Lecture Notes on Vector Algebra and Quaternions

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Abstract

These are a series of notes which are meant to pedagogically present the algebra of *Quaternions* as an elegant and unified system for representing multiplication between vectors in three dimensions.

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Introduction

My purpose in writing these notes is to present the algebra of *Quaternions* as an elegant and unified alternative for representing multiplication between vectors in three dimensions. The standard vector algebra which we teach in classes today (which we call *Gibbs' vectors*) was constructed by Gibbs and Heaviside in the late 1880's, and was at the time intended to be a pragmatic reduction of the Quaternionic algebra.

There are basically four reasons why I think that Quaternions are a better vectorial system for the modern student and researcher: 1) they provide a fluent and powerful formalism for manipulating vectors and deriving vector identities, thus facilitating the comprehension and solution of difficult problems, 2) they are more notationally economical then Gibbs' vectors, 3) they establish many important ideas which end up being extremely useful in quantum mechanics, modern differential geometry, and even computer graphics, and 4) they provide a compelling justification for the origin and significance of the dot and cross products. Particularly in favor of the last point, many students feel puzzled when they first memorize the definitions of the dot and cross products. To uninitiated eyes, the products seem peculiar and arbitrary—they are non-associative, non-commutative, and possess no multiplicative identity. Only after years of applying the dot and cross products can one begin to develop a feeling for their true significance.

On the whole, I believe that the use of Quaternions develops a confidence and understanding that cannot be gained by using ordinary vector methods. We have such beautiful mathematics at our fingertips, why don't we use it?

I have written these notes so that a student with no previous knowledge of vectors can follow the development from the beginning, though in discussion I frequently relate and compare Quaