Preprint typeset in JHEP style - HYPER VERSION hep-th/0406199

# Locality, Causality, and an Initial Value Formulation for Open String Field Theory

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Abstract: In this paper, we explore the questions of time, locality and causality in the framework of covariant open bosonic string field theory. We show that if an open string field is expressed as a certain local function on spacetime—in particular, a function of the lightcone component of the midpoint and the transverse center of mass degrees of freedom—that cubic string field theory is nonsingular and local in lightcone time. In particular, the theory has a well defined initial value formulation resembling that of an ordinary second order relativistic field theory in lightcone frame. This description can be achieved by a nonsingular unitary transformation on the Fock space, and we demonstrate explicitly that the theory is gauge invariant and the interaction vertex is local in this basis. With an initial value formulation at hand, we are able to construct an explicit second quantized operator formalism for the theory using the Hamiltonian BRST formalism. We also explore issues of causality by considering a singular limit of the theory where all spacetime coordinates are taken to the midpoint. At any stage in this limit, the theory is well-defined and arbitrarily close to being completely local and manifestly causal. We argue that the this limit must account for the macroscopic causality of the string S-matrix.

Keywords: String Field Theory, Tachyon Condensation, BRST quantization.

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## 1. Introduction

Causality is a basic requirement of any acceptable physical theory; causes should always precede their effects, and if the theory is Lorentz invariant, spacelike separated events should be uncorrelated.

In quantum field theory, causality follows from a basic fact: quantum field theory is *local*. In non-gravitational theories, where the space-time manifold is fixed and non-dynamical, causality means that the theory has localizable observables which commute at spacelike separations, since space-like separated measurements should not interfere. This follows directly from locality, since observables are functions of local fields that, as quantum operators, commute at space-like separations (up to a gauge transformation) as a consequence of Lorentz invariance and locality of the theory's interactions. However, in many theories of interest the only known observables are S-matrix elements describing scattering experiments, rather than local observables. This is certainly true in gravitational theories, where local operators are not diffeomorphism (gauge) invariant, and it seems to be the case in string theory, even open string theory. For such theories, causality requires certain analytic properties of the S-matrix which ensure that, in a scattering experiment, two incoming wave-packets will collide *before* the outgoing wave-packets emerge. These analytic properties also follow from locality: one expresses the scattering amplitude in terms of Greens functions of interpolating fields, either elementary or composite, which

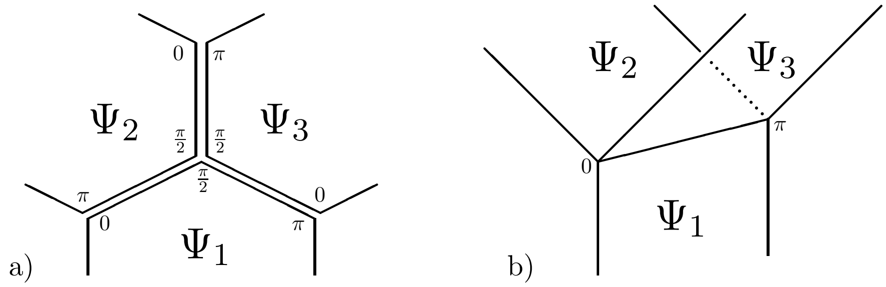



Figure 1: a) The Witten vertex $\langle\Psi_1, \Psi_2 * \Psi_3\rangle$. b) A hypothetical, but obviously incorrect, "local" interaction of strings.

create asymptotic particle states when acting on the vacuum. The locality of the commutators of these interpolating fields leads to the analytic properties of a causal S-matrix. This measure of macroscopic causality follows from locality, but is weaker, since on-shell S-matrix amplitudes are not enough to probe micro-causality. So it seems that locality is an indispensable theoretical mechanism for ensuring that our theories are sensible and causal.

How does this discussion extend to string theory? At first sight it would appear that strings, being extended objects, could not be described by a local, casual theory. It seems natural to address this question directly in the framework of covariant string field theory. In the case of open bosonic strings we have a particularly simple formulation, due to Witten[1], in terms of a cubic spacetime action,

$$S = -\langle\Psi, Q_B\Psi\rangle - \frac{2}{3}g\langle\Psi, \Psi * \Psi\rangle \tag{1.1}$$

where $\langle,\rangle$ is the BPZ inner product, $Q_B$ is the BRST operator corresponding to a choice of conformal background, and the string field $\Psi$ is a vector in the state space of a particular boundary conformal field theory (see [2] for a nice review). The cubic interaction term is given by the Witten vertex, which can be described as follows: Writing the string field as a functional of an open string configuration $\Psi[x(\sigma)] = \langle x(\sigma)|\Psi\rangle, \quad \sigma \in [0, \pi]$ (ignoring ghosts), to calculate $\langle\Psi_1, \Psi_2 * \Psi_3\rangle$ we must glue the left half of $x(\sigma)$ in $\Psi_1$ to the right half of $x(\sigma)$ in $\Psi_2$ and so on around cyclically, as shown in figure 1a. Certainly, this procedure is very nonlocal. By comparison, a naive "local" interaction $\Psi[x(\sigma)]^3$ (figure 1b) clearly has nothing to do with string interactions as we know them.

Perhaps then it comes as a surprise that critical string theory produces an analytic S-matrix consistent with macroscopic causality. In absence of any other known theoretical mechanism which might explain this, despite appearances one is lead to believe that string interactions must be, in some sense, local.

The problems with nonlocality and causality in string theory become almost fatal when we realize that the theory appears to be nonlocal in time. To see the problems more deeply, suppose we truncate the cubic action eq.1.1 to include only the tachyon field $\Psi = \int dp\phi(p)|p\rangle$:

$$S_{\text{tachyon}} = \int dx \left[\phi(\tfrac{1}{2}\,\partial^2 + 1)\phi - \frac{2}{3}\kappa g[e^{\frac{1}{2}V_{00}\partial^2}\phi]^3\right] \tag{1.2}$$

Due to the nasty differential operator $e^{\frac{1}{2}V_{00}\partial^2}$ this action contains an infinite number of derivatives in both time and space (if we included more fields we would get an action with similar, but more complicated appearance). The Lagrangian eq.1.2 is completely nonlocal, as can be seen with the formula[1],

$$e^{\frac{1}{2}V_{00}\partial^2}\phi(x) = \frac{1}{2\pi V_{00}}\int dx' e^{\frac{1}{2V_{00}}(x-x')^2}\phi(x')$$

Apparently, the tachyon at a point $x$ couples not only to itself, but also to its values arbitrarily far in the future, in the past, and even at space-like separations! Such a theory could not be meaningfully causal. Nor could one imagine that it has a sensible initial value formulation. Without an initial value formulation, we cannot proceed to the Hamiltonian formalism and canonically quantize, so it is not clear that a quantum theory for the action eq.1.2 even exists. Perhaps the most frightening aspect of eq.1.2, however, is that any theory whose Lagrangian depends nontrivially on any more than first time derivatives has a completely unstable Hamiltonian[3, 4]. The instabilities presumably present in eq.1.2, however, could not manifest themselves in the perturbative S-matrix, since the higher derivatives enter only at the level of the interaction. Certainly, we hope that string theory has a causal, stable Hamiltonian and a sensible quantum mechanical definition beyond perturbation theory; yet it has been a mystery how string theory really manages to escape these sicknesses.

It is therefore clear that any acceptable physical theory should be local in time. If the theory is Lorentz invariant, presumably this means that it is (in some sense) local in space as well, and this should be sufficient to ensure macroscopic causality of the S-matrix.

In this paper we attempt to face up to these facts in the framework of open bosonic string field theory. We find that string theory avoids problems with nonlocality in a surprising way. In particular, we find that the Witten vertex is "local enough" to allow for a nonsingular description of the theory which is completely local along a single null direction. Specifically, if we regard the string field as a certain function on spacetime—a function of the lightcone component of the midpoint and the transverse center of mass degrees of freedom—the cubic action is local and first order in lightcone time derivatives. Therefore string field theory has a well-defined initial value formulation, a sensible canonical quantum theory, and a Hamiltonian free of higher derivative instabilities. However, since our choice of spacetime coordinate is not Lorentz invariant, microscopic causality is not manifest and

[1] Actually, there is even some subtlety in defining the differential operator $e^{\frac{1}{2}V_{00}\partial^2}$ [12]. Our formula gives a definition which converges only for a certain class of functions.

the theory remains nonlocal in the transverse spatial directions. The remaining nonlocality, we find, cannot be removed while maintaining a nonsingular description of the theory. This perhaps should be expected, since the known ultraviolet properties of string theory, in particular ultraviolet finiteness and polynomial boundedness of the S-matrix, seem to contradict properties of a completely local quantum field theory. The picture we find is therefore similar to that in lightcone string field theory: the theory is local in (lightcone) time but still nonlocal in space. However, unlike lightcone string field theory, it is clear that cubic string field theory at least has a local limit where all spacetime coordinates are taken to the midpoint. We investigate this limit with a careful choice of regulator and show that at any stage the theory is nonsingular but arbitrarily close to being local and manifestly causal. We believe that the existence of this limit, though singular, must account for the macroscopic causality of the string S-matrix. Thus, string theory is local enough to avoid the inconsistencies of a theory which is acausal and nonlocal in time, but is nonlocal enough to make string theory different from quantum field theory.

To motivate our particular perspective, it seems appropriate to discuss earlier attempts to understand the role of locality, causality and time in string theory, and explain why we feel these approaches do not adequately address the problems just raised. To start with, we mention some discussions of causality in the framework of free lightcone[5] and covariant[6] string field theory. It has been argued that the natural generalization of the commutativity of quantum fields at space-like separations is that string fields should commute outside the so-called *string light cone*,

$$\int d\sigma[x(\sigma) - x'(\sigma)]^2 > 0 \quad \rightarrow \quad [\Psi(x(\sigma)), \Psi(x'(\sigma))] = 0 \tag{1.3}$$

This is a very strange condition. It is not reparameterization invariant, and although correct for free strings, is is violated once interactions are included[7].

It must be said that the meaning and necessity of eq.1.3 is far from clear. String fields are not observable nor is it clear how to construct observables from them; there is no reason why their commutators should have any locality properties at all. What is required for the establishment of macroscopic causality is the analytic properties of the scattering amplitudes, which is related directly to locality properties of the correlation functions. Correlation functions of string fields expressed as functionals of $x(\sigma)$ correspond to path integrals over Riemann surfaces with holes and half-disks removed; such objects are highly singular and ill-suited for constructing scattering amplitudes for asymptotic particle states in string theory. To construct a more appropriate basis of interpolating fields one should decompose the string field into a mode basis of ordinary local fields, $\phi^i(x)$, one for each observable asymptotic particle described by the string. These can serve as a basis of interpolating fields for the purpose of constructing the S-matrix, and if they satisfy causal commutation relations

$$[\phi^i(x), \phi^j(y)] = 0 \qquad (x-y)^2 > 0$$

(though perhaps in a singular limit) then macrocauslity should be valid. At any rate, for free strings, the Lagrangian is simply a sum of free Lagrangians for the individual string modes, and trivially the component fields commute at space-like separation. Causality is only an issue once interactions are included. Therefore we feel that eq.1.3 is not a good starting point for discussing issues of locality and causality in string theory.

Apparently, string field theory should not be regarded as specifying an action for some singular functional $\Psi[x(\sigma)]$, but rather as specifying a action for a countable number of local spacetime fields $\phi^i(x)$, one for each mode of the string. To resolve the difficulties of nonlocality and causality in string theory, we need to show that there is some choice of $\phi^i(x)$ which at the very least renders the theory local in time, and perhaps in some singular limit local in space as well. A natural question then arises: what is the label $x$ in all of these fields? In quantum field theory, the meaning of $x$ is clear: it refers to the location $\vec{x}$ of a point particle at time $x^0$. In our case, however, the string is not a point particle; whatever $x$ describes must depend on how we choose to "break up" the string into particle-like constituents. While we do not have a unique notion of "position" in string theory, we do have a well-defined notion of momentum: the conserved charge $p_\mu$ associated with translations of $x(\sigma)$. It is natural to require that an acceptable choice of $x$ satisfies,

$$[x^\mu, p_\nu] = i\delta^\mu_\nu.$$

This condition follows for any $x$ given by,

$$x^\mu = \int_0^\pi d\sigma f^\mu_\nu(\sigma) x^\nu(\sigma) \qquad \int_0^\pi d\sigma f^\mu_\nu(\sigma) = \delta^\mu_\nu \tag{1.4}$$

Thus, having decided that local fields are what interest us, we must still decide, subject to eq.1.4, what our fields are local in.

In most studies of string field theory to date, the standard choice for $x$ has been the string center of mass $x_{\rm cm} = \frac{1}{\pi}\int d\sigma x(\sigma)$. The center of mass has the advantage of being the natural spacetime coordinate for the mass eigenstates of the free string. However, the disadvantage of $x_{\rm cm}$ is that the action appears extremely nonlocal in both space and time, as we saw in eq.1.2.

The problems with the center of mass $x$ have recently become unavoidable in the context of the tachyon condensation problem in cubic string field theory. Let us recall the basic story. After the seminal work of Sen[8] it has been realized that the tachyon of the open bosonic string can be interpreted as an instability of the space-filling D25-brane on which the open string ends. If the brane is allowed to decay, one is presumably left with a vacuum without any D-branes or open strings, i.e. the vacuum of the closed bosonic string. Strong evidence for the validity of this conjecture comes from the level truncation scheme, where one truncates the cubic action to include only a finite number of lightest mass fields. At zero momentum, the Lagrangian is reduced to a quadratic plus cubic polynomial potential of scalar fields whose minimum is an approximation to the closed string vacuum. Remarkably, level truncation converges rapidly[9], and yields an excellent approximation to the known (or conjectured) exact result for the difference in vacuum energies between the unstable and stable solutions.

It is natural given a potential of this form to study time dependent solutions, starting close to the unstable maximum and rolling down to the stable vacuum. Sen in fact proposed a boundary conformal field theory describing such a process, whereby the tachyon rolls homogenously towards the closed string vacuum but does not cross over in finite time[10]. Attempts to identify such a solution using level truncation of the mass eigenstates in open string field theory, however, have run into serious problems. Solutions seem to have very erratic behavior, in complete qualitative disagreement with Sen's rolling tachyon solution: the string field seems to pass quickly through the closed string vacuum and then quite far up the steep side of the potential, after which a sequence of oscillations of diverging amplitude ensues[12, 11]. In retrospect, the pathological runaway behavior of these solutions is not much of a surprise, given the higher derivative instabilities expected from the extreme nonlocality of the cubic action when truncated in mass eigenstates[2].

To restore sanity one is tempted to consider the lightcone string field theory, which although gauge fixed and not manifestly Lorentz invariant, has the advantage of being ghost free and completely local in lightcone time $x^+$. The difficulty with this approach, however, is that we have no concrete evidence that lightcone string field theory contains nonperturbative information, and in particular no stable closed string vacuum solution is known. In fact, some[4] have argued that that the theory's locality in $x^+$ is solely an artifact of perturbation theory. Indeed it is not difficult to see that there is something of a paradox with the theory's locality in $x^+$: the lightcone string field theory only assigns one physical phase space degree of freedom per component field, whereas the covariant theory, containing an infinite number of time derivatives, assigns an infinite number. The authors of ref.[4] offered a possible resolution to this paradox in terms of the mechanism of "localization," which can be understood as follows. In studying a theory whose higher time derivatives enter only in interaction, one can always identify two types of solutions: perturbative solutions, which pass over in the weak coupling limit to solutions of the free theory, and nonperturbative solutions which do not. The nonperturbative solutions generically carry negative energies and display unstable, "runaway" behavior. For such theories, it turns out to be possible to find an "equivalent" theory without higher derivatives whose solution space contains only the perturbative "stable" solutions, but not the runaway ones[3]. Possibly this is how lightcone string field theory achieves its locality in $x^+$. If this were true, then nonperturbatively lightcone string field theory would be inequivalent to covariant string field theory, and in particular probably fails to be Lorentz invariant. One can hope that

[2]For some additional studies of the nonlocal properties of truncated open string field theory and p-adic string theory, see ref.[13]

[3]To see how localization works[4], consider a simple model of a particle at position $q$ satisfying the equation of motion,

$$0=\left(\frac{d^2}{dt^2}+\omega^2\right)q+\frac{g(1-g)}{\omega^2}\frac{d^4}{dt^4}q$$

In this model, the quartic time derivative is the "interaction," whose strength is measured in $g$. It is simple to see that equation has four linearly independent solutions, two positive energy solutions with frequencies $\pm\frac{\omega}{\sqrt{1-g}}$ and two negative energy solutions with frequencies $\pm\frac{\omega}{\sqrt{g}}$. Only the positive energy solutions are well defined in the $g\to 0$ limit, and they can be described just as well with the second order equation of motion,

$$0=\left(\frac{d^2}{dt^2}+\frac{\omega^2}{1+g}\right)q$$

Thus, perturbatively the theory is equivalent to one without higher derivatives. For many more generic and complicated examples, we refer the reader to ref.[4].

this is not the case, but clearly it would be much more reassuring to establish some sort of locality (particularly in $x^+$) directly in covariant string field theory.

Thus, other approaches having proved inadequate, we are forced to return to covariant string field theory and consider whether we can do better by choosing a position coordinate *other* than the string center of mass to label the spacetime dependence of our component fields. Keep in mind that, as far as the closed string vacuum is concerned, any choice of $x$ is equally valid, since the vacuum is translationally invariant and hence independent of $x$.

A little thought about the Witten vertex reveals that the only choice of $x$ which has a hope of giving us a local action for the component fields is the string midpoint, $x(\frac{\pi}{2})$. Indeed, there were several papers in the early days[14, 15, 16] which attempted to formulate string field theory directly in terms of fields which were local functions of the midpoint. Unfortunately such a formulation appears to be extremely singular: the component fields all carry infinite energy and the vertex is generically afflicted with anomalies which spoil locality. The central insight of this paper, however, is that we do not need a completely local and nonsingular formulation in terms of component fields; all we need is locality in a single direction which can be identified as time. This is possible: if we choose $x$ so that its light cone component $x^+$ coincides with the string midpoint, the component fields can have finite energy and the vertex is manifestly local in lightcone time.

This paper is organized as follows. In section 2 we explain why it is not possible to have a completely local formulation of string field theory in terms of the midpoint coordinate. Specifically, the kinetic terms in the action are infinite and the vertex generally fails to be local, contrary to naive expectations. In section 3 we identify a much less singular basis (the "tilde basis") where only lightcone time $x^+$ is identified with the string midpoint, and explain how this can be achieved by a nonsingular unitary transformation on open string the Fock space. We also introduce a convenient basis for the ghost fields where the zero mode of $c(\sigma)$ is taken to be $c(\frac{\pi}{2})$. With the help of the tilde basis we elucidate the initial value formulation and identify constraints which must be imposed on the initial conditions. In section 4 we construct an explicit second-quantized operator formalism for the interacting string field theory. We identify a remarkably simple spacetime BRST charge and prove that it is both nilpotent and commutes with the string Hamiltonian. As an added bonus, we show that our choice of basis in the ghost sector allows one to see explicitly that the classical master action for string field theory automatically satisfies the *quantum* BV master equation, thus providing a field theoretic proof that the Feynmann diagrams of the cubic action automatically provide a complete single cover of the moduli space of open Riemann surfaces. In section 5 we explore the issue of causality, in particular whether we can show given our initial value formulation that information propagates only in the future light cone. Since lightcone time plays a preferred role in the tilde basis, causality is not manifest. However we explore the singular limit where all components of $x$ are taken to the string midpoint. With a careful choice of regulator, we show that at any stage in the limit the theory is nonsingular and arbitrarily close to being manifestly Lorentz invariant and local, and hence presumably causal. In appendix A we carefully study the interaction vertex in the tilde basis, and demonstrate both analytically and numerically that the theory is both gauge invariant and completely local in lightcone midpoint time for well-behaved string fields. We have placed this discussion in an appendix since it is somewhat technical, but it is crucial for establishing the credibility of our results—experience shows that string field theory is a delicate framework which easily disintegrates if arguments are overly formal. In appendix B we prove some identities used in appendix A, and in appendix C we list some useful formulas. In section 6 we offer some conclusions.

