= 2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{(2m+2n-1)\sqrt{2n-2m+1}}{\sqrt{(2m-1)2n}}v_{2m-1}(k)v_{2n-2m+1}(k_1)v_{2n}(k_2) \\
&\qquad \text{for } k_1<0,\ k_2>0 \\
&= 2\sqrt{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(2m-2n-1)\sqrt{2n+2m-1}}{\sqrt{(2m-1)2n}}v_{2m-1}(k)v_{2n+2m-1}(k_1)v_{2n}(k_2) \\
&\qquad \text{for } k_1>0,\ k_2<0 \\
&= 0 \quad \text{for } k_1,k_2<0
\end{aligned}
$$

$$
K^{\rm oo}_{L^{\rm o}}(k_1,k_2,k) = K^{\rm ee}_{L^{\rm o}}(k_1,k_2,k)=0 \tag{83}
$$

Comparing the form of these equations with eq.(65), it is clear that the above kernels all diverge.

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