After much of this work was completed, we realized that the basic features of the lightcone midpoint formalism had already been discovered many years ago by Maeno[17], though his work seems to be completely unknown. We feel that these ideas are sufficiently fundamental to be brought again to the attention of the community. At any rate, our work goes substantially beyond Maeno's in providing an modern and detailed discussion, an explicit second quantized operator formalism for the theory[4], some exploration of the role of causality, and a careful demonstration of both gauge invariance and locality.

## 2. Why the midpoint doesn't work

Before moving on to the body of our paper, it seems necessary to explain why *only* the lightcone component of the midpoint can be understood as defining a time coordinate in which cubic string field theory is local and no more than second order in time derivatives. Indeed, since the cubic interaction identifies the midpoint coordinates of all three string fields locally, one would imagine that any timelike component of the midpoint could be used to construct a well-defined initial value formulation of the theory. In fact, one might even propose that the string field should be viewed as a spacetime function of all components of midpoint, in which case the theory should be completely local and second order in the midpoint coordinate. This idea was originally proposed by Witten[18] and was subsequently explored in references [14, 15, 16]. Recently the idea has reappeared in the context of the operator/Moyal formalism in ref.[19, 20]. Our discussion follows that of references [14, 15, 16, 19]. The idea is to implement a unitary transformation,

$$U = \exp\left[-p_\mu \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}(\alpha^\mu_{2n} - \alpha^\mu_{-2n})\right]. \tag{2.1}$$

on the mode basis of the state space of the boundary conformal field theory. Under this transformation the string center of mass $x$ becomes the string midpoint position $x(\frac{\pi}{2})$:

$$Ux^\mu U^{-1} = x^\mu(\tfrac{\pi}{2}).$$

The tachyon vacuum state $|k\rangle$ transforms to a new vacuum state $|k\rangle'$ labelled by momenta $k$ which are now interpreted as conjugate to the midpoint position, rather than the string center of mass. The spacetime fields corresponding to the modes in this basis should satisfy field equations whose kinetic term is second order and whose quadratic nonlinear term contains no derivatives.

[4]By contrast, Maeno constructs the Hamiltonian path path integral for the theory, related in a fairly straightforward way to the usual Lagrangian path integral of Thorn[26].

The trouble is that the unitary operator eq.2.1 is singular, as can be seen by viewing its normal ordered form,

$$U = \exp\left[-\tfrac{1}{2}\,p^2 \sum_{n=1}^{\infty} \frac{1}{2n}\right] \exp\left[p_\mu \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \alpha^\mu_{-2n}\right] \exp\left[-p_\mu \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \alpha^\mu_{2n}\right]$$

The first factor here involves a divergent sum. Even more troubling is the appearance of $L_0$ in this basis,

$$L_0 = \tfrac{1}{2}\,p^2 + \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n + p_\mu \sum_{n=1}^{\infty} (-1)^n (\bar{\alpha}^\mu_{2n} + \bar{\alpha}^\mu_{-2n}) + p^2 \sum_{n=1}^{\infty} 1 \tag{2.2}$$

where $\bar{\alpha} = U\alpha U^{-1}$. The last term is infinite, so it seems that $L_0$ is undefined in this basis. We interpret this divergence as meaning that the fields in this basis, though (naively) satisfying local field equations, have infinite energy and are unphysical. In fact, due to the singular nature of $U$ locality of the interaction is even a subtle issue. One approach[14] to regulating these divergences is to replace $(-1)^n/2n$ in eq.2.1 with $\omega_{2n}/2n$ with $\omega_{2n} = \lambda^n(-1)^n$ and take the limit $\lambda \to 1^-$. In this regularization the vertex turns out *not* to be local in the limit $\lambda \to 1^-$ contrary to reasonable expectations.

Another approach explored in ref.[16] involves $\zeta$-function regularization, where one sets $\omega_{2n} = (2n)^{-s}(-1)^n$ and takes the limit $s \to 0^+$. In this approach they were able to demonstrate the locality of the vertex; moreover the divergent sum in $L_0$ is now reinterpreted as $-\frac{1}{2}$, by the magic of $\zeta$ function regularization. However, though the theory appears nonsingular and local in this regularization, it fails to be gauge invariant at $s = 0$ due to ambiguities in defining $Q_B^2$; moreover the physical perturbative states are undefined in this basis due to the singular nature of $U$ at $s = 0$. While these troubles can be avoided at nonzero $s$, the theory is only local at $s = 0$ so it seems that we still do not have a well defined local formulation of the theory. In our opinion the $\zeta$-function regulator is probably not sensible anyway, since we do not believe it is correct make the analytic continuation $\sum 1 \to -\frac{1}{2}$. The kinetic term $\langle\Psi|c_0(L_0 + L_0^{gh} - 1)|\Psi\rangle$ is not an analytic function of $s$ since for complex $s$ the transformation generated by $U$ on the Fock space is not unitary, so clearly the kinetic term for a particular component field will depend both on $s$ and $s^*$. At any rate, the divergence of $L_0$ in the midpoint basis probably has a physical origin and should not be argued away.

One point should be kept in mind when discussing these regulators: at any stage, the uncertainty in the expectation value of $x(\frac{\pi}{2})$ remains infinite. To see this, consider the transformed tachyon state at a point $x$:

$$|x,\omega\rangle = U_\omega|x\rangle$$

where $\omega$ refers to $\omega_{2n}$, our chosen midpoint regulator. The "uncertainty" in the expectation value of $x(\frac{\pi}{2})$ can be defined in terms of the root mean square deviation,

$$\langle\Delta x(\tfrac{\pi}{2})\rangle^2(\omega) \equiv \frac{1}{\delta(0)}\langle x,\omega|x(\tfrac{\pi}{2})^2|x,\omega\rangle - \left[\frac{1}{\delta(0)}\langle x,\omega|x(\tfrac{\pi}{2})|x,\omega\rangle\right]^2$$

Calculating this we find,

$$\langle \Delta x(\tfrac{\pi}{2})\rangle^2(\omega) = D\sum_{n=1}^{\infty}\frac{((-1)^n-\omega_{2n})^2}{2n}$$

Unless the midpoint limit has already been reached, for any acceptable regulator $\lim_{n\to\infty}\omega_{2n} = 0$. Therefore, this sum is logarithmically divergent. It would therefore seem dubious that these regulators actually "approach" the midpoint[5].

## 3. Lightcone basis

Let us now describe explicitly the choice of basis which renders the theory local in lightcone time. Consider a spacetime vector $v^\mu = (v^0, v^1, v^2, ...)$. Define lightcone components,

$$v^+ \equiv -v_- = \tfrac{1}{\sqrt{2}}(v^0+v^1) \qquad v^- \equiv -v_+ = \tfrac{1}{\sqrt{2}}(v^0-v^1) \tag{3.1}$$

It is useful to introduce two vectors $\lambda$ and $\chi$ satisfying,

$$\begin{aligned} \lambda_\mu\lambda^\mu &= \chi_\mu\chi^\mu = 0 \quad \lambda_\mu\chi^\mu = -1 \\ \lambda_\mu v^\mu &= v^+ \qquad\qquad\; \chi_\mu v^\mu = v^-. \end{aligned} \tag{3.2}$$

In addition define

$$\begin{aligned} v^\mu_{\mathrm{M}} &\equiv (\delta^\mu_\nu + \lambda^\mu\chi_\nu)v^\nu = (\tfrac{1}{\sqrt{2}}v^+,\, \tfrac{1}{\sqrt{2}}v^+, v_2, ...) \\ v^\mu_{\mathrm{P}} &\equiv (\delta^\mu_\nu + \chi^\mu\lambda_\nu)v^\nu = (-\tfrac{1}{\sqrt{2}}v^-,\, \tfrac{1}{\sqrt{2}}v^-, v_2, ...) \\ v^\mu_{\perp} &\equiv (\delta^\mu_\nu + \lambda^\mu\chi_\nu + \chi^\mu\lambda_\nu)v^\nu = (0, 0, v_2, ...). \end{aligned} \tag{3.3}$$

$v_{\mathrm{M}}$ denotes $v$ with it's minus component set to zero, $v_{\mathrm{P}}$ is $v$ with its plus component set to zero, and $v_\perp$ is $v$ with both its plus and minus components set to zero.

Consider the state space $\mathcal{H}_{\mathrm{BCFT}}$ of the boundary conformal field theory describing an open bosonic string living on a space-filling D25 brane. The usual basis for $\mathcal{H}_{\mathrm{BCFT}}$ is given by the mode oscillators $\alpha^\mu_n, b_n, c_n$ acting on the vacuum $|k\rangle$ describing the open string tachyon at momentum $k$ ($\alpha^\mu_0 = p^\mu$). We consider a change of basis generated by the unitary transformation,

$$U = \exp\left[-p_+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}(\alpha^+_{2n}-\alpha^+_{-2n})\right]. \tag{3.4}$$

Under this change of basis the matter oscillators and zero-modes transform as,

$$\begin{aligned} \tilde{\alpha}^\mu_n &\equiv U\alpha^\mu_n U^{-1} = \alpha^\mu_n - \cos\tfrac{n\pi}{2}\,p_+\lambda^\mu \qquad n\neq 0 \\ \tilde{p}_\mu &\equiv Up_\mu U^{-1} = p_\mu \\ \tilde{x}^\mu &\equiv Ux^\mu U^{-1} = x^\mu - i\sqrt{2}\chi^\mu\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}(\alpha^+_{2n}-\alpha^+_{-2n}). \end{aligned} \tag{3.5}$$

[5]In the midpoint lightcone basis, the cross terms which generate this divergence are absent. Therefore, at least in a light-like direction, we can meaningfully converge to the midpoint

The ghosts are unaffected. In particular, the plus component of $\tilde{x}$ lies at the string midpoint while the other components lie on the center of mass:

$$\tilde{x}^+ = x^+(\tfrac{\pi}{2}) \qquad \tilde{x}^\mu_{\rm P} = x^\mu_{\rm P} \tag{3.6}$$

Naively, then, we expect string field theory to be local and first order in $\partial_+$. The vacuum $|k\rangle$ transforms into a state $|k\rangle'$:

$$|k\rangle' \equiv \exp\left[k_+\sum_{n=1}^\infty \frac{(-1)^n}{2n}\alpha^+_{-2n}\right]|k\rangle, \tag{3.7}$$

Since the transformation is unitary, this basis satisfies the usual properties,

$$\begin{aligned}[][\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_{-n}] &= m\eta^{\mu\nu}\delta_{mn} \qquad [b_m, c_{-n}] = \delta_{mn} \\ \tilde{\alpha}^\mu_n|k\rangle' &= b_n|k\rangle' = c_n|k\rangle' = 0 \qquad n>0 \\ p_\mu|k\rangle' &= k_\mu|k\rangle' \quad b_0|k\rangle' = 0 \end{aligned} \tag{3.8}$$

The zeroth Virasoro generator takes the form:

$$\begin{aligned} L_0 &= \tilde{L}_0|_0 + \tfrac{1}{2}p^2 + p_+\sum_{n=1}^\infty(-1)^n(\tilde{\alpha}^+_{2n}+\tilde{\alpha}^+_{-2n}) \\ &= \tilde{L}_0|_0 + \tfrac{1}{2}p_\perp^2 + p_+P^+(\tfrac{\pi}{2}) \end{aligned} \tag{3.9}$$

$P(\frac{\pi}{2})$ is the momentum of the string midpoint (see appendix C for a definition of $P(\frac{\pi}{2})$ and other midpoint coordinates in following equations). Our notation is that a tilde over an operator denotes that operator with the replacement $\alpha \to \tilde{\alpha}$, and $|_0$ means we set the zero modes to vanish: $p = b_0 = c_0 = 0$. Though $L_0$ is nondiagonal in this basis, it is finite and well defined. This means, in particular, that states created by acting a finite number of $\tilde{\alpha}, b$, and $c$s on the new vacuum $|k\rangle'$ have finite energy. Hence, we seem to have a nonsingular and physical basis for describing string fields with the crucial advantage that the theory in this basis should appear local and first order in lightcone time derivatives.

At this point one might make an objection to our approach: by isolating one particular lightcone component of the position and translating it to the midpoint, we have spoiled manifest Lorentz invariance. This is really not the case, since cubic string field theory is Lorentz invariant and all we have done is chosen a particular basis for describing it. What we really mean when we say "manifest Lorentz invariance" is that Lorentz transformations are generated by *linear* transformations of the fields in the theory. This is as true in our basis as in any basis. In particular, the generator of Lorentz transformations in the old basis is,

$$J^{\mu\nu} = x^{[\mu}p^{\nu]} + i\sum_{n=1}^\infty \frac{1}{n}\alpha^{[\mu}_{-n}\alpha^{\nu]}_n. \tag{3.10}$$

In the tilde basis it is:

$$J^{\mu\nu} = \tilde{J}^{\mu\nu} + \chi^{[\mu}p^{\nu]}\bar{X}^+ + p_+\lambda^{[\mu}\bar{X}^{\nu]} \tag{3.11}$$

where

$$\bar{X}^\mu = i\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}(\tilde{\alpha}^\mu_{2n} - \tilde{\alpha}^\mu_{-2n}) = x^\mu(\tfrac{\pi}{2}) - x^\mu. \tag{3.12}$$

What is different about the tilde basis is that Lorentz transformations not only transform the position coordinates and the spacetime indices, as in usual field theory, but also transform between different spacetime fields corresponding to different modes in the basis. However, it is important to realize that if one were to try to approximate the theory by truncating fields beyond some level in the tilde excitations, Lorentz symmetry is lost. In the old basis, Lorentz invariance is preserved at any order in the level truncation scheme.

Let us see how some important operators in the theory appear in the tilde basis. The Virasoro operators are:

$$\begin{aligned} L_{2n} &= \tilde{L}_{2n}|_0 + p_{\mathrm{M}}\cdot\tilde{\alpha}_{2n} + \pi(-1)^n p_+ P^+(\tfrac{\pi}{2}) \\ L_{2n-1} &= \tilde{L}_{2n-1}|_0 + p_{\mathrm{M}}\cdot\tilde{\alpha}_{2n-1} - i(-1)^n p_+ x^+(\tfrac{\pi}{2})' \end{aligned} \tag{3.13}$$

In writing the BRST operator in the tilde basis, it is useful first to separate explicitly its dependence on zero modes:

$$Q_B = c_0(\tfrac{1}{2}\,p^2 + L_0|_0 + L_0^{gh} - 1) - b_0 A + p\cdot B + Q_B|_0 \tag{3.14}$$

where,

$$\begin{aligned} A &= 2\sum_{n=1}^{\infty} n c_{-n} c_n \\ B^\mu &= \sum_{n=1}^{\infty}(c_n\alpha^\mu_{-n} + c_{-n}\alpha^\mu_n) \end{aligned} \tag{3.15}$$

The BRST operator in the tilde basis can then be written,

$$\begin{aligned} Q_B &= \pi c(\tfrac{\pi}{2})p_+ P^+(\tfrac{\pi}{2}) - i\pi p_+ x^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2}) \\ &\quad + c_0(\tfrac{1}{2}\,p_\perp^2 + \tilde{L}_0|_0 + L_0^{gh} - 1) - b_0 A + p_M\cdot\tilde{B} + \tilde{Q}_B|_0 \end{aligned} \tag{3.16}$$

Of particular importance is the first term of this equation. Our interest in this term stems from the fact that it can be interpreted as responsible for the dynamics of the string field. It is the only term in the string field theory action where $p_+$ appears multiplied by $p_-$, and Fourier transforming $p_+p_- = -\partial_+\partial_- = \frac{1}{2}\,(\partial_0^2 - \partial_1^2)$ which contains the familiar second time derivative generating time evolution. The thing to notice about this term is that it is multiplied by $c(\frac{\pi}{2})$ while the corresponding $p_+p_-$ term in the old basis is multiplied by $c_0$, as can be seen by inspecting eq.3.14. The relevance of this fact is as follows. The string field can be written as the sum of two terms:

$$|\Psi\rangle = |\phi\rangle + |\psi\rangle$$

where $b_0|\phi\rangle = 0$ and $c_0|\psi\rangle = 0$. The familiar choice of Siegel gauge corresponds to setting $|\psi\rangle = 0$. The kinetic term in the string field theory action can be written in terms of $|\phi\rangle$

and $|\psi\rangle$

$$
\begin{aligned}
S &= -\langle\Psi, Q_B\Psi\rangle \\
&= -\langle\phi, c_0(\tfrac{1}{2}p^2 + L_0|_0 + L_0^{gh} - 1)\phi\rangle - \langle\psi, b_0 A\psi\rangle - 2\langle\psi, (p\cdot B + Q_B|_0)\phi\rangle
\end{aligned}
$$

Note that, because $c_0$ annihilates $|\psi\rangle$, second time derivatives only appear acting on $|\phi\rangle$, and so $|\phi\rangle$ is the only component of the string field which is truly dynamical. From a Hamiltonian perspective $|\psi\rangle$ represents gauge degrees of freedom since its conjugate momentum vanishes identically. However, from eq.3.16 we can see that with respect to the midpoint lightcone time $|\psi\rangle$ is no longer non-dynamical, since $c(\frac{\pi}{2})$ does not annihilate $|\psi\rangle$. This suggests that, to separate the dynamical and gauge degrees of freedom, it is more useful to decompose the string field as,

$$
|\Psi\rangle = |\phi\rangle' + |\psi\rangle'
$$

where $b_0|\phi\rangle' = 0$ and $c(\frac{\pi}{2})|\psi\rangle' = 0$.

Therefore it seems useful to perform yet another change of basis on the ghost sector so that the zero mode of the $c$ ghost is $c(\frac{\pi}{2})$ rather than $c_0$. This change of basis can be implemented by the unitary transformation,

$$
U_{gh} \equiv \exp\left[-b_0\sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n})\right] \tag{3.17}
$$

The ghost oscillators transform as,

$$
\begin{aligned}
\tilde{b}_n &\equiv U_{gh} b_n U_{gh}^{-1} = b_n - b_0\cos\tfrac{n\pi}{2} \qquad n \neq 0 \\
\tilde{c}_n &\equiv U_{gh} c_n U_{gh}^{-1} = c_n \qquad n \neq 0 \\
\tilde{b}_0 &\equiv U_{gh} b_0 U_{gh}^{-1} = b_0 \\
\tilde{c}_0 &\equiv U_{gh} c_0 U_{gh}^{-1} = c_0 + \sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n}) = c(\tfrac{\pi}{2})
\end{aligned} \tag{3.18}
$$

The vacua transform as

$$
|-,k\rangle' \equiv |k\rangle' = U_{gh}|k\rangle' \qquad |+,k\rangle' \equiv U_{gh}c_0|k\rangle' \tag{3.19}
$$

In particular, the $|k\rangle'$ vacuum does not transform (we will usually suppress the $-$ when denoting this vacuum). Since the transformation is unitary, the basis satisfies the expected properties,

$$
\begin{aligned}
&[\tilde{b}_m, \tilde{c}_{-n}] = \delta_{mn} \\
&\tilde{b}_n|k\rangle' = \tilde{c}_n|k\rangle' = 0 \qquad n > 0 \\
&\tilde{b}_n|+,k\rangle' = \tilde{c}_n|+,k\rangle' = 0 \qquad n > 0 \\
&b_0|k\rangle' = 0, \qquad \tilde{c}_0|+,k\rangle' = 0
\end{aligned} \tag{3.20}
$$

The ghost Virasoros take the form,

$$\begin{aligned}
L_0^{gh} &= -\pi b_0 \pi_b(\tfrac{\pi}{2})' + \tilde{L}_0^{gh}|_0 \\
L_{2n}^{gh} &= 2n\tilde{b}_{2n}(\tilde{c}_0 - \bar{C}) - \pi(-1)^n b_0\pi_b(\tfrac{\pi}{2})' + \tilde{L}_{2n}^{gh}|_0 \\
L_{2n-1}^{gh} &= (2n-1)\tilde{b}_{2n-1}(\tilde{c}_0 - \bar{C}) - (2n-1)\pi(-1)^n b_0\pi_b(\tfrac{\pi}{2}) \\
&\quad +(-1)^n b_0 c(\tfrac{\pi}{2})' + \tilde{L}_{2n-1}^{gh}|_0
\end{aligned} \tag{3.21}$$

where,

$$\bar{C} \equiv \sum_{n=1}^{\infty}(-1)^n(\tilde{c}_{2n} + \tilde{c}_{-2n}) = c(\tfrac{\pi}{2}) - c_0$$

The BRST operator takes the somewhat complicated form,

$$\begin{aligned}
Q_B &= \pi p_+ \left[\tilde{c}_0 P^+(\tfrac{\pi}{2}) - ix^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2})\right] + (\tilde{c}_0 - \bar{C})\left[\tfrac{1}{2}\,p_\perp^2 + \tilde{L}_0|_0\right] \\
&\quad +\tilde{c}_0\tilde{L}_0^{gh} - \tfrac{1}{2}\,(\bar{C}\tilde{L}_0^{gh} + \tilde{L}_0^{gh}\bar{C}) + \tfrac{\pi}{2}\pi_b(\tfrac{\pi}{2})'[\tilde{c}_0\tilde{b}_0 - \tilde{b}_0\tilde{c}_0] \\
&\quad +p_{\mathrm{M}}\cdot\tilde{B} + \tilde{Q}_B|_0
\end{aligned} \tag{3.22}$$

Note that now when we write a tilde over an operator, this means we replace all oscillators with their tilde'd counterparts in both matter and ghost sectors. Also $|_0$ means we set the zero modes to vanish in the tilde basis. Remember however that in eq.3.16 the tilde denotes replacing *only* the oscillators in the matter sector. From here on, when we talk about the tilde basis we mean making the unitary transformations eq.3.4 and eq.3.17 in both matter and ghost sectors.

The BPZ inner product in the tilde basis takes exactly the same form as it does in the old basis. In particular, we have the familiar relations,

$$\begin{aligned}
\langle\tilde{\alpha}_{-m}^\mu\Psi,\Phi\rangle &= (-1)^{m+1}\langle\Psi,\tilde{\alpha}_m^\mu\Phi\rangle \\
\langle\tilde{b}_{-m}\Psi,\Phi\rangle &= (-1)^m(-1)^\Psi\langle\Psi,\tilde{b}_m\Phi\rangle \\
\langle\tilde{c}_{-m}\Psi,\Phi\rangle &= (-1)^{m+1}(-1)^\Psi\langle\Psi,\tilde{c}_m\Phi\rangle
\end{aligned} \tag{3.23}$$

where $(-1)^\Psi$ denotes the Grassmann parity of $\Psi$. The two string vertex $\langle V_2|$ is as in the old basis after the replacement of the oscillators and vacua with their tilded counterparts.

It is worth noting that all numerical calculations in string field theory performed in Siegel gauge with $p_+ = 0$ translate directly in our formalism, since in this context the tilde basis is identical to the old basis. This includes the vast majority of successful calculations in the theory performed to date. For calculations outside of Siegel gauge, such as those calculating the spectrum of fluctuations around various D-brane vacua, or for calculations of time independent spatially inhomogeneous solutions which happen to depend on $x^1$, our basis will of course yield different results. It is worth exploring the level truncation scheme in the tilde basis to see whether it represents a usefully convergent approximation to the theory. As mentioned earlier, the only explicitly time dependent solution produced in the level expansion sofar seems to display strange behavior due to the infinite number of time derivatives, and it is not altogether clear whether the solution should be taken seriously. Our hope is that the tilde basis provides a better framework for studying time dependent solutions in the theory. Investigations along these lines are currently under way and will be published in ref.[21].

We expect that in the tilde basis the matter part of the three string vertex will be local in lightcone time. What should happen to the ghost part of the vertex? The ghost vertex[22] satisfies the following overlap condition:

$$c^{(A)}(\sigma)|V_3\rangle = -c^{(A+1)}(\pi-\sigma)|V_3\rangle \qquad \sigma \in [0, \tfrac{\pi}{2}] \tag{3.24}$$

where the index $A = 1, 2, 3, \mathrm{mod} 3$ denotes the Hilbert space on which the $c$ ghost acts. Suppose that this equation holds strictly even in the boundary case $\sigma = \frac{\pi}{2}$. The vertex should then satisfy:

$$\begin{aligned} c^{(1)}(\tfrac{\pi}{2})|V_3\rangle &= -c^{(2)}(\tfrac{\pi}{2})|V_3\rangle = c^{(3)}(\tfrac{\pi}{2})|V_3\rangle \\ &= -c^{(1)}(\tfrac{\pi}{2})|V_3\rangle \end{aligned}$$

This implies the surprising result,

$$c^A(\tfrac{\pi}{2})|V_3\rangle = 0$$

This means, at least naively, that any string field $|\psi\rangle'$ satisfying $\tilde{c}_0|\psi\rangle' = 0$ will have vanishing star product with any other field:

$$\psi' * A = 0 \tag{3.25}$$

In fact, in an earlier paper Okuyama[23] considered a basis for the the ghost state space equivalent to our tilde basis, and he showed that fields annihilated by $\tilde{c}_0$ do in fact satisfy this property. For completeness in the next section we study this property explicitly, together with the locality of the vertex, and attempt to understand the circumstances under which it can be expected to hold. An immediate consequence of eq.3.25 is that the star product should have no identity element. This is surprising since there has long been known a string field $|I\rangle$ which seems to behave as an identity element when multiplied with well-behaved string fields. Apparently, the field $|\psi\rangle'$ is not well behaved by this criterion. In fact, the expression for the identity string field involves explicitly a midpoint insertion of the $c$ ghost momentum:

$$\pi_c(\tfrac{\pi}{2}) = \frac{1}{\pi}\left[b_0 + \sum_{n=1}^{\infty}(-1)^n(b_{2n} + b_{-2n})\right] \tag{3.26}$$

A brief look at eq.3.18 reveals that this operator is undefined in the tilde basis, as must be identity string field. Similar problems with the identity are encountered in vacuum string field theory[24] where the kinetic operator $\mathcal{Q} = c(\frac{\pi}{2})$ has no well defined action on $|I\rangle$. Fortunately, though in early formulations $|I\rangle$ was used to define the integral $\int$ in the string field theory action, the identity turns out to be unnecessary in the fundamental formulation of the theory— the old integral is now replaced by the BPZ inner product $\langle , \rangle$ which is completely nonsingular in the tilde basis. Still, one might be bothered by the fact that our choice of basis has rendered such an important element of the algebra singular.

However, we emphasize that in order to give string field theory a well defined initial value formulation it is not necessary to transform the ghosts, and for a particular application one may choose not to for the sake of salvaging the identity string field[6]. Still, even if we do not transform the ghosts, we would need to regulate expressions involving both the identity string field and the BRST operator since the BRST operator in the tilde basis depends explicitly on $c(\frac{\pi}{2})$.

At this point we should explain precisely in what sense the tilde basis allows us to define the initial value problem for open string field theory. Since locality in time can only be achieved in the lightcone frame and the theory possesses gauge invariance, the initial value problem is somewhat more complicated than in nondegenerate second order systems where time evolution is uniquely determined by specifying the coordinates and velocities at $t = 0$. First off, to solve for the evolution of the string field uniquely we must fix a gauge. A natural choice is Siegel gauge, which although afflicted with Gribov problems[25] is sufficient for our discussion. The Siegel gauge equations of motion are,

$$\left[\partial_+\partial_- - \partial_\perp^2 - i\partial_+\sum_{n=1}^{\infty}(-1)^n(\tilde{\alpha}_{2n}^+ + \tilde{\alpha}_{-2n}^+) + \tilde{L}_0|_0 + L_0^{gh} - 1\right]\Psi(x) + gb_0\Psi * \Psi(x) = 0 \tag{3.27}$$

To see how this equation of motion appears for a typical component field in the tilde basis, it is useful to consider a simplified model with all the relevant features. Consider a $1+1$ dimensional field theory for two scalars $\Phi(x,t) = (\phi(x,t), B(x,t))$ (writing $t = x^+$ and $x = x^-$ to avoid index clutter) with the equation of motion,

$$(-\partial_t\partial_x + 1)\phi(x,t) + \partial_t B(x,t) + \int_{-\infty}^{\infty} dydz\Phi(y,t)\cdot G(x,y,z)\Phi(z,t) = 0 \tag{3.28}$$

where $G$ is some coupling matrix depending on three copies of $x$. We interpret this as an equation of motion for $\phi$ which is first order in $t = x^+$ but due to the interaction is not local in $x = x^-$. The fact that the evolution of $\phi$ is coupled the time derivative of $B$ corresponds to the $-i\partial_+\sum_{n=1}^{\infty}(-1)^n(\tilde{\alpha}_{2n}^+ + \tilde{\alpha}_{-2n}^+)$ term in eq.3.27, which couples each component field in the tilde basis to the time derivatives of an infinite number of other fields which differ from it by one fewer $\tilde{\alpha}_{2n}^+$ excitation and one greater $\tilde{\alpha}_{2n}^-$ excitation. To determine the evolution of $\phi$ we integrate this equation with respect to $x^-$:

$$\dot{\phi}(x,t) = \dot{\phi}(c,t) + \int_c^x dw\left[\phi(w,t) + \dot{B}(w,t) + \int_{-\infty}^{\infty} dydz\Phi(y,t)\cdot G(w,y,z)\Phi(z,t)\right] \tag{3.29}$$

Let us assume for the moment that $B$ and its time derivative are known functions. From this equation, we can see that if we specify the initial configuration of $\phi$ on the lightlike surface $t = 0$ and the value of $\dot{\phi}$ on the orthogonal lightlike surface $x = c$, we can determine the time evolution of the field uniquely. Note that the nonlocality of the interaction in $x = x^-$ plays no role in this statement. However, one might be skeptical that the term

[6] Analogously, in the matter sector the midpoint lightcone + component of the momentum $P_+(\frac{\pi}{2})$ is singular in the tilde basis. Fortunately, we are not aware of any important operators or fields in string field theory where $P_+(\frac{\pi}{2})$ enters in a crucial way.

"initial value formulation" really applies to this system, since we not only need information about the field at $t = 0$, but at all times on the surface $x = c$. This is true, but it is worth noting that $c$ can be chosen arbitrarily; in particular, we can choose $c = -\infty$ corresponding to early times and large spatial distances. For physically reasonable solutions we expect $\phi$ to vanish out there, so $\dot{\phi}(-\infty, t) = 0$ and the only remaining boundary condition is $\phi$ at $t = 0$, which is indeed an "initial" condition.

Of course this prescription assumes that $\dot{B}$ is already known. In practice this is tricky, since to solve for the time derivative of any one field in the tilde basis, one has to know the time derivatives of an infinite number of other fields, each of which in turn is determined by the time derivatives of an infinite number of yet other fields. This mess can be disentangled with the help of the level truncation scheme. At level $N$, what one can do is consider the fields of level $N$ containing no $\tilde{\alpha}^+_{2n}$ excitations. Since these fields can only be sourced by the time derivatives of fields at a higher level number, in their truncated equations of motion the $B$ term in eq.3.28 is absent. One can then determine the time derivatives of these fields and then plug them into the equations of motion of the fields they source, determine the time derivatives of these other fields, and proceed this way recursively. Sending the level number to infinity, the initial value formulation for each field is as in the previous paragraph. Another approach to this problem is to note that the $p_+$ dependence in the Siegel gauge equation of motion occurs in the form $p_+ P^+(\frac{\pi}{2})$; one can imagine making another unitary change of basis which diagonalizes $P^+(\frac{\pi}{2})$ so that the equation of motion for a given field will involve the time derivative of only that field. Such a change of basis can be achieved by the unitary transformation,

$$V = \exp\left[-ix^- \sum_{n=1}^{\infty} (-1)^n (\alpha^+_{2n} + \alpha^+_{-2n})\right]$$

Oddly, operators in this basis depend explicitly on $x^-$. String field theory is still translationally invariant, presumably, but now translations along $x^-$ transform between different modes in the basis. This seems less palpable to us than the idea that Lorentz transformations translate between different modes, so we have not explored this basis seriously.

Solving the dynamical equations of motion in Siegel gauge does not guarantee that we have fully solved the string field equation. In addition, there are constraints on the initial conditions at $x^+(\frac{\pi}{2}) = 0$. To see what these are, multiply the string field equation by 1 in the form,

$$\begin{aligned} 0 = Q_B\Psi + g\Psi * \Psi &= (P^+(\tfrac{\pi}{2}))^{-1}[b_0, c(\tfrac{\pi}{2})P^+(\tfrac{\pi}{2}) - i\pi_b(\tfrac{\pi}{2})x^+(\tfrac{\pi}{2})']\,(Q_B\Psi + g\Psi * \Psi) \\ &= (P^+(\tfrac{\pi}{2}))^{-1}\Big[(c(\tfrac{\pi}{2})P^+(\tfrac{\pi}{2}) - i\pi_b(\tfrac{\pi}{2})x^+(\tfrac{\pi}{2})')\Big[(L_0 + L_0^{gh} - 1)\Psi + gb_0\Psi * \Psi\Big] \\ &\quad + b_0\,(c(\tfrac{\pi}{2})P^+(\tfrac{\pi}{2}) - i\pi_b(\tfrac{\pi}{2})x^+(\tfrac{\pi}{2})')\,[Q_B\Psi + g\Psi * \Psi]\Big] \end{aligned}$$

The first term in this equation is proportional to the Siegel gauge equations of motion. The second term has a surprising simplification due to the fact that $c(\frac{\pi}{2})P^+(\frac{\pi}{2}) - i\pi_b(\frac{\pi}{2})x^+(\frac{\pi}{2})'$ annihilates interaction vertex, essentially for the same reason $c(\frac{\pi}{2})$ does (see section 5). Therefore, a solution to eq.3.27 is also a complete solution of the string field equations if and only if,

$$b_0 \frac{c(\frac{\pi}{2})P^+(\frac{\pi}{2}) - i\pi_b(\frac{\pi}{2})x^+(\frac{\pi}{2})'}{P^+(\frac{\pi}{2})} Q_B\Psi = 0 \tag{3.30}$$

The thing to notice about this equation is that it is independent of $p_+$—the $p_+$ dependence cancels because $[c(\frac{\pi}{2})P^+(\frac{\pi}{2}) - i\pi_b(\frac{\pi}{2})x^+(\frac{\pi}{2})']^2 = 0$. Therefore eq.3.30 contains no time derivatives and can be interpreted as a constraint on the initial conditions, like Gauss's law in electrodynamics. It is remarkable that this constraint is linear and independent of the string coupling.

Summarizing, the initial value formulation of cubic string field theory can be described as thus: Fixing Siegel gauge, we can specify the initial value of the string field at $x^+(\frac{\pi}{2}) = 0$ subject to the constraint eq.3.30. Subsequent time evolution is then simply determined by integrating the Siegel gauge equations of motion eq.3.27.

## 4. Hamiltonian BRST Quantization

We now turn to an important application of our formalism: canonical quantization. Before launching into technicalities, however, it is important to understand why having a local and first order formulation of the theory is a crucial element for defining a consistent and meaningful quantum theory.

For this purpose it is useful to consider how one would canonically quantize string field theory in the old basis, in which the Lagrangian depends on an infinite number of time derivatives. Our discussion reviews that of ref.[4], and for more details we refer the reader to that reference. Canonical quantization, of course, proceeds by translating the theory to the Hamiltonian formalism and replacing the classical Poisson bracket algebra with an analogous operator algebra acting on a suitably defined state space. In the case of string field theory, however, this procedure is complicated by the fact that the usual Hamiltonian formalism is defined only for theories whose Lagrangian depends only on coordinates and their first time derivatives—not an infinite number of their time derivatives. However there exists a generalization of the Hamiltonian formalism, due to Ostrogradski[3, 4], which applies to Lagrangians depending on coordinates and time derivatives up to any order $N$. For each configuration space coordinate $q$ in this formalism, there are $2N$ phase space coordinates $Q_1, Q_2, ..., Q_N, P_1, P_2, ..., P_N$ representing the $2N$ initial conditions necessary to specify a solution to the Euler-Lagrange equations. For string field theory, therefore, every component field $\phi$ yields $2N$ phase space coordinates with $N \to \infty$, and each pair $Q_n, P_n$ is associated with a distinct particle excitation, i.e. each component field generates an infinite spectrum of particle species. Further, since $Q_n = (\frac{d}{dt})^{n-1}q$ and $n$ can be arbitrarily large, field operators at different times commute; $[q(0), q(t)] = 0$ for any finite $t$.

A perhaps even more serious difficulty comes from higher derivative instabilities. Viewing the Hamiltonian,

$$H = \sum_{n=1}^{N-1} P_n Q_{n+1} + P_N \dot{Q}_N - L(Q_1, ..., Q_N, \dot{Q}_N)$$

one can see that, because $P_n$ and $Q_{n+1}$ are independent phase space variables, the first term of this equation can be made arbitrarily negative and the Hamiltonian is unbounded from below[7]. Thus the theory is unstable, possessing well-known "runaway" solutions generic in higher derivative theories. These instabilities in turn wreak havoc in the quantum theory. One can attempt to remove the instabilities at the quantum level by reinterpreting negative energy states with positive norm as positive energy states with negative norm; but then one either looses the probabilistic interpretation or one looses unitarity by removing the negative norm states (which don't decouple) by hand.

These problems are unacceptable in string field theory. It has been widely felt that somehow the canonical formalism goes awry in this case, though it has been far from clear how the theory escapes these sicknesses. Earlier approaches to quantization have proceeded by defining a configuration space path integral[26]. Since the Hilbert space structure and unitarity of the underlying theory are not manifest in this approach, potential problems with the canonical formalism are obscured, though presumably nevertheless present. It might be said, however, that the situation is helped by the fact that the higher derivatives only appear in the interaction. At the free level the theory has a standard Hilbert space representation describing the familiar perturbative string states. The interaction, since it is included only perturbatively, only adds small corrections to solutions in an otherwise local and second order field theory, and in this way problems associated with the higher derivative nature of the theory do not immediately manifest themselves.

Our basis, of course, brings new light to the situation. In midpoint lightcone time, the Lagrangian is local and first order. The phase space is described in the usual way by the component fields and their momenta, and the theory should be free of higher derivative instabilities and related problems at the non-perturbative interacting level.

So let us construct the canonical quantum theory with the help of our basis. Since string field theory is a gauge theory there are many canonical quantization schemes available to us: one may attempt to quantize without fixing a gauge, as in the Dirac method; one may attempt to find a unitary gauge, manifestly free of negative norm states, which would yield a formalism analogous (though not identical[8]) to lightcone string field theory; or, one may fix a covariant gauge and quantize via the Hamiltonian BRST formalism. Here we follow the last approach, since it is the one seemingly most natural in covariant string field theory and most closely tied to earlier quantization schemes which proceeded via the path integral[26].

In fixing a covariant gauge, we must in general introduce Fadeev-Popov ghosts and define a gauge-fixed action with BRST symmetry. Since cubic string field theory has a complicated reducible gauge invariance, it is helpful[26] to discuss ghosts and BRST symmetry in the context of the Batalin-Vilkovisky (BV) formalism[28]. For useful reviews of this formalism, see references [26, 27, 29, 30]. The point of departure in the BV formalism is the master action, a generalization of the usual gauge invariant action which in addition possesses unphysical ghost fields and antifields and a BRST symmetry relating them.

[7] By the assumption of nondegeneracy, we can write $\dot{Q}_N$ as an invertible function of $Q_1, ..., Q_N, P_N$.

[8] Interactions in the lightcone string field theory are described with a different choice of interaction vertex, so this formalism cannot be derived by fixing a gauge in the cubic string field theory alone.

Gauge fixing the master action yields the analogy of the Fadeev-Popov gauge fixed action with a residual BRST symmetry. In the case of cubic string field theory, at least at tree level, master action takes the same form as the usual action[26],

$$S = -\langle\Psi, Q_B\Psi\rangle - \tfrac{2}{3}g\langle\Psi, \Psi * \Psi\rangle \tag{4.1}$$

only now, the Grassmann odd string field $\Psi$ contains as components not only the physical string field at worldsheet ghost number 1, but unphysical ghost fields and antifields at worldsheet ghost numbers $\leq 0$ and $\geq 2$ respectively. The statement that $S$ is BRST invariant at tree level is expressed by the classical master equation,

$$\{S, S\} = 0$$

where $\{,\}$ is a Poisson-like bracket on the superspace of fields and antifields called the "antibracket" (see footnote). At the quantum level BRST invariance is ensured provided that the master action satisfies the "quantum" master equation,

$$\{S, S\} = 2i\hbar\Delta S$$

where $\Delta$ is a "symplectic Laplacian" on the superspace of fields and antifields[9].

To proceed we must introduce a basis of states in $\mathcal{H}_{\rm BCFT}$, allowing us to decompose the string field into an infinite collection of component spacetime fields. Anticipating the importance of the tilde basis, we define:

$$\begin{aligned}
&\Phi_i \equiv \tilde{\alpha}^{\mu_1}_{-l_1}...\tilde{\alpha}^{\mu_L}_{-l_L}\tilde{b}_{-m_1}...\tilde{b}_{-m_M}\tilde{c}_{-n_1}...\tilde{c}_{-n_N}|-\rangle' \quad l, m, n \geq 1\\
&\Psi_i \equiv \tilde{c}_0\Phi_i
\end{aligned} \tag{4.2}$$

The index $i$ is to be interpreted as a list of indices necessary to specify the right hand side of eq.4.2. In this basis an arbitrary string field can be decomposed as,

$$|A(x)\rangle = A^i(x)\Phi_i + B^i(x)\Psi_i$$

Note that we are working in the position representation, so $\Phi_i, \Psi_i$ are in the momentum independent component of $\mathcal{H}_{\rm BCFT}$. This basis has definite worldsheet ghost number, Grassmann parity, and reality properties:

*Worldsheet ghost number:*

$$gh(\Phi_i) \equiv gh(i)\Phi_i \qquad\qquad gh(\Psi_i) = (gh(i)+1)\Psi_i \tag{4.3}$$

[9] Explicit formulas for the antibracket and $\Delta$ are as follows. Given a theory with fields $\varphi^A$ and antifields $\varphi^*_A$ we have

$$\{F, G\} = \int dx F\left(\frac{\overleftarrow{\delta}}{\delta\varphi^A(x)}\frac{\overrightarrow{\delta}}{\delta\varphi^*_A(x)} - \frac{\overleftarrow{\delta}}{\delta\varphi^*_A(x)}\frac{\overrightarrow{\delta}}{\delta\varphi^A(x)}\right)G$$

$$\Delta F = \int dx\frac{\overrightarrow{\delta}}{\delta\varphi^*_A(x)}F\frac{\overleftarrow{\delta}}{\delta\varphi^A(x)}$$

*Grassmann parity:*

$$\epsilon(\Phi_i) \equiv (-1)^{\epsilon(i)+1}\Phi_i \qquad \epsilon(\Psi_i) = (-1)^{\epsilon(i)}\Psi_i \qquad \epsilon(i) = (gh(i)+1)\mathrm{mod}2 \tag{4.4}$$

*Reality:*

$$(\Phi_i, A) \equiv (-1)^{\Im(i)}\Phi_i^+ A \qquad (\Psi_i, A) \equiv (-1)^{\Im(i)+\epsilon(i)}\Psi_i^+ A \tag{4.5}$$

Above $(,)$ denotes the momentum independent component of the BPZ inner product, defined so that

$$\langle A, B\rangle \equiv \int dx (A(x), B(x))$$

$(,)$ defines an invertible bilinear form,

$$G_{ij} \equiv (\Psi_i, \Phi_j) \tag{4.6}$$

satisfying

$$\begin{aligned} G_{ij} = (-1)^{\epsilon(j)}G_{ji} = G_{ji}(-1)^{\epsilon(i)} \qquad \bar{G}_{ij} = (-1)^{\Im(i)+\Im(j)}G_{ji} \\ gh(i)G_{ij} = G_{ij}(2-gh(j)) \end{aligned} \tag{4.7}$$

As a final bit of notation, consider an operator $\mathcal{O}$ of Grassmann parity $(-1)^{\mathcal{O}}$ and independent of ghost zero modes. We can define an equivalent matrix $\mathcal{O}_j^i$ so that,

$$\mathcal{O}\Phi_i \equiv \mathcal{O}_i^j\Phi_j \qquad \mathcal{O}\Psi_i = (-1)^{\mathcal{O}}\mathcal{O}_i^j\Psi_j \tag{4.8}$$

All this notation will be useful for describing the formalism that follows.

Using the basis $\Phi_i, \Psi_i$ we can decompose the master string field $\Psi$ in terms of its component fields and antifields. Write $\Psi$ as a sum of two terms,

$$\Psi = \Psi_- + \Psi_+$$

where $\Psi_-$ contains the physical field and ghosts at worldsheet ghost numbers $\leq 1$ and $\Psi_+$ contains the antifields at worldsheet ghost numbers $\geq 2$. We write,

$$\begin{aligned} \Psi_-(x) &= \Phi_i\phi^i(x) + \Psi_i\psi^i(x) \\ \Psi_+(x) &= \Psi_i G^{ij}\phi_j^*(x)(-1)^{\epsilon(j)+1} + \Phi_i G^{ij}\psi_j^*(x)(-1)^{\epsilon(j)} \end{aligned} \tag{4.9}$$

Here we define $\phi^i = \phi_i^* = 0$ for $gh(i) \geq 2$ and $\psi^j = \psi_j^* = 0$ for $gh(j) \geq 1$. The string field must be real and Grassmann odd:

$$\langle \Psi, A\rangle = \Psi^+ A \qquad \epsilon(\Psi) = -\Psi$$

This implies,

$$\begin{aligned} \bar{\phi}^i &= (-1)^{\Im(i)+\epsilon(i)}\phi^i & \epsilon(\phi^i) &= (-1)^{\epsilon(i)}\phi^i \\ \bar{\psi}^i &= (-1)^{\Im(i)+1}\psi^i & \epsilon(\psi^i) &= (-1)^{\epsilon(i)+1}\psi^i \\ \bar{\phi}_i^* &= (-1)^{\Im(i)+\epsilon(i)+1}\phi_i^* & \epsilon(\phi_i^*) &= (-1)^{\epsilon(i)+1}\phi_i^* \\ \bar{\psi}_i^* &= (-1)^{\Im(i)}\psi_i^* & \epsilon(\psi_i^*) &= (-1)^{\epsilon(i)}\psi_i^* \end{aligned} \tag{4.10}$$

In particular, note that the fields have opposite Grassmann parity from their antifields. Finally, we introduce the notion of (spacetime) ghost number $\mathcal{G}$, defined:

$$\begin{aligned} \mathcal{G}(\phi^i) &= (1-gh(i))\phi^i \qquad & \mathcal{G}(\psi^i) &= -gh(i)\psi^i \\ \mathcal{G}(\phi_i^*) &= (gh(i)-2)\phi_i^* \qquad & \mathcal{G}(\psi_i^*) &= (gh(i)-1)\psi_i^* \end{aligned} \tag{4.11}$$

Given that the string field $\Psi$ and its components satisfy these properties, following ref.[26] it is straightforward to show that the master action in eq.4.1 has ghost number zero and satisfies the classical master equation.

Of course, we are ultimately interested in the quantum mechanics, so it is important to ask whether $S$ as written in eq.4.1 is BRST invariant at the quantum level. However, transformation to the tilde basis brings a surprise: The action eq.4.1 is automatically BRST invariant at the quantum level since $\Delta S = 0$. This is apparent because $S$ has no term where any component field is multiplied by its antifield. In the kinetic term this is obvious since the action has ghost number zero. In the interaction, this is true because a field associated with a state $\Phi_i$ is always paired with an antifield associated with the state $\Psi_j G^{ji}$, or conversely a field associated with $\Psi_i$ is paired with an antifield associated with $\Phi_j G^{ji}$. The point is that one of the pair is always associated with the state $\Psi_i$ which always annihilates the vertex since it is proportional to $c(\frac{\pi}{2})$. Hence, in the tilde basis fields never couple to their antifields. In the old basis, however, $\Delta S \neq 0$ and the action apparently must receive quantum corrections in powers of $\hbar$. These quantum corrections would manifest themselves in extra Feynmann diagrams contributing to the loop amplitudes derived from eq.4.1 and in principle ensure gauge invariance of the path integral. The role of these extra diagrams has been somewhat of a puzzle in light of well-known arguments[31, 32] that the Feynmann diagrams derived from the cubic action eq.4.1 provide a complete and single cover of the moduli space of open Riemann surfaces, and so in themselves must give the correct open string theory. Therefore, the absence of quantum corrections in our formalism provide another justification of the usefulness of the tilde basis.

We now would like to fix a particular gauge. A convenient covariant gauge choice is Feynmann-Siegel gauge $b_0\Psi = 0$, analogous to the Landau gauge in Yang-Mills theories. To arrive at the gauge fixed action, we must add a BRST trivial term to eq.4.1 containing additional fields and antifields and then eliminate the antifields using a gauge fermion which imposes delta function gauge fixing conditions in the Siegel gauge. For details, see ref.[26]. In the end, we find the following Lagrangian,

$$L = -\int dx_{\rm P}(\Psi, Q_B\Psi) - \tfrac{2}{3}g\int dx_{\rm P}(\Psi, \Psi*\Psi) + 2\int dx_{\rm P}(\beta, \tilde{b}_0\Psi) \tag{4.12}$$

where $\int dx_P$ denotes the integral over $x_\perp$ and $x^-$. Since the interaction is local in lightcone time, the Lagrangian depends only on the fields and their first time derivatives. Integrating out the auxiliary field $\beta$ clearly imposes the Siegel gauge condition. The fields $\Psi$ and $\beta$ can be expanded in terms of components,

$$\Psi(x) = \Phi_i\phi^i(x) + \Psi_i\psi^i(x) \qquad \beta(x) = \Psi_i\beta^i(x)$$

The sums over $i$ here now go over all ghost numbers; for ghost numbers less than or equal to 1 $\phi^i, \psi^i$ are the same physical string field and ghosts as in eq.4.9; for ghost numbers greater than 1, $\phi^i, \psi^i$ are unphysical antighosts introduced in the gauge fixing procedure. At any rate, $\phi^i, \psi^i$ satisfy the same ghost number, Grassmann parity, and reality properties as before. The action $S_{gf} = \int dx^+ L$ possesses a gauge fixed BRST symmetry[26],

$$s\Psi_- = (Q_B\Psi + \Psi * \Psi)_- \qquad s\Psi_+ = -(b_0\beta)_+ \qquad s\beta = 0 \tag{4.13}$$

where $A_-, A_+$ denotes the component of $A$ with worldsheet ghost numbers less than or equal to 1 and greater than 1, respectively. Via the Noether procedure we can calculate the conserved spacetime BRST charge, which takes the simple form,

$$\Omega = \int dx_{\mathrm{P}}(\Psi, [\tilde{c}_0(\partial_- - iD) - iX]Q_B\Psi) \tag{4.14}$$

where we have for shorthand defined,

$$D \equiv \pi P^+(\tfrac{\pi}{2}) - p^+ \qquad X \equiv -i\pi x^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2})$$

In the quantum theory, of course, $\Omega$ should be nilpotent and define the physical states through its cohomology.

The Lagrangian eq.4.12 will be our starting point for canonical quantization. The first step is to define momenta $\pi^i, \sigma^i, \chi^i$ canonically conjugate to the component fields $\phi^i, \psi^i, \beta^i$ respectively. These satisfy the usual Poisson bracket relations[10],

$$\begin{aligned}
\left[\phi^i(x), \pi_j(x')\right]_{\mathrm{PB}} &= -(-1)^{\epsilon(i)}\left[\pi_j(x'), \phi^i(x)\right]_{\mathrm{PB}} = (-1)^{\epsilon(i)}\delta^i_j\delta(x - x') \\
\left[\psi^i(x), \sigma_j(x')\right]_{\mathrm{PB}} &= (-1)^{\epsilon(i)}\left[\sigma_j(x'), \psi^i(x)\right]_{\mathrm{PB}} = -(-1)^{\epsilon(i)}\delta^i_j\delta(x - x') \\
\left[\beta^i(x), \chi_j(x')\right]_{\mathrm{PB}} &= (-1)^{\epsilon(i)}\left[\chi_j(x'), \beta^i(x)\right]_{\mathrm{PB}} = -(-1)^{\epsilon(i)}\delta^i_j\delta(x - x')
\end{aligned} \tag{4.15}$$

Due to our use of lightcone time, it turns out that none of the momenta are invertible functions of the velocities—i.e. we have constraints:

$$\begin{aligned}
\varphi^1_i &= G_{jk}[(\partial_- + iD)^j_i\phi^k + iX^k_i\psi^j] - \pi_i \approx 0 \\
\varphi^2_i &= iG_{jk}X^j_i\phi^k + \sigma_i \approx 0 \\
\varphi^3_i &= \chi_i \approx 0 \\
\varphi^4_i &= G_{ij}\beta^j \approx 0 \\
\varphi^5_i &= G_{ij}\psi^j \approx 0
\end{aligned} \tag{4.16}$$

where we have fixed the gauge $\beta^i = 0$. Solving these constraints we see that the only independent phase space degrees of freedom are the $\phi^i$s, and further since the constraints are second class, the $\phi^i$s have a well-defined Poisson bracket with respect to the induced symplectic structure on the constraint surface, i.e. the Dirac bracket:

$$[\phi^i(x), \phi^j(x')]_{\mathrm{DB}} = \tfrac{1}{2}\left(\frac{1}{\partial_- - iD}\right)^i_k G^{kj}\delta(x - x') \tag{4.17}$$

[10]From here on, we will write $x = x_{\mathrm{P}}$ to avoid unnecessary notational clutter and write $d = 26 - 1$ for the number of transverse dimensions.

It is the $\phi^i$s and this bracket which represents the true canonical structure of the theory.

The operator appearing in front of the delta function in eq.4.17 is essentially the inverse of $P^+(\frac{\pi}{2})$ and is singular due to the fact that $P^+(\frac{\pi}{2})$ has a continuous spectrum around 0. This is analogous to the situation of a relativistic scalar field formulated in lightcone frame, where on the right hand side we find the inverse of $\partial_-$. The ambiguity in defining the inverse of $\partial_-$ is fixed by choosing a principal value contour prescription,

$$\frac{1}{\partial_-}\delta(x-x') \equiv \int \frac{dk}{(2\pi)^d}\mathcal{P}\left(\frac{e^{ik\cdot(x-x')}}{ik_-}\right) = \tfrac{1}{2}\,\mathrm{sign}(x^- - x^{-'})\delta(x_\perp - x'_\perp)$$

which is the only choice consistent with the symmetry properties of the Poisson bracket. In our case, we may define the inverse of $P^+(\frac{\pi}{2})$ by making the Taylor expansion,

$$\left(\frac{1}{\partial_- - iD}\right)^i_j \delta(x-x') = \frac{1}{\partial_-}\sum_{n=0}^{\infty}\frac{i^n(D^n)^i_j}{\partial_-^n}\delta(x-x')$$

where again we take the principal value contour prescription,

$$\frac{1}{\partial_-^n}\delta(x-x') \equiv \int \frac{dk}{(2\pi)^d}\mathcal{P}\left(\frac{e^{ik\cdot(x-x')}}{(ik_-)^n}\right) = \frac{1}{2(n-1)!}\mathrm{sign}(x^- - x'^{-'})(x^- - x^{-'})^{n-1}\delta(x_\perp - x'_\perp)$$

With this definition the right hand side of eq.4.17 is consistent with the symmetry of the Poisson bracket.

We now elevate the classical component fields $\phi^i$ to quantum operators satisfying the Hermiticity property,

$$(\phi^i)^+ = (-1)^{\epsilon(i)+\Im(i)}\phi^i \tag{4.18}$$

and having ghost number and Grassmann parity,

$$\mathcal{G}(\phi^i) = (1 - gh(i))\phi^i \qquad \epsilon(\phi^i) = (-1)^{\epsilon(i)}\phi^i$$

The correspondence rule says that the $\phi^i$s satisfy graded commutation relations,

$$[\phi^i(x), \phi^j(x')] = \frac{i}{2}\left(\frac{1}{\partial_- - iD}\right)^i_k G^{kj}\delta(x-x') \tag{4.19}$$

where,

$$[A,B] \equiv AB - (-1)^{\epsilon(A)\epsilon(B)}BA = -(-1)^{\epsilon(A)\epsilon(B)}[B,A]$$

Dynamical evolution of the operators $\phi^i$ is determined by Heisenberg's equations of motion, which at the linear level amount to,

$$\left[\partial_+(\partial_- - iD)^i_j + (M - \tfrac{1}{2}\,\partial_\perp^2)^i_j\right]\phi^j(x,\tilde{x}^+) = 0 \tag{4.20}$$

where we have defined $M \equiv \tilde{L}_0|_0 + \tilde{L}_0^{gh} - 1$. The solution to these equations can be expanded in Fourier modes,

$$\phi^i(x,\tilde{x}^+) = \int \frac{dk}{(2\pi)^{d/2}}e^{ik\cdot x}\exp\left[i\tilde{x}^+(k_- - D)^{-1}(\tfrac{1}{2}\,k_\perp^2 + M)\right]^i_j a^j(k) \tag{4.21}$$

A little calculation shows that the mode operators $a^i(k)$ satisfy the reality properties,

$$a^i(k)^+ = (-1)^{\epsilon(i)+\Im(i)} a^i(-k) \tag{4.22}$$

and commutation relations

$$[a^i(k), a^j(k')] = \tfrac{1}{2} \left(\frac{1}{k_- - D}\right)^i_k G^{kj} \delta(k+k') \tag{4.23}$$

It is a worthwhile exercise to prove that the reality properties of both sides of this equation are consistent. Taking the Hermitian conjugate of the left hand side gives,

$$\begin{aligned}
([a^i(k), a^j(k')])^+ &= (-1)^{\Im(i)+\Im(j)+\epsilon(i)+\epsilon(j)} [a^j(-k'), a^i(-k)] \\
&= (-1)^{\Im(i)+\Im(j)+\epsilon(i)+\epsilon(j)} \tfrac{1}{2} \left(\frac{1}{-k'_- - D}\right)^j_k G^{ki} \delta(k+k')
\end{aligned}$$

The right hand side gives,

$$\left[\tfrac{1}{2}\left(\frac{1}{k_- - D}\right)^i_k G^{kj}\delta(k+k')\right]^+ = \tfrac{1}{2}\left(\frac{1}{k_- - \bar{D}}\right)^i_k (-1)^{\Im(k)} G^{kj} (-1)^{\Im(j)+\epsilon(j)} \delta(k+k')$$

To proceed we must make use of a few properties of $D$. Since $D$ is Hermitian, Grassmann even, and satisfies $(A, DB) = -(DA, B)$, it has the properties

$$G_{ik} D^k_j = -D^k_i G_{kj} \qquad (-1)^{\Im(i)} \bar{D}^i_j = -D^i_j (-1)^{\Im(j)} \qquad (-1)^{\epsilon(i)} D^i_j = D^i_j (-1)^{\epsilon(j)}$$

Thus,

$$\begin{aligned}
\left[\tfrac{1}{2}\left(\frac{1}{k_- - D}\right)^i_k G^{kj}\delta(k+k')\right]^+ &= \tfrac{1}{2}(-1)^{\Im(i)+\Im(j)+\epsilon(j)} \left(\frac{1}{k_- + D}\right)^i_k G^{kj}\delta(k+k') \\
&= \tfrac{1}{2}(-1)^{\Im(i)+\Im(j)+\epsilon(j)} G^{ik} \left(\frac{1}{k_- - D}\right)^j_k \delta(k+k') \\
&= \tfrac{1}{2}(-1)^{\Im(i)+\Im(j)+\epsilon(j)+\epsilon(i)} \left(\frac{1}{-k'_- - D}\right)^j_k G^{ki}\delta(k+k')
\end{aligned}$$

So the equation works out consistently. It is also worth checking that the ghost number properties of eq.4.23 are consistent. The left hand side has $\mathcal{G} = 0$, whereas the right hand side is,

$$\begin{aligned}
\mathcal{G}([a^i(k), a^j(k')]) &= (\mathcal{G}(a^i) + \mathcal{G}(a^j))[a^i(k), a^j(k')] \\
&= (2 - gh(i) - gh(j))\tfrac{1}{2}\left(\frac{1}{k_- - D}\right)^i_k G^{kj}\delta(k+k') \\
&= [2 - gh(i) - (2 - gh(i))]\tfrac{1}{2}\left(\frac{1}{k_- - D}\right)^i_k G^{kj}\delta(k+k') = 0
\end{aligned}$$

using eq.4.7 and the fact that $D$ is ghost number zero. As expected, oscillators only have nontrivial commutation relations when the sum of their ghost numbers vanish.

The final ingredient in constructing the quantum theory is defining a suitable representation of the algebra eq.4.23. Note that we can rewrite eq.4.23 in the familiar form,

$$[a^i(k), a^j(k')^+] = \tfrac{1}{2}\left(\frac{1}{k_- - D}\right)^i_k G^{kj}(-1)^{\epsilon(j)+\Im(j)}\delta(k-k') \tag{4.24}$$

Apparently the $a^i$s satisfy a harmonic oscillator algebra and we can define the usual Fock space representation by acting creation operators on a vacuum. However, there is a complication due to the fact that the creation and annihilation operators are not independent; the creation operator at momentum $k$ is the same as the annihilation operator at momentum $-k$. Apparently, it is not consistent to define a vacuum annihilated by all the $a^i(k)$ for all $k$, since these operators are non-commuting. Rather, we define the vacuum to be a state of ghost number zero satisfying,

$$a^i(k)|0\rangle = 0 \quad k_- > 0, \qquad \mathcal{G}(|0\rangle) = 0 \tag{4.25}$$

so that for example $a^i(-k)|0\rangle, k_- > 0$ describes a single string in state $i$ and momentum $k$. In this way, we have a Fock space capable of describing multiple string states whose minus component of momentum is strictly positive. The positivity of $k_-$ can be understood as a consequence of the requirement that the lightcone energy $k_+$ be positive, since for a state of mass $m$ we have $k_+ = \frac{\frac{1}{2}k_\perp^2 + m^2}{k_-}$.

Since the righthand side of eq.4.23 is not a positive definite bilinear form it is clear that our theory has negative norm states. To establish unitarity we must consider the role of the BRST charge eq.4.14. At the quantum level, the BRST charge should be nilpotent and define the physical states as elements of its cohomology. Fortunately, the BRST charge is simply quadratic in the fields and so there is no ordering ambiguity in lifting it to the quantum level. In terms of modes, we can write the BRST operator as,

$$\Omega = -i\int dk a^i(k)\Omega_{ij}(k)a^j(-k) \tag{4.26}$$

where,

$$\Omega_{ij}(k) \equiv (\Phi_i, [\tilde{c}_0(k_- + D) + X]Q_B\Phi_j) \tag{4.27}$$

where $Q_B$ in this expression is evaluated at momentum $-k$ with $k_+ = 0$. Let us now establish explicitly the $\Omega$ is nilpotent. Calculating,

$$\begin{aligned}\Omega^2 &= \tfrac{1}{2}[\Omega,\Omega] \\ &= -\tfrac{1}{2}\left[\int dk a^i(k)a^l(-k)\Omega^2_{il}(k) + \int dk a^j(-k)a^l(k)(-1)^{\epsilon(j)}\Omega^2_{lj}(k)\right]\end{aligned}$$

Where,

$$\Omega^2_{il}(k) = \Omega_{ij}(k)\left(\frac{1}{-k_- - D}\right)^j_m G^{mk}\Omega_{kl}(k)$$

Calculating,

$$\Omega^2_{il}(k) = (\Phi_i, [\tilde{c}_0(k_- + D) + X]Q_B\frac{1}{-k_- - D}\Phi_m G^{mk}(\Phi_k, [\tilde{c}_0(k_- + D) + X]Q_B\Phi_l))$$

Now note that for any string field $\Psi$ we have the property,

$$\Phi_m G^{mk}(-1)^{\epsilon(k)}(\Phi_k, \Psi) = \tilde{b}_0 \Psi \tag{4.28}$$

Using this fact we can simplify the right hand side,

$$\begin{aligned}
\Omega^2_{il}(k) &= (\Phi_i, [\tilde{c}_0(k_+ - D) + X] Q_B \frac{\tilde{b}_0}{-k_- - D}[-\tilde{c}_0(k_- + D) + X] Q_B \Phi_l)(-1)^{\epsilon(l)+1} \\
&= (\Phi_i, [\tilde{c}_0(k_- + D) + X] Q_B [-\tilde{c}_0(k_- + D) - X] \frac{\tilde{b}_0}{-k_- - D} Q_B \Phi_l)(-1)^{\epsilon(l)+1} \\
&\quad +(\Phi_i, [\tilde{c}_0(k_- + D) + X] Q_B [\tilde{b}_0, \tilde{c}_0(k_- + D)] \frac{1}{-k_- - D} Q_B \Phi_l)(-1)^{\epsilon(l)+1}
\end{aligned}$$

Since $Q_B$ is nilpotent and can be written as $Q_B = k_+(\tilde{c}_0(k_- + D) + X) + Q_B|_{k_+=0}$ when $k_+ \neq 0$, it follows that

$$[Q_B, \tilde{c}_0(k_- + D) + X] = 0$$

and therefore,

$$\begin{aligned}
\Omega^2_{il}(k) &= (\Phi_i, [\tilde{c}_0(k_- + D) + X]^2 Q_B \frac{\tilde{b}_0}{-k_- - D} Q_B \Phi_l)(-1)^{\epsilon(l)+1} \\
&\quad +(\Phi_i, [\tilde{c}_0(k_- + D) + X] Q_B^2 \Phi_l)(-1)^{\epsilon(l)+1}
\end{aligned}$$

which vanishes as a consequence of $Q_B^2 = [\tilde{c}_0(k_- + D) + X]^2 = 0$. Therefore, $\Omega$ is nilpotent and defines a cohomology; by the usual prescription, we identify the physical Hilbert space of the theory with the state cohomology of the BRST operator at ghost number zero, $\mathcal{H}_{phys} = H_0(\Omega)$, and likewise quantum mechanical observables with the operator cohomology at ghost number zero.

A consistency requirement, of course, is that the physical Hilbert space should be preserved under dynamical evolution. This would be guaranteed by the fact that $\Omega$ is a conserved charge, so it is important to verify this explicitly. The free Hamiltonian takes the form,

$$H_0 = \int dk a^i(k) H_{ij}(k) a^j(-k) \tag{4.29}$$

where,

$$H_{ij}(k) = (\Psi_i, [k_\perp^2 + M]\Phi_j)$$

In normal ordered form this can be written,

$$H_0 = 2\int_{k_->0} dk a^i(-k) H_{ij}(k) a^j(k) + \tfrac{1}{2}\int_{k_->0} dk \delta(0)(M + k_\perp^2)^m_k \left(\frac{1}{k_- - D}\right)^k_m (-1)^{\epsilon(m)}$$

The divergent constant term could presumably describe the vacuum energy of a space-filling D25 brane, though really the value of this constant is a matter of definition. Note that this is the only ultraviolet divergence we expect to find in the quantum string field theory, since nonlocality of the interaction presumably saves us from further ultraviolet divergences at the interacting level. This of course is what we expect from string theory. Anyway, to prove that $\Omega$ is a conserved charge we must calculate[11],

$$[H_0, \Omega] = -\frac{i}{2}\left[\int dk a^i(k) a^l(-k) I_{il}(k) + \int dk a^j(-k) a^l(k) I_{jl}(k)\right]$$

where

$$I_{il}(k) = H_{ij}(k)\left(\frac{1}{-k_- - D}\right)^j_m G^{mk}\Omega_{kl}(k) + \Omega_{ij}(k)\left(\frac{1}{-k_- - D}\right)^j_m G^{mk} H_{kl}(k)$$

This quantity can be calculated as follows,

$$\begin{aligned} I_{il}(k) &= (\Psi_i, [k_\perp^2 + M][-k_- - D]^{-1}\Phi_m) G^{mk}(\Phi_k, [\tilde{c}_0(k_- + D) - X]Q_B\Phi_l) \\ &\quad +(\Phi_i, [\tilde{c}_0(k_- + D) + X]Q_B[-k_- - D]^{-1}\Phi_m) G^{mk}(\Psi_k, [k_\perp^2 + M]\Phi_l) \\ &= -(\Phi_i, [k_\perp^2 + M][-k_- - D]^{-1}[\tilde{c}_0(k_- + D) + X]Q_B\Phi_l) \\ &\quad +(\Phi_i, [\tilde{c}_0(k_- + D) + X]Q_B[-k_- - D]^{-1}[k_\perp^2 + M]\Phi_l) \\ &= -\left(\Phi_i, \left[(k_\perp^2 + M)\frac{1}{-k_- - D}[\tilde{c}_0(k_- + D) + X]Q_B \right.\right. \\ &\qquad \left.\left. + Q_B[\tilde{c}_0(k_- + D) + X]\frac{1}{-k_- - D}(k_\perp^2 + M)\right]\Phi_l\right) \end{aligned}$$

To proceed note that

$$(k_\perp^2 + M)\Phi_i = \tilde{b}_0 Q_B \Phi_i$$

where as before $Q_B$ is understood to be evaluated at $k_+ = 0$. Thus we may write,

$$\begin{aligned} I_{il}(k) &= -\left(\Phi_i, \left[Q_B\tilde{b}_0\frac{1}{-k_- - D}[\tilde{c}_0(k_- + D) + X]Q_B \right.\right. \\ &\qquad \left.\left. + Q_B[\tilde{c}_0(k_- + D) + X]\frac{1}{-k_- - D}\tilde{b}_0 Q_B\right]\Phi_l\right) \\ &= -\left(\Phi_i, Q_B[\tilde{c}_0(k_- + D) + X, \tilde{b}_0]\frac{1}{-k_- - D}Q_B\Phi_l\right) = -(\Phi_i, Q_B^2\Phi_l) = 0 \end{aligned}$$

as expected, establishing that $\Omega$ is a conserved charge.

## 5. Causality

Having found a suitable local time coordinate for open string field theory, the next natural question to ask is whether the theory is causal. One might imagine, in particular, asking whether a localized perturbation of the initial conditions at $x^+(\frac{\pi}{2}) = 0$ only affects the subsequent time evolution inside the future lightcone of the perturbation. At first glance the answer would seem to be "no," since the vertex in the tilde basis is nonlocal in the transverse directions. A perturbation at $\tilde{x}$ will therefore affect the value of the string field

[11] From the fact that $\Omega$ is independent of the open string coupling it is apparent that it should commute with both the free and interacting Hamiltonians. $\Omega$ commutes with the interaction simply because the operator $\tilde{c}_0(k_- + D) + X$ annihilates the vertex (see section 5).

at $\tilde{y}$ even when $\tilde{x}$ and $\tilde{y}$ are "spacelike" separated: $(\tilde{x}-\tilde{y})^2 > 0$. Curiously, however, we cannot conclude from this that the theory is acausal, since $(\tilde{x}-\tilde{y})^2$ is not actually a Lorentz invariant quantity. This of course makes sense: The various components of $\tilde{x}$ have different interpretations in terms of either the midpoint or center of mass degrees of freedom, and these components are not related to each other only by a Lorentz transformation. Thus, in a sense the tilde basis has made the nature of time transparent at the cost of making causality obscure; in particular, while there is a well-defined notion of *time ordering* in the tilde basis, there is no sense in which fields in the tilde basis evaluated at distinct spacetime points have any definite causal relation to one another.

To discuss causality it seems necessary to use a basis where the component fields do have definite causal relationship. Specifically, this entails expanding the string field in a basis or eigenstates of some position operator $\xi$ satisfying $[\xi, p] = i$ where $\xi \cdot \xi$ is Lorentz invariant. Of course, a natural choice of $\xi$ is the string center of mass. However, since the field equations for this $\xi$ are nonlocal in both space and time, specifying appropriate initial conditions at $t = 0$ in some sense only determines time evolution arbitrarily far in the future, and the past lightcone of the evolution includes arbitrarily large sections of the initial value surface. Therefore, one might claim in some trivial sense that localized perturbations only effect the time evolution in the future lightcone, but it is hard to really take this argument seriously since the initial value problem is not under control; it is not clear to what extent the field for $t > 0$ is either independent or dynamically determined by the initial conditions at $t = 0$. Apparently, to make a really convincing argument for causality we must choose a covariant $\xi$ for which the initial value problem is well-defined. There is only one such $\xi$: the midpoint.

This brings good and bad news. The good news is that the theory in the midpoint basis is Lorentz invariant and local, so causality is manifest. The bad news is that the midpoint basis is singular: the component fields carry infinite energy and the vertex is only local with a careful choice of regularization. Confronted with these problems, it seems we can only argue for causality in string field theory by regulating the midpoint basis, and showing that as the regulator is removed the theory becomes arbitrarily close to being manifestly causal. Our task therefore is to find a consistent regulator. Unfortunately neither or the regulators $\omega_{2n} = \lambda^n(-1)^n,\ \ \lambda \to 1^-$ or $\omega_{2n} = (2n)^{-s}(-1)^n,\ \ s \to 0^+$ mentioned in section 2 are acceptable. The $\lambda$ regulator suffers from an anomaly in locality vertex[15, 14], and fields in the $s$ regulator acquire infinite energy at $s = 1$, even before the midpoint $s = 0$ is reached.

We will construct a regulator which is closer in spirit to the approach we have taken to the lightcone midpoint basis. For this purpose, we must introduce some notation. Consider a complex basis of spacetime vectors $\lambda^i, \chi^i$ with $i = 0, ..., D/2 - 1$ satisfying,

$$\begin{aligned} \lambda^0_\mu v^\mu &= \frac{1}{\sqrt{2}}(v^0 + v^1) & \chi^0_\mu v^\mu &= \frac{1}{\sqrt{2}}(v^0 - v^1) \\ \lambda^j_\mu v^\mu &= \frac{1}{\sqrt{2}}(v^{2j} + iv^{2j+1}) & \chi^j_\mu v^\mu &= \frac{1}{\sqrt{2}}(v^{2j} - iv^{2j+1}) \qquad j \geq 1 \end{aligned} \tag{5.1}$$

These satisfy,

$$\lambda^i \cdot \lambda^j = \chi^i \cdot \chi^j = 0 \qquad \lambda^i \cdot \chi^j = \eta^{ij} \qquad \eta_{ij} = \mathrm{diag}(-1, 1, ...1) \tag{5.2}$$

We will use $\eta_{ij}$ to raise and lower the $i, j$ indices. Since the $\lambda^i, \chi^i$ form a basis, they furnish a resolution of the identity,

$$\delta^\mu_\nu = \lambda^\mu_i \chi^i_\nu + \chi^\mu_i \lambda^i_\nu$$

and satisfy the reality properties,

$$\bar{\lambda}^0 = \lambda^0 \qquad \bar{\chi}^0 = \chi^0 \qquad \bar{\lambda}^i = \chi^i \qquad \bar{\chi}^i = \lambda^i \qquad i \geq 1$$

We consider a basis defined by the similarity transformation,

$$U_\sigma = \exp\left[-p \cdot \lambda^i \chi_i \cdot \sum_{n=1}^\infty \frac{(-\sigma^2)^n}{2n}(\alpha_{2n} - \alpha_{-2n}) - p \cdot \chi^i \lambda_i \cdot \sum_{n=1}^\infty \frac{(-1)^n}{2n}(\alpha_{2n} - \alpha_{-2n})\right] \tag{5.3}$$

where $\sigma < 1$. This operator is not unitary, but for $\sigma \to 1$ it becomes unitary and equal to the operator eq.2.1 defining the completely midpoint basis. Note that the sum over the $\lambda$ oscillators is not regulated, but the sum over the $\chi$ oscillators is. The position operator defined by eq.5.3 will therefore not be covariant, but as $\sigma \to 1$ it will be. Transforming with $U_\sigma$ in the usual way we define,

$$\begin{aligned} \check{\alpha}_n &= U_\sigma \alpha_n U_\sigma^{-1} = \alpha_n - \cos\tfrac{n\pi}{2}\, p \cdot (\lambda^i \chi_i \sigma^{|n|} + \chi^i \lambda_i) \qquad n \neq 0 \\ \check{p} &= U_\sigma p U_\sigma^{-1} = p \\ \check{x} &= U_\sigma x U_\sigma^{-1} = x + i \sum_{n=1}^\infty \frac{(-1)^n}{2n}(\lambda_i \chi^i \sigma^n + \chi_i \lambda^i) \cdot (\alpha_{2n} - \alpha_{-2n}) \\ |\check{k}\rangle &= U_\sigma |k\rangle \end{aligned} \tag{5.4}$$

This basis satisfies the usual properties,

$$\begin{aligned} [\check{\alpha}^\mu_m, \check{\alpha}^\nu_{-n}] &= n\delta_{mn}\eta^{\mu\nu} \qquad \mathrm{bpz}(\check{\alpha}_n) = \check{\alpha}_{-n} \\ \check{\alpha}_n|\check{k}\rangle &= 0 \quad n \geq 1 \qquad [\check{x}^\mu, p_\nu] = i\delta^\mu_\nu \end{aligned}$$

but, because $U_\sigma$ is not unitary, $(\check{\alpha}_n)^+ \neq \check{\alpha}_{-n}$. This means in particular that spacetime fields in this basis are subject to a rather complicated, nonlocal reality condition. While this is somewhat bothersome it does not pose a problem as far as causality is concerned; in the limit $\sigma \to 1$ $U_\sigma$ becomes unitary and the nonlocality of the reality condition disappears. Calculating $L_0$ we find,

$$L_0 = \left(\tfrac{1}{2} + \frac{\sigma^2}{1-\sigma^2}\right) p^2 + p \cdot \sum_{n=1}^\infty (-1)^n [\lambda^i \chi_i \sigma^{2n} + \chi^i \lambda_i] \cdot (\check{\alpha}_{2n} + \check{\alpha}_{-2n}) + \check{L}_0|_0 \tag{5.5}$$

For any $\sigma < 1$ $L_0$ is well-defined, but as expected there is a pole in the $p^2$ term at $\sigma = 1$.

We now turn to the vertex and establish that it is local in the $\sigma \to 1$ limit. Transforming to the czech basis we find,

$$\langle V_3^m| = \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3)\langle +,\check{k}^1|\langle +,\check{k}^2|\langle +,\check{k}^3| \times \exp\left[-\tfrac{1}{2} V_{00}^{AB} k^A \cdot k^B - V_{m0}^{AB} \check{a}_m^A \cdot k^B - \tfrac{1}{2} V_{mn}^{AB} \check{a}_m^A \cdot \check{a}_n^B\right] \times \exp\left[-k^A \cdot \lambda^i \chi_i \cdot \beta_n(\sigma)(\check{a}_n^{+A} - \check{a}_{-n}^{+A}) - k^A \cdot \chi^i \lambda_i \cdot \beta_n(\check{a}_n^{+A} - \check{a}_{-n}^{+A})\right] \quad (5.6)$$

where $\beta_n(\sigma) = \frac{1}{\sqrt{n}} \cos \frac{n\pi}{2} \sigma^n$. Pulling the unregulated factor in $U_\sigma$ through the vertex first, the quadratic momentum dependence and the coupling of the $\lambda$ oscillators to the momentum disappears completely; the calculation is exactly analogous to the one in appendix A. Pulling the regulated factor through the vertex we find the expression,

$$\langle V_3^m| = \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3)\langle +,\check{k}^1|\langle +,\check{k}^2|\langle +,\check{k}^3| \times \exp\left\{-[V_{m0}^{AB} + (V_{mn}^{AB} + \delta^{AB}\delta_{mn})\beta_n(\sigma)]\check{a}_m^A \cdot \chi^i \lambda_i \cdot k^B - \tfrac{1}{2} V_{mn}^{AB} \check{a}_m^A \cdot \check{a}_n^B\right\}$$

The momentum dependent factor of course vanishes in the limit $\sigma \to 1$ as a consequence of the identity eq.A.2 found in appendix A. Therefore, we have identified a nonsingular set of component spacetime fields evolving according to field equations which are arbitrarily close to being local.

Morally, one expects that since the interaction is arbitrarily close to being local, the theory should appear causal in the $\sigma \to 1$ limit. When we say that it "appears" causal, we really mean the following: One can imagine finding two time dependent solutions $\Psi$ and $\Psi'$ for $\sigma < 1$, where the initial conditions for $\Psi'$ differ from those of $\Psi$ in some compact region $R$ on the null plane $x^+(\frac{\pi}{2}) = 0$. The expectation is that,

$$\lim_{\sigma\to 1} \Psi(x) = \lim_{\sigma \to 1} \Psi'(x) \qquad (x-y)^2 > 0 \ \text{ for } \ y \in R \quad (5.7)$$

at least up to a gauge transformation, since the locality of the theory in this limit would seem to preclude any information from reaching $x$ from $R$. However, the limit $\sigma \to 1$ is so singular that it is admittedly premature to claim that locality of the theory in this limit truly implies eq.5.7. Indeed, it is not clear that the limit in eq.5.7 even exists, and probably it can only be interpreted in a distributional sense. Still the existence of a local limit provides some evidence, for whatever it's worth, that the theory is in some sense causal.

Another argument in favor of causality can be mounted with the help of the Moyal formalism developed by Bars and collaberators[20]. In this framework, the string field in the matter sector is considered as a local function of the midpoint and higher mode coordinates: $\Psi = \Psi[x(\frac{\pi}{2}), x_{2n}, p_{2n}]$ where $x_{2n}$ are the even Fourier modes of $x(\sigma)$ and $p_{2n}$ are a particular linear combination of the odd Fourier modes of $p(\sigma)$. Witten's star product is formally given by computing the Moyal product of the fields, where $[x_{2m}, p_{2n}]_\star = \delta_{mn}$, and the midpoint coordinates are identified locally. Of course, we know that the theory in such a language must be singular, but Bars *et al* have developed a convenient and reliable regulator whereby one essentially truncates the theory to include only a finite number $N$

of $x_{2n}, p_{2n}$ and regulates the linear operator relating $p_{2n}$ to the odd momentum Fourier modes in a particular way. The truncated action in this formalism is finite for any $N < \infty$ and furthermore is both Lorentz invariant and local in $x(\frac{\pi}{2})$. Therefore, at any stage as we take $N \to \infty$ it is clear that we are dealing with a causal theory, and it is natural to suppose that it continues to be so when the limit is saturated. This argument has the advantage that causality is manifest at every stage, but has the disadvantage that the Moyal regulator deforms the structure of the theory. Furthermore, the gauge invariance of the truncated theory is not understood. As we have seen gauge invariance and the initial value formulation in the exact theory seem to imply that the star algebra is degenerate, and so probably does not admit a well-defined representation in terms of a Moyal product.

We should say a few more words about the physical interpretation of the the causal limit eq.5.7. Causality requires that spacelike separated *physical* events should not be correlated. The string field however is not an observable. There need be no *a priori* constraint on causal propagation of the string field itself, only on the gauge invariant physical degrees of freedom it represents. Implicit in this language, however, is that classical open string field theory possesses local gauge invariant observables which propagate. Our results imply that open string field theory has observables which are localized in (lightcone) time—formally, they correspond to classical BRST cohomology classes of functionals of the initial conditions—however it seems unlikely that the theory possesses observables which are local in all spacetime coordinates. What, then, is the physical significance of the causal limit we have so carefully constructed? The answer to this question lies in the S-matrix, which is the only physical observable we explicitly understand in string theory. Our proposal is that the S-matrix can be meaningfully formulated in terms of Green's functions of interpolating component fields in the czech basis. Due to the locality of the theory in the $\sigma \to 1$ limit, the component fields should satisfy local and causal commutation relations in this limit if we fix a covariant gauge. This limit then will make manifest locality properties of the Green's functions which lead directly to the analytic properties of a causal S-matrix.

## 6. Conclusion

In this paper we have shown that, if time is identified with the lightcone component of the string midpoint, cubic string field theory is local and first order in lightcone time. Further, since the cubic vertex provides a complete single cover of the moduli space of open Riemann surfaces[31, 32], the fact that the action is local in time at tree level is sufficient to guarantee that locality is not spoiled by quantum corrections. We have taken care to prove that this result is not a formal artifact; we can expand the string field in a basis of eigenstates of $x^+(\frac{\pi}{2})$, and for such eigenstates the cubic action is well-defined, gauge invariant, and local in time. We have also identified a singular midpoint limit where the action is well defined but arbitrarily close to being local and manifestly causal.

The existence of a local time coordinate in open string field theory seems to rely crucially on our choice of interaction vertex. Certainly, the Witten vertex is the simplest choice but gives only one way of slicing up the moduli space of open Riemann surfaces. In a more general decomposition[33], we would need not only a cubic vertex, but an an infinite sequence of higher order vertices at tree level and beyond to recover the correct moduli space. In general it is not obvious that there exists a unitary transformation analogous to eq.3.4 rendering all of these vertices local in some time coordinate. This is an interesting question of principle, though our results indicate that at any rate observables in open string theory can be described in terms of at least one set of underlying gauge degrees of freedom for which the initial value problem is well-defined. In this sense, nonlocality in time in any formulation of open string field theory must be "pure gauge," regardless of whether the nonlocality can be removed by an invertible field redefinition. The situation with respect to closed string theory, on the other hand, is far from clear. In closed string field theory there is no simple choice of overlap type vertex or vertices like Witten's which covers the moduli space of Riemann surfaces to all genus[34]. The fact that the Witten vertex is an overlap was certainly instrumental for us, though it is not completely clear that overlap type vertices are either a necessary or sufficient condition for the existence of a local time coordinate. In the case of closed string theory one might be encouraged by the fact that both open and closed string interactions can be formulated (at least perturbatively) in the lightcone gauge string field theory, where by construction the interactions are local in lightcone time. However, as mentioned in the introduction, it is not clear whether locality in this context derives from the underlying gauge invariant string field theory, or whether it emerges via the process of "localization[4]," whereby one restricts attention to perturbative solutions and derives a perturbatively equivalent local action from an underlying nonlocal one. Lightcone string field theory may then be inherently perturbative, and the fact that lightcone time is well-defined in this context does not guarantee that it is meaningful nonperturbatively. If this is the case, then it is possible that time in closed string theory can only be defined "holographically" through open string degrees of freedom.

It is important to understand that implicit in our formalism is a proposal for what states should be rightfully considered elements of the algebra of string fields. The criterion is that the cubic action for a physical string field should be the same regardless of whether the field is expressed in a basis of eigenstates of the string center of mass or a basis of eigenstates of $\tilde{x}$. The two most prominent examples of string fields we know should be "physical" are the perturbative string states and the numerical solution representing the closed string vacuum. Both of these are well defined in the tilde basis—we have seen this explicitly for the perturbative string states, and the closed string vacuum is trivially well-defined (assuming that it is defined in the center of mass basis, which it seems to be), since it is a state at zero momentum. For solutions representing lower dimensional branes the situation is less clear, but we have no reason to believe that they are so singular at the midpoint that they don't admit well-defined representation in the tilde basis. The most prominent example of a state which should *not* be considered "physical" in our framework is the identity string field, since it is crucial for consistency and gauge invariance in the tilde basis that there are fields which have vanishing star product with any other string field.

It is interesting to observe that, sofar, it has only seemed possible to find Hamiltonian formulations of string theory in the lightcone frame, as for example in light cone string field theory or Matrix/membrane theory. One might be tempted to claim that our result fits into this general pattern, and that all of these formulations indicate that time in string theory can only be understood in the lightcone. While this may be true, we cannot at this point conclude that the appearance of lightcone time in all of these formulations is more than a coincidence. Lightcone time in membrane theory and lightcone string field theory emerges for a very different reason: the lightcone allows a tractable solution of the theory's constraints. In lightcone string theory, lightcone time is useful even before strings are allowed to interact. We of course are not trying to solve any constraints, and lightcone time only appears prominent at the interacting level by considerations of energetics. However, the common use of lightcone time is suggestive, and it would be interesting to establish a theoretical connection. Actually, the connection to lightcone string field theory may not be so far off, since in fixing lightcone gauge $x^+(\sigma) =$ const. one is already using a formalism where the midpoint (or any $x^+(\sigma)$) plays the role of time.

There are many interesting directions one might explore beyond our current work. It is worthwhile to continue our preliminary analysis of the canonical quantum theory, for example to calculate perturbative amplitudes and establish unitarity. One hope is that the canonical formalism may provide a novel perspective on the role of closed strings in open string field theory, though in this connection it is probably necessary to consider the superstring due to well-known difficulties with the closed-string tachyon and BRST invariance in the bosonic theory[35]. Of course, our results for the bosonic theory should carry over directly to the superstring, in either the RNS[18] or the Berkovitz[36] formalism, since their interactions can be formulated directly in terms of Witten-type vertices. It might also be useful to have a more explicit understanding of the gauge invariance of the theory from a Hamiltonian perspective, in particular to work out the constraint algebra and to explore other quantization schemes and gauge fixing procedures.

We have spent some time discussing the role of causality in string field theory, though what we have had to say on this subject is clearly far from the final word. To begin, it would be nice to have a more explicit argument relating macroscopic causality of the S-matrix to the midpoint limit. Since this limit is quite singular it is not *a priori* obvious that such an argument can be made in a convincing way. However, at a deeper level, it is interesting to observe that we had to appeal to intuition from local field theory as a "crutch" in order even to discuss causality in string field theory, forcing us to describe the theory in a singular and probably inappropriate fashion. Perhaps it is possible to formulate a more general and less singular criterion of microscopic causality that could be applied to string theory. These issues deserve more thought, but we believe that the ideas presented in this paper, particularly the initial value formulation, provide a good starting point.

Another crucial avenue to explore is the construction of time dependent solutions. Some preliminary work in this direction is underway and will be reported in ref.[21], though it might be said that finding and interpreting such solutions presents a formidable numerical problem. Already at level (2,4) in the tilde basis there are eight spacetime fields and more than a hundred nonlinear terms with complicated momentum dependence in the equations of motion. In addition one must worry about constraints on the initial conditions and their consistency with time evolution of the truncated equations of motion. The hope is that one can find some evidence for the existence of Sen's "rolling tachyon" boundary conformal field theory solution[10], where the tachyon rolls down from the unstable maximum homogenously towards the closed string vacuum, but does not cross over in finite time. In the lightcone formalism, however, homogenous solutions are less natural. Furthermore, for generic initial conditions, the many fields in the tilde basis will undergo complicated chaotic motion after they fall off the unstable maximum, and in general should not be expected to approach the closed string vacuum at all. Clearly, there are many challenging problems in this direction which need to be explored.

We would like to thank Wati Taylor for some discussions. The work of TE was supported by the National Science Foundation under Grant No. PHY00-98395, and that of DG by National Science Foundation under Grant No. PHY99-07949.

## A. Interaction Vertex and Gauge Invariance

The whole point in our change of basis eq.3.4 is that the interaction in this language should contain no lightcone time derivatives, and hence the dynamical structure of the theory is defined purely by the kinetic term, which (as in usual field theory) is first order in $\partial_+$. We now study the interaction explicitly and show that, with a few reasonable caveats, this is in fact the case. The interaction in cubic string field theory is defined by the three string vertex $\langle V_3| \in \mathcal{H}^*_{\rm BCFT} \otimes \mathcal{H}^*_{\rm BCFT} \otimes \mathcal{H}^*_{\rm BCFT}$:

$$-\frac{g}{3}\langle \Psi, \Psi * \Psi\rangle = -\frac{g}{3}\langle V_3||\Psi\rangle|\Psi\rangle|\Psi\rangle$$

where,

$$\begin{aligned}
\langle V_3| &= \langle V_3^m| V_3^{gh} \\
\langle V_3^m| &= \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3)\langle +, k^1|\langle +, k^2|\langle +, k^3| \\
&\quad \times \exp\left[-\tfrac{1}{2}\, V_{00}^{AB} k^A \cdot k^B - V_{m0}^{AB} a_m^A \cdot k^B - \tfrac{1}{2}\, V_{mn}^{AB} a_m^A \cdot a_n^B\right] \\
V_3^{gh} &= \exp\left[X_{m0}^{AB} c_m^A b_0^B + X_{mn}^{AB} c_m^A b_n^B\right]
\end{aligned} \tag{A.1}$$

Here, $A = 1, 2, 3$ denotes which copy of $\mathcal{H}^*_{\rm BCFT}$ the oscillator acts on and $m, n = 1, 2, ...\infty$ denotes the mode number of the oscillator, repeated indices summed. Here we use normalized oscillators in the matter sector $a_n^\mu = \frac{1}{\sqrt{|n|}}\alpha_n^\mu, n \neq 0$ and $\alpha' = \frac{1}{2}$. Explicit expressions for the constants $\kappa, V$, and $X$ (the latter two are called "Neumann coefficients") were calculated in ref.[22] and can be found for convenient reference for example in ref.[2]. Note that the matter part of the vertex contains a factor

$$\exp[\tfrac{1}{2}\, V_{00}^{AB} k_0^A k_0^B - V_{m0}^{AB} a_m^{0A} k_0^B]$$

and since in position space $p_0 = -i\frac{\partial}{\partial t}$ the vertex contains time derivatives to an arbitrarily high order which seem to render the initial value problem for string field theory ill-defined.

Let us now transform the vertex to the tilde basis. We have in the matter sector,

$$\begin{aligned}
\langle V_3^m| &= \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3)\langle +, k^1|'\langle +, k^2|'\langle +, k^3|' \\
&\quad \times \exp\left[-\tfrac{1}{2}\, V_{00}^{AB} k^A \cdot k^B - V_{m0}^{AB} \tilde{a}_m^A \cdot k^B - \tfrac{1}{2}\, V_{mn}^{AB} \tilde{a}_m^A \cdot \tilde{a}_n^B\right] \\
&\quad \times \exp\left[-k_+^A \beta_n (\tilde{a}_n^{+A} - \tilde{a}_{-n}^{+A})\right] U_{gh}^1 U_{gh}^2 U_{gh}^3
\end{aligned}$$

where $\beta_n = \frac{1}{\sqrt{n}} \cos \frac{n\pi}{2}$. Pulling the factor on the third line through the factor on the second line, we find:

$$\begin{aligned}\langle V_3^m| = \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3) \langle +, k^1|' \langle +, k^2|' \langle +, k^3|' \\ \times \exp\left[(V_{00}^{AB} + V_{m0}^{AB}\beta_m) k_+^A k_-^B - (V_{m0}^{AB} + (V_{mn}^{AB} + \delta^{AB}\delta_{mn})\beta_n)\tilde{a}_m^{+A} k_+^B\right] \\ \exp\left[-\tfrac{1}{2} V_{00}^{AB} k_\perp^A \cdot k_\perp^B - V_{m0}^{AB} \tilde{a}_m^A \cdot k_{\mathrm{M}}^B - \tfrac{1}{2} V_{mn}^{AB} \tilde{a}_m^A \cdot \tilde{a}_n^B\right] U_{gh}^1 U_{gh}^2 U_{gh}^3\end{aligned}$$

The factor on the second line contains all of the lightcone time derivatives in the vertex. Presumably, if the following identities are satisfied:

$$\begin{aligned} 0 &= V_{00}^{AB} + V_{m0}^{AB}\beta_m \\ 0 &= V_{m0}^{AB} + (V_{mn}^{AB} + \delta^{AB}\delta_{mn})\beta_n \end{aligned} \tag{A.2}$$

where $B$ is contracted with a conserved quantity, the vertex contains no derivatives with respect to lightcone time. Now consider the ghost component of the vertex.

$$\langle V_3| = \langle V_3^m|(U_{gh}^1 U_{gh}^2 U_{gh}^3)^{-1} \exp\left[X_{m0}^{AB} \tilde{c}_m^A b_0^B + X_{mn}^{AB} \tilde{c}_m^A \tilde{b}_n^B\right] \exp\left[-b_0^A \gamma_n (\tilde{c}_n^A + \tilde{c}_{-n}^A)\right]$$

where $\gamma_n = \cos \frac{n\pi}{2}$. Pulling the third factor though the second factor,

$$\langle V_3| = \langle V_3^m|(U_{gh}^1 U_{gh}^2 U_{gh}^3)^{-1} \exp\left[(X_{m0}^{AB} + (X_{mn}^{AB} + \delta^{AB}\delta_{mn})\gamma_n) \tilde{c}_m^A b_0^B + X_{mn}^{AB} \tilde{c}_m^A \tilde{b}_n^B\right]$$

Suppose that

$$0 = X_{m0}^{AB} + (X_{mn}^{AB} + \delta^{AB}\delta_{mn})\gamma_n. \tag{A.3}$$

In this case, the dependence on $b_0$ disappears from the vertex. In particular, we have the property anticipated earlier,

$$\langle V_3|\tilde{c}_0^A = 0$$

since $\tilde{c}_0$ passes unimpeded through the exponential and acts directly on the $\langle +, k|'$ vacuum which it annihilates. Altogether, the three string vertex in the tilde basis takes the form,

$$\begin{aligned}\langle V_3| = \kappa \int dk^1 dk^2 dk^3 \delta(k^1+k^2+k^3) \langle +, k^1|' \langle +, k^2|' \langle +, k^3|' \\ \exp\left[-\tfrac{1}{2} V_{00}^{AB} k_\perp^A \cdot k_\perp^B - V_{m0}^{AB} \tilde{a}_m^A \cdot k_{\mathrm{M}}^B - \tfrac{1}{2} V_{mn}^{AB} \tilde{a}_m^A \cdot \tilde{a}_n^B + X_{mn}^{AB} \tilde{c}_m^A \tilde{b}_n^B\right]\end{aligned} \tag{A.4}$$

assuming the identities eq.A.2 and eq.A.3. Observe that the vertex is still nonlocal in the transverse and $x^-$ directions (in particular we have a term $k_-^A \tilde{a}^{-B}$). Because the interaction contains an infinite number of derivatives with respect to $x^-$ the theory is still nonlocal in ordinary time $t = \frac{1}{\sqrt{2}}(x^+ + x^-)$. Only in $\tilde{x}^+$ have we achieved a well-defined initial value formulation.

Apparently, our entire approach rests on the validity of the identities eq.A.2 and eq.A.3. These identities are nontrivial and at best can be expected to hold only over an appropriately defined domain. It is therefore worthwhile to study these relations carefully and demonstrate whether or not they should be taken seriously. Before we launch into a more explicit and technical analysis, however, it is worth noting that the relations eq.A.2 and eq.A.3 can both be derived straightforwardly as a consequence of the overlap conditions:

$$\langle V_3|(X^A(\tfrac{\pi}{2}) - X^{A+1}(\tfrac{\pi}{2})) = 0 \qquad \langle V_3|c^A(\tfrac{\pi}{2}) = 0.$$

Since the vertex was explicitly constructed in ref.[22] so that half string overlap conditions (of which this is a special case) would be satisfied, barring unanticipated subtleties we expect eq.A.2 and eq.A.3 to hold.

We now turn to a more detailed study of eq.A.2 and eq.A.3. For this purpose, it is useful to rewrite the identities in a slightly different form. Define

$$\begin{aligned}
M^{AB}_{mn} &\equiv (-1)^m V^{AB}_{mn} \\
\tilde{M}^{AB}_{mn} &\equiv -(-1)^m \frac{1}{\sqrt{m}} X^{AB}_{mn}\sqrt{n} \\
m^{AB}_{n} &\equiv (-1)^n V^{AB}_{n0} \\
\tilde{m}^{AB}_{n} &\equiv -(-1)^n \frac{1}{\sqrt{n}} X^{AB}_{n0}
\end{aligned} \tag{A.5}$$

These satisfy the relations,

$$M^{AB} = M^{A+1,B+1} \qquad M^{AB}_{mn} = (-1)^m M^{BA}_{mn} \qquad M_{mn} + M^{12}_{mn} + M^{21}_{mn} = \delta_{mn}$$

and

$$m^{AB} = m^{A+1,B+1} \qquad m^{AB}_{n} = (-1)^n m^{BA}_{n} \qquad m_n + m^{12}_n + m^{21}_n = 0$$

with corresponding relations in the ghost sector[12]. Here we write $M \equiv M^{11}$ and $m \equiv m^{11}$ and the sum $A+1$ is taken mod 3. Reexpressing eq.A.2 and eq.A.3 in terms of $M, m, \tilde{M}, \tilde{m}$ and using these relations, we find five independent identities:

$$0 = 3m_{2m} + (3M_{2m,2n} + \delta_{2m,2n})\beta_{2n} \tag{A.6}$$

$$0 = m^{12}_{2m-1} + M^{12}_{2m-1,2n}\beta_{2n} \tag{A.7}$$

$$0 = \ln\frac{27}{16} + 3m_{2n}\beta_{2n} \tag{A.8}$$

$$0 = \tilde{m}_{2m} + (\tilde{M}_{2m,2n} - \delta_{2m,2n})\beta_{2n} \tag{A.9}$$

$$0 = \tilde{m}^{12}_{2m-1} + \tilde{M}^{12}_{2m-1,2n}\beta_{2n} \tag{A.10}$$

We will call these equations the "midpoint identities." Some of these relations have appeared elsewhere in the literature. For instance, eq.A.9 plays an important role in vacuum string field theory, where it was used to prove[37] that the pure ghost kinetic operator first proposed by Hata and Kuwano[38] based on their Siegel gauge solution is equivalent to the midpoint value of the $c$ ghost.

We now present detailed analytic evidence in support of the midpoint identities. Our approach uses the spectrum of the Neumann coefficients as described in ref.[39]. The

[12]Note that these relations fix the ambiguity in the zero mode part of the matter Neumann coefficients due to momentum conservation. In particular, $V^{AB}_{00} = \frac{1}{2}\ln\frac{27}{16}\delta^{AB}$.

spectrum of the Neumann coefficients is defined by an orthonormal basis of eigenfunctions $v_n(\kappa)$,

$$\sum_{n=1}^{\infty} v_n(\kappa)v_n(\kappa') = \delta(\kappa-\kappa') \qquad \int_{-\infty}^{\infty} v_m(\kappa)v_n(\kappa) = \delta_{mn} \tag{A.11}$$

Up to normalization, the $v_n(\kappa)$s are a complete set of orthogonal polynomials of degree $m-1$ and satisfy $v_n(-\kappa) = (-1)^{n+1}v_n(\kappa)$[13]. They can be defined implicitly via the generating function,

$$\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} v_n(\kappa) = \frac{1}{\kappa N(\kappa)}(1-e^{-\kappa\tan^{-1}z}) \tag{A.12}$$

where,

$$N(\kappa) = \sqrt{\frac{1}{\kappa}\sinh\frac{\pi\kappa}{2}}$$

The $v_n(\kappa)$s satisfy the crucial properties[39, 40],

$$\begin{aligned}
M_{mn}v_n(\kappa) &= -\frac{1}{1+2\cosh\frac{\pi\kappa}{2}}v_m(\kappa)\\
M^{12}_{mn}v_n(\kappa) &= -\frac{1+\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}v_m(\kappa)\\
M^{21}_{mn}v_n(\kappa) &= -\frac{1+\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}v_m(\kappa)\\
\tilde{M}_{mn}v_n(\kappa) &= \frac{1}{2\cosh\frac{\pi\kappa}{2}-1}v_m(\kappa)\\
\tilde{M}^{12}_{mn}v_n(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}-1}{2\cosh\frac{\pi\kappa}{2}-1}v_m(\kappa)\\
\tilde{M}^{12}_{mn}v_n(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}-1}{2\cosh\frac{\pi\kappa}{2}-1}v_m(\kappa)
\end{aligned} \tag{A.13}$$

and are thus eigenvectors of the $M$s.

Using $v_n(\kappa)$ to transform the midpoint identities eq.A.6-A.10 into the diagonal basis, we run into an immediate problem: $\beta_{2m}$ is undefined in this basis. Explicit calculation shows:

$$\beta(\kappa) \equiv \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n}}v_{2n}(\kappa) = \frac{1}{\kappa N(\kappa)}(1-\cos\infty)$$

Divergences of this sort are by now well understood in the $\kappa$ basis[41] and are related to the delicate role of the midpoint in the vertex, for example with anomalies in the associativity of the star product. Since our formalism makes use of the midpoint structure of the vertex in a crucial way, the appearance of such a divergence is not surprising. However, clearly we need to keep track of this singularity and show that it does not endanger the validity of eq.A.6-A.10 and the proposed form of the vertex. We can regulate $\beta$ as follows:

$$\beta_\omega(\kappa) \equiv \sum_{n=1}^{\infty}\frac{(i\tanh\omega)^{2n}}{\sqrt{2n}}v_{2n}(\kappa) = \frac{1}{\kappa N(\kappa)}(1-\cos\omega\kappa) \tag{A.14}$$

[13]For a convenient reference on the various properties of these functions, see the appendix of ref.[41].

In the limit $\omega \to \infty$ we should recover $\beta_{2n}$, but $\beta_\omega(\kappa)$ does not converge in the sense of functions. It does, however, converge in the sense of distributions,

$$\lim_{\omega\to\infty} \beta_\omega(\kappa) = \frac{1}{N(\kappa)}\mathcal{P}\left(\frac{1}{\kappa}\right) \tag{A.15}$$

where $\mathcal{P}$ denotes the principal value. Transforming our identities to the $\kappa$ basis (with the exception of eq.A.8) we find,

$$0 = 3m(\kappa) + 2\frac{\cosh\frac{\pi\kappa}{2} - 1}{1 + 2\cosh\frac{\pi\kappa}{2}}\frac{1}{N(\kappa)}\mathcal{P}\left(\frac{1}{\kappa}\right) \tag{A.16}$$

$$0 = m^{12}(\kappa) + \frac{\sinh\frac{\pi\kappa}{2}}{1 + 2\cosh\frac{\pi\kappa}{2}}\frac{1}{N(\kappa)}\mathcal{P}\left(\frac{1}{\kappa}\right) \tag{A.17}$$

$$0 = \tilde{m}(\kappa) + 2\frac{1 - \cosh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2} - 1}\frac{1}{N(\kappa)}\mathcal{P}\left(\frac{1}{\kappa}\right) \tag{A.18}$$

$$0 = \tilde{m}^{12}(\kappa) + \frac{\sinh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2} - 1}\frac{1}{N(\kappa)}\mathcal{P}\left(\frac{1}{\kappa}\right) \tag{A.19}$$

where,

$$m(\kappa) = \sum_{n=1}^{\infty} v_{2n}(\kappa) m_{2n} \tag{A.20}$$

and likewise for $m^{12}(\kappa), \tilde{m}(\kappa)$, and $\tilde{m}^{12}(\kappa)$. Notice that each midpoint identity in the $\kappa$ basis appears with a principal value distribution multiplied by a function which vanishes at $\kappa = 0$. Since there is no pole there would seem to be no need to specify a contour prescription. The first terms in these equations, in fact, do not involve the principal value. For example, upon calculating $m^{12}(\kappa)$ one finds that eq.A.17 reads,

$$0 = \frac{1}{N(\kappa)(1 + 2\cosh\frac{\pi\kappa}{2})}\left[\sinh\frac{\pi\kappa}{2}\,\mathcal{P}\left(\frac{1}{\kappa}\right) - \frac{\sinh\frac{\pi\kappa}{2}}{\kappa}\right] \tag{A.21}$$

Since there is no pole at $\kappa = 0$ it is very tempting to claim that this equation is exactly true. However, we should more precisely think of this equation as holding in the sense of distributions—when the right hand side is integrated against a smooth test function, the resulting integral should always vanish. If we were to choose a test function which is not smooth at $\kappa = 0$, the validity of eq.A.21 is much more dubious. Suppose we multiply eq.A.21 by a delta function.

$$\frac{1}{3\sqrt{\pi}}\left[\delta(\kappa)\sinh\frac{\pi\kappa}{2}\,\mathcal{P}\left(\frac{1}{\kappa}\right) - \frac{\pi}{2}\delta(\kappa)\right]$$

This expression involves a product of distributions and is hence ill-defined. In particular, if we happen to multiply the $\sinh\frac{\pi\kappa}{2}$ with the principal value distribution first, then $\sinh\frac{\pi\kappa}{2}\mathcal{P}(1/\kappa) = \sinh\frac{\pi\kappa}{2}/\kappa$ and the expression vanishes as expected. If however we are unwise enough to multiply $\sinh\frac{\pi\kappa}{2}$ with the delta function first, the first term vanishes and cannot cancel the second term; we get a nonzero answer, $-\frac{\sqrt{\pi}}{6}\delta(\kappa)$. In the mode basis, the delta function corresponds to the vector,

$$\delta_n = \frac{1}{\sqrt{\pi}}\frac{\sin\frac{n\pi}{2}}{\sqrt{n}}, \qquad \sum_{n=1}^{\infty}\delta_n v_n(\kappa) = \delta(\kappa) \tag{A.22}$$

Multiplying eq.A.21 by a delta function corresponds to evaluating,

$$\delta_{2m-1} m^{12}_{2m-1} + \delta_{2m-1} M^{12}_{2m-1,2n}\beta_{2n} =? \tag{A.23}$$

If our identities were exactly true, this expression would vanish. In reality, however, it is ill-defined because the double sum is ambiguous—the answer depends on which order the summation is carried out. These problems can be avoided by a simple restriction of domains. In particular, the midpoint identities should be interpreted as relations between distributions in the topological dual of an appropriately defined Hilbert space. For equations A.7 and A.10, we can take the Hilbert space to be simply that of square summable sequences $\ell^2$, since $\delta_n$ does not have finite norm under this inner product. For equations A.6 and A.9, we would need a slightly more singular distribution to cause ambiguities, for example the derivative of a delta function. In the mode number basis, the derivative of a delta function corresponds to,

$$\delta'_n = -\frac{1}{\sqrt{\pi}}\frac{\cos\frac{n\pi}{2}}{\sqrt{n}}\sum_{m=1}^{n/2}\frac{1}{2m-1} \qquad \sum_{n=1}^{\infty}\delta'_n v_n(\kappa) = \delta'(\kappa)$$

Apparently it is sufficient to take the Hilbert space of equations A.6 and A.9 to be $\ell^2$ as well, though we could probably slightly weaken this requirement since $\delta'_n$ has a norm even more divergent than $\delta_n$.

It is important to realize that our restriction on the Hilbert spaces translates directly into a restriction on the operators and string fields for which the vertex can be expected to take the proposed form eq.A.4. Let us consider an example of an operator whose action on the vertex in the old basis does not agree with with its action on the transformed vertex: the + component of the half string momentum:

$$\begin{aligned} P^{1/2}_+ &= \int_0^{\pi/2} d\sigma P_+(\sigma) = \tfrac{1}{2}p_+ + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}(\alpha^-_{2n-1}+\alpha^-_{-2n+1}) \\ &= \tfrac{1}{2}p_+ - \delta_n(a^-_n + a^-_{-n}) = \tilde{P}^{1/2}_+ \end{aligned} \tag{A.24}$$

If we don't assume the identity eq.A.7, the vertex takes the form,

$$\begin{aligned} \langle V_3| &= \langle V_3^{(us)}|A \\ A &= \exp\left[(m^{12}_{2m-1} + M^{12}_{2m-1,2n}\beta_{2n})(\tilde{a}^{+1}_{2m-1}p^2_+ - \tilde{a}^{+1}_{2m-1}p^3_+ + \text{cyclic})\right] \end{aligned}$$

where $\langle V_3^{(us)}|$ is the vertex of eq.A.4 and "cyclic" denotes the sum of $\tilde{a}^{+1}_{2m-1}p^2_+ - \tilde{a}^{+1}_{2m-1}p^3_+$ with the state space indices cyclically permuted. If the midpoint identities and postulated vertex are exactly correct, then $A = 1$ and $P^{1/2}_+$ should commute with $A$. However, calculation shows

$$[A, P^{1/2}_+] = (\delta_{2m-1}m^{12}_{2m-1} + \delta_{2m-1}M^{12}_{2m-1,2n}\beta_{2n})(p^2_+ - p^3_+)A$$

We find the same ambiguous double sum as in eq.A.23. If the sum over $n$ is performed first the commutator vanishes as expected, but if the sum over $m$ is performed first, the

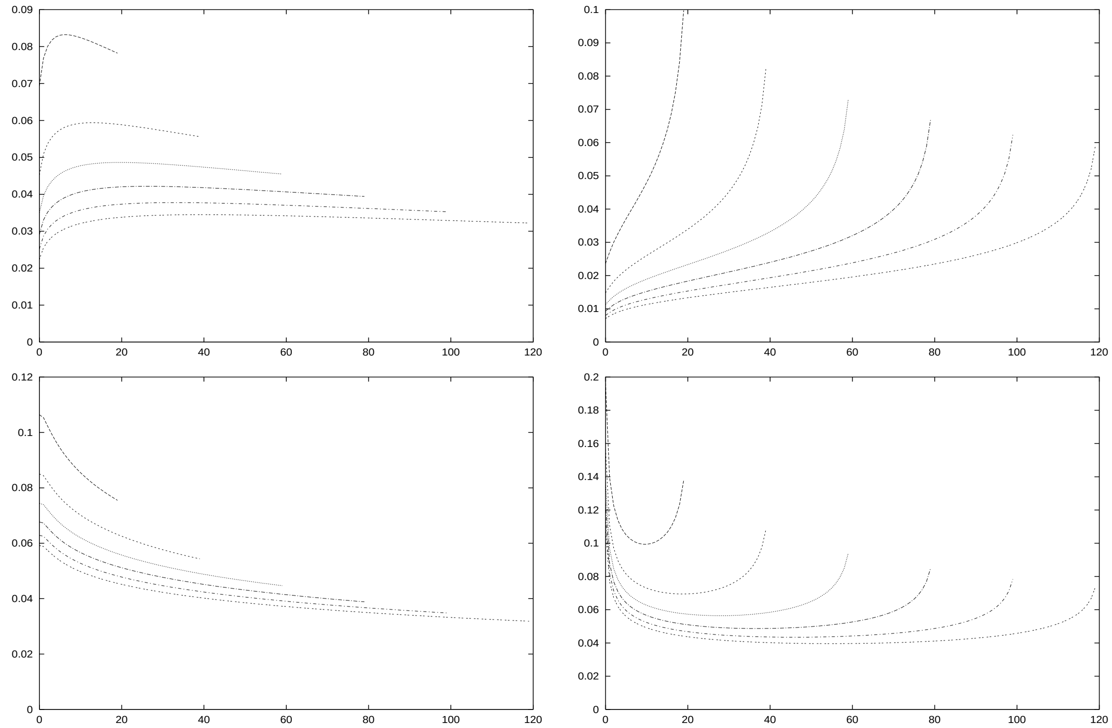
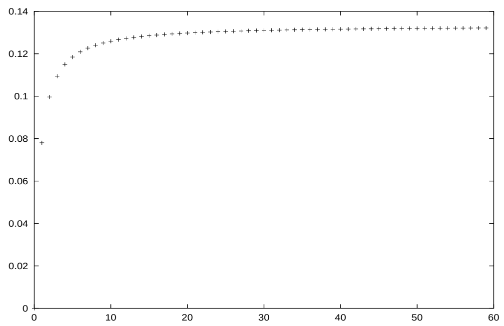


Figure 2: The components of $(-1)^n v_n^L$ graphed as a function of $n$ for $L = 20, 40, ..., 120$. Clockwise we have $(-1)^n v_n^L$ for equations A.6, A.7, A.10, A.9

commutator does not vanish. This is an indication that the action of $P_+^{1/2}$ on the vertex does not commute with the transformation into the tilde basis. Actually, the difficulty with $P_+^{1/2}$ is not surprising since in an early paper[42] Horowitz and Strominger showed that the operator $P^{1/2}$ could be used to generate translations of the string center of mass (and hence $\tilde{x}$) of a string field using the star product. In particular, $P_+^{1/2}$ could be used to translate $\tilde{x}^+$. This, however, is inconsistent with the conjectured locality of the vertex in lightcone time: Evaluating the star product of two fields at a particular $\tilde{x}^+$ will always generate another field at the *same* $\tilde{x}^+$. In the ghost sector, similar difficulties are encountered with $b(\frac{\pi}{2})$. We are fortunate that these operators do not seem to play a crucial role in the theory—they do not appear in the BRST operator, BPZ inner product, or the Virasoros expressed in the tilde basis. We therefore expect that the proposed vertex eq.A.4 should be valid for a sufficiently general and physically interesting class of string fields.

Modulo issues of domains, we still need to prove the that the $m(\kappa)$s cancel the principal value terms in eq.A.16-A.19 as expected. This is easily done along the lines of Okuyama in his proof[37] that the kinetic operator $\mathcal{Q}$ of vacuum string field theory is $c(\frac{\pi}{2})$. For completeness we have included a sample computation in appendix B, along with a separate derivation of eq.A.8.

It is also instructive to see how the midpoint identities converge in the level truncation scheme. To this end we can calculate the $L$-component vector $v_n^L$ derived by truncating the right hand side of one of the equations A.6-A.10 to level $L$ (by which we mean replace the a)




b)

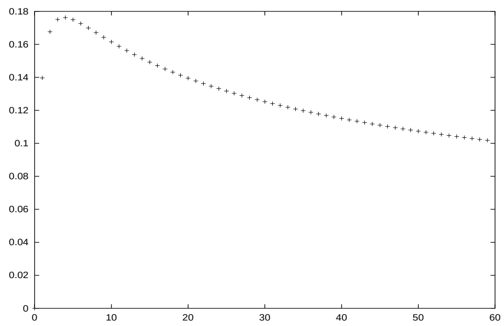



Figure 3: a) The norm of $v_n^L$ graphed as a function of $L$ for identity eq.A.6. b) The inner product $l \cdot v^L$ for $l_n = \frac{(-1)^n}{2n}$ graphed as a function of $L$ for identity eq.A.6.

$M$s by truncated $L \times L$ matrices and the $m$s and $\beta$ by $L$ component vectors). Presumably as $L \to \infty$ the components of $v_n^L$ should vanish. In figure 2 we have graphed $|v_m(L)|$ for each identity eq.A.6-A.10 at levels $L = 20, 40, ..., 120$, and indeed the components seem to fall towards zero, albeit slowly. Perhaps a little worrisome is the fact that, for fixed $L$, $v_n^L$ is roughly constant for large values of the mode number label $n$. This implies that, up to a vanishing factor as $L \to \infty$, the norm of $v_n^L$ diverges linearly. We have plotted the norm of $v_n^L$ for identity eq.A.6 as a function of $L$ in figure 3a. As $L$ increases, the norm does not vanish but seems to approach a constant value, implying that the components $v_n^L$ fall to zero roughly as $1/\sqrt{L}$. Because of the distributional nature of the midpoint identities, we should not necessarily have expected the norm of $v^L$ to vanish at $L = \infty$, even though its components do. What we really need to check is that the midpoint identities vanish in a distributional sense when summed against a vector in $\ell^2$. As a particular example we can consider the vector $l_n = \frac{(-1)^n}{2n} \in \ell^2$. We have graphed $l \cdot v^L$ for identity eq.A.6 in figure 3b. For large values of $L$ the inner product seems to be decreasing towards zero as expected, though quite slowly—the falloff goes approximately as $f(L) = (\ln(L)+\gamma)/\sqrt{L}$ ($\gamma$ = Euler's constant). Using a fitting function $a+bf(L)+cf(L)^2+df(L)^3$ we can extrapolate to $L = \infty$ finding $l \cdot v^L \approx -.00088$, within about half a percent of zero relative to the maximum value of $l \cdot v^L$. All of this indicates that the vertex does take the proposed form for well-behaved string fields as the level of truncation in the theory is increased. However, the convergence is rather slow, so at low orders in the level expansion in the old basis one would not expect to immediately see the essential locality and stability of the theory in lightcone time.

So far, we have not discussed the crucial issue of gauge invariance of our formalism. The gauge invariance of cubic string field theory is based on the following axioms[1]: 1) $Q_B$ is nilpotent. 2) The star algebra is associative. 3) The BPZ inner product is BRST invariant. 4) $Q_B$ acts as a derivation of the star algebra. Properties 1)-3) can be easily verified in the tilde basis with a little calculation. Property 4), however, is notoriously subtle[22], and though we have argued that the action of the BRST operator on the interaction vertex should commute with the transformation to the tilde basis, the BRST operator does involve terms which act quite singularly at the string midpoint and it would be nice to have an explicit argument that these operators do not bring additional subtleties. In the case where we transform only the matter sector to the tilde basis, the derivation property of $Q_B$ is particularly easy to see and is worth illustrating. In terms of the interaction vertex, the derivation property of $Q_B$ is expressed by the formula,

$$\sum_{A=1}^{3} \langle V_3|Q_B^A = 0 \tag{A.25}$$

We will show that when the matter part of the vertex is taken to be as in eq.A.4 and the BRST operator is expressed as in eq.3.16, this property holds for a large class string fields which are not too singular at the string midpoint. We write,

$$\begin{aligned}\langle V_3| &= \Big[\langle\tilde{V}_3|\Big]_{k_+=0} \exp(-i\tilde{x}^{+,A}k_+^A)\\ Q_B &= \pi c(\tfrac{\pi}{2})k_+P^+(\tfrac{\pi}{2}) - ik_+x^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2}) + \tilde{Q}_B|_{k_+=0}\end{aligned} \tag{A.26}$$

where $\tilde{Q}_B$ and $\langle\tilde{V}_3|$ are given by their corresponding expressions in the old basis after the replacement of the oscillators and vacua with their tilde'd counterparts. Recalling that $c(\frac{\pi}{2})$ annihilates the vertex (as argued before), eq.A.25 becomes,

$$\sum_{A=1}^{3}\langle V_3|Q_B^A = -i\sum_{A=1}^{3}\langle V_3|[k_+x^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2})]^A + \left[\sum_{A=1}^{3}\langle\tilde{V}_3|\tilde{Q}_B^A\right]_{k_+=0} e^{-i\tilde{x}^{+,B}k_+^B}$$

Except for the fact that the oscillators and vacua are tilde'd, the second term of this equation is exactly the computation of eq.A.25 in the old basis, and vanishes for the same reason, as established for example in ref.[22]. This leaves,

$$\sum_{A=1}^{3}\langle V_3|Q_B^A = -i\sum_{A=1}^{3}\langle V_3|[k_+x^+(\tfrac{\pi}{2})'\pi_b(\tfrac{\pi}{2})]^A \tag{A.27}$$

The right hand side of this equation now vanishes assuming the overlap condition,

$$\langle V_3|\pi_b^A(\tfrac{\pi}{2}) = 0$$

This condition is equivalent to the identities,

$$\begin{aligned}0 &= \delta_{2m-1} - \tilde{M}_{2m-1,2n-1}\delta_{2n-1}\\ 0 &= \tilde{M}^{12}_{2m,2n-1}\delta_{2n-1}\end{aligned}$$

where $\delta_{2m-1}$ is as in eq.A.22. Transforming to the kappa basis, these identities are

$$0 = \frac{\sinh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2}-1}\delta(\kappa) \qquad 0 = \left[1 - \frac{1}{2\cosh\frac{\pi\kappa}{2}-1}\right]\delta(\kappa)$$

Assuming we consider string fields such that the right hand side of these equations are integrated against test functions which are smooth at $\kappa = 0$ (i.e. they are not too singular at the midpoint), $Q_B$ acts as a derivation in the tilde basis and there is no anomaly in gauge invariance. This is equivalent to saying that these identities are relations between distributions in the topological dual of $\ell^2$. One might wonder how the operator $x^+(\frac{\pi}{2})'$, which multiplies $\pi_b(\frac{\pi}{2})$ in eq.A.27, factors into this argument (or similarly, $P^+(\frac{\pi}{2})$ which multiplies $c(\frac{\pi}{2})$). It turns out that the action of $x^+(\frac{\pi}{2})'$ on the vertex is not as well-defined as that of $\pi_b(\frac{\pi}{2})$. Formally, $x^+(\frac{\pi}{2})'$ also annihilates the vertex, but the "identities" one derives by imposing $\langle V_3|x^{+,A}(\frac{\pi}{2})' = 0$ involve sums like $\sum(-1)^n$ which just barely diverge. Transforming to the kappa basis, we find that these identities involve the complex delta function $\delta(\kappa - 2i)$, which is a more singular distribution than the real delta function, but nevertheless is well-defined[41] in the topological dual of the space of test functions which are analytic on the strip $|\Im(\kappa)| \leq 2i$[14]. In mode number language, the identities $\langle V_3|x^{+,A}(\frac{\pi}{2})' = 0$ relate distributions in the topological dual of a half order Sobolev space. Taking extra care, we might restrict the space of string fields so that the complex delta function $\delta(\kappa - 2i)$ is always integrated against an analytic test function, though this is probably overkill: the fact that $\pi_b(\frac{\pi}{2})$ annihilates the vertex is probably sufficient to ensure the vanishing of the right hand side of eq.A.27. Presumably, similar considerations follow for the case when we transform both matter and ghost sectors to the tilde basis, though in this case the analysis is more involved and we will not go into it here.

## B. Derivation of Identities implying Locality

In this appendix we prove the identities eq.A.6-A.10 using the spectroscopy of the Neumann coefficients. We will prove eq.A.16 following Okuyama[37] and leave eq.A.17-A.19 as exercises for the skeptical reader (in fact eq.A.18 was already proved in ref.[37]). To verify eq.A.16 we must evaluate the sum eq.A.20. For this we need the explicit form of $m_{2n}$:

$$m_{2m} = -\frac{2}{3}\frac{A_{2n}}{\sqrt{2n}}$$

where the constants $A_{2n}$ are defined implicitly through the expansion[22],

$$\left(\frac{1+iz}{1-iz}\right)^{1/3} = \exp\left(\frac{2i}{3}\tan^{-1} z\right) = 1 + \sum_{n=1}^{\infty} A_{2n} z^{2n} + i\sum_{n=1}^{\infty} A_{2n-1} z^{2n-1}$$

To compute $m(\kappa)$ insert the identity in the form,

$$\begin{aligned} 3m(\kappa) &= -\frac{1}{\pi i}\oint \frac{dz}{z}\sum_{n=1}^{\infty} v_{2n}(\kappa) z^{-2n}\sum_{m=1}^{\infty} A_{2m} z^{2m} \\ &= -\frac{1}{\pi i}\oint \frac{dz}{z}\frac{1-\cosh\kappa\tan^{-1} z^{-1}}{\kappa N(\kappa)}\exp\left(\frac{2i}{3}\tan^{-1} z\right) \end{aligned} \qquad \text{(B.1)}$$

where we used equations A.12. Setting $|z| = 1$,

$$3m(\kappa) = -\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} d\theta \frac{1-\cosh\kappa\tan^{-1} e^{-i\theta}}{\kappa N(\kappa)}\exp\left(\frac{2i}{3}\tan^{-1} e^{i\theta}\right)$$

[14] The sums don't converge in the mode basis comes because $v_n(\kappa)$ has a branch cut singularity at $\kappa = 2i$.

Note that $\tan^{-1}$ has branch points at $\pm i$ so care must be taken to ensure we are integrating on the correct branch. This integral is made much more transparent upon making the substitutions,

$$\tan^{-1} e^{i\theta} = \tfrac{\pi}{4} + ix \qquad \tan^{-1} e^{-i\theta} = \tfrac{\pi}{4} - ix \qquad d\theta = \frac{2dx}{\cosh 2x} \qquad \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad x \in [-\infty, \infty]$$
$$\tan^{-1} e^{i\theta} = -\tfrac{\pi}{4} - ix \qquad \tan^{-1} e^{-i\theta} = -\tfrac{\pi}{4} + ix \qquad d\theta = \frac{2dx}{\cosh 2x} \qquad \theta \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}], \quad x \in [-\infty, \infty]$$

We have,

$$\begin{aligned}
3m(\kappa) &= -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dx}{\cosh 2x} \frac{1-\cosh\kappa(\tfrac{\pi}{4} - ix)}{\kappa N(\kappa)} \exp\left[\frac{2i}{3}(\tfrac{\pi}{4} + ix)\right] \\
&\quad -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dx}{\cosh 2x} \frac{1-\cosh\kappa(\tfrac{\pi}{4} - ix)}{\kappa N(\kappa)} \exp\left[\frac{2i}{3}(-\tfrac{\pi}{4} - ix)\right] \\
&= -\frac{4}{\pi}\int_{-\infty}^{\infty} \frac{dx}{\cosh 2x} \frac{1-\cosh\kappa(\tfrac{\pi}{4} - ix)}{\kappa N(\kappa)} \cosh\left[\frac{2i}{3}(\tfrac{\pi}{4} + ix)\right] \\
&= -\frac{4}{\pi}\frac{1}{\kappa N(\kappa)}\int_{0}^{\infty} \frac{dx}{\cosh 2x} \Big[\sqrt{3}\cosh\tfrac{2}{3}x - \cosh(\tfrac{\pi\kappa}{4} + \tfrac{i\pi}{6})\cosh(i\kappa x + \tfrac{2}{3}x) \\
&\qquad - \cosh(\tfrac{\pi\kappa}{4} - \tfrac{i\pi}{6})\cosh(i\kappa x - \tfrac{2}{3}x)\Big]
\end{aligned}$$

This integral can be evaluated directly with the help of the formula,

$$\int_0^\infty dx \frac{\cosh ax}{\cosh bx} = \frac{\pi}{2b}\frac{1}{\cos\frac{\pi a}{2b}} \qquad |\Re(a)| < \Re(b)$$

We find,

$$\begin{aligned}
3m(\kappa) &= -\frac{1}{\kappa N(\kappa)}\left[\frac{\sqrt{3}}{\cos\frac{\pi}{6}} - \frac{\cosh(\frac{\pi\kappa}{4} + \frac{i\pi}{6})}{\cos(\frac{\pi}{6} + \frac{i\pi\kappa}{4})} - \frac{\cosh(\frac{\pi\kappa}{4} - \frac{i\pi}{6})}{\cos(\frac{\pi}{6} - \frac{i\pi\kappa}{4})}\right] \\
&= -\frac{2}{\kappa N(\kappa)}\frac{\cosh\frac{\pi\kappa}{2} - 1}{1 + 2\cosh\frac{\pi\kappa}{2}}
\end{aligned} \tag{B.2}$$

after some algebra. This proves the identity eq.A.16.

We must now consider the remaining identity, eq.A.8, which is not of the form eq.A.16-A.19. To prove this formula we must evaluate the sum/integral,

$$3m_{2n}\beta_{2n} = \int_{-\infty}^{\infty} d\kappa m(\kappa)\beta(\kappa) = 2\int_{-\infty}^{\infty} d\kappa \frac{1-\cosh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}\frac{1}{2\kappa\sinh\frac{\pi\kappa}{2}}$$

Here we do not need to worry about the fact that $\beta(\kappa)$ is proportional to the principal value distribution since the integrand is smooth at $\kappa = 0$. Closing the contour in the upper half plane, we find an infinite succession of poles with corresponding residues,

$$\begin{aligned}
\mathrm{Res}(2i(2n-1)) &= \frac{1}{i\pi}\frac{1}{2n-1} \\
\mathrm{Res}(2i(2n-\tfrac{4}{3})) &= -\frac{1}{2\pi i}\frac{1}{2n-\frac{4}{3}} \\
\mathrm{Res}(2i(2n-\tfrac{2}{3})) &= -\frac{1}{2\pi i}\frac{1}{2n-\frac{2}{3}}
\end{aligned}$$

for $n = 1, 2, ..., \infty$. We therefore find the sum,

$$3m_{2n}\beta_{2n} = 2\sum_{n=1}^{\infty}\left[\frac{2}{2n-1} - \frac{1}{2n-\frac{2}{3}} - \frac{1}{2n-\frac{4}{3}}\right] \tag{B.3}$$

This sum can be evaluated with the help of the formulas,

$$\sum_{n=1}^{\infty}\frac{1}{n(2n+1)} = 2 - 2\ln 2 \qquad \sum_{n=1}^{\infty}\frac{1}{n(9n^2-1)} = \frac{3}{2}(\ln 3 - 1)$$

Writing eq.B.3 in terms of these summations[15], we find

$$3m_{2n}\beta_{2n} = -\ln\frac{27}{16}$$

proving the last identity eq.A.8.

## C. Useful Formulas

For convenient reference and to set conventions, here we write some important operators used in the paper. We use $\alpha' = \frac{1}{2}$ so that $\alpha_0^\mu = p^\mu$, and metric signature $(-, +, +...)$.

Virasoros and $Q_B$:

$$\begin{aligned}
L_m &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \\
L_m^{gh} &= \sum_{n=-\infty}^{\infty}(m-n)b_{m+n}c_{-n} \\
Q_B &= \sum_{n=-\infty}^{\infty} : c_n(L_{-n} + \tfrac{1}{2}L_{-n}^{gh} - \delta_{n0}) :
\end{aligned}$$

String coordinates:

$$\begin{aligned}
x^\mu(\sigma) &= x^\mu + i\sum_{n=1}^{\infty}\frac{1}{2n}(\alpha_n^\mu - \alpha_{-n}^\mu)\cos n\sigma \\
\pi P^\mu(\sigma) &= p^\mu + \sum_{n=1}^{\infty}(\alpha_n^\mu + \alpha_{-n}^\mu)\cos n\sigma \\
b(\sigma) &= -\sum_{n=1}^{\infty}(b_n - b_{-n})\sin n\sigma \\
\pi\cdot\pi_c(\sigma) &= b_0 + \sum_{n=1}^{\infty}(b_n + b_{-n})\cos n\sigma \\
c(\sigma) &= c_0 + \sum_{n=1}^{\infty}(c_n + c_{-n})\cos n\sigma \\
\pi\cdot\pi_b(\sigma) &= \sum_{n=1}^{\infty}(c_n - c_{-n})\sin n\sigma.
\end{aligned}$$

[15]Thanks to N. Mann for doing these sums faster than one of the authors (T. Erler) could.

Midpoint coordinates:

$$
\begin{aligned}
x^\mu(\tfrac{\pi}{2}) &= x^\mu + i\sum_{n=1}^{\infty} \tfrac{(-1)^n}{2n}\,(\alpha^\mu_{2n} - \alpha^\mu_{-2n}) \\
\pi\cdot P^\mu(\tfrac{\pi}{2}) &= p^\mu + \sum_{n=1}^{\infty}(-1)^n(\alpha^\mu_{2n} + \alpha^\mu_{-2n}) \\
x^\mu(\tfrac{\pi}{2})' &= i\sum_{n=1}^{\infty}(-1)^n(\alpha^\mu_{2n-1} - \alpha^\mu_{-2n+1}) \\
\pi\cdot P(\tfrac{\pi}{2})' &= \sum_{n=1}^{\infty}(2n-1)(-1)^n(\alpha^\mu_{2n-1} + \alpha^\mu_{-2n+1}) \\
b(\tfrac{\pi}{2}) &= \sum_{n=1}^{\infty}(-1)^n(b_{2n-1} - b_{-2n+1}) \\
b(\tfrac{\pi}{2})' &= -\sum_{n=1}^{\infty}2n(-1)^n(b_{2n} - b_{-2n}) \\
\pi\cdot \pi_c(\tfrac{\pi}{2}) &= b_0 + \sum_{n=1}^{\infty}(-1)^n(b_{2n} + b_{-2n}) \\
\pi\cdot \pi_c(\tfrac{\pi}{2})' &= \sum_{n=1}^{\infty}(2n-1)(-1)^n(b_{2n} + b_{-2n}) \\
c(\tfrac{\pi}{2}) &= c_0 + \sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n}) \\
c(\tfrac{\pi}{2})' &= \sum_{n=1}^{\infty}(2n-1)(-1)^n(c_{2n-1} + c_{-2n+1}) \\
\pi\cdot \pi_b(\tfrac{\pi}{2}) &= -\sum_{n=1}^{\infty}(-1)^n(c_{2n-1} - c_{-2n+1}) \\
\pi\cdot \pi_b(\tfrac{\pi}{2})' &= \sum_{n=1}^{\infty}2n(-1)^n(c_{2n} - c_{-2n})
\end{aligned}
$$

We have written these in terms of oscillators in the usual basis, but whenever we refer to them in the paper (with the exception of eq.3.26) they happen to take the same form in the tilde basis.

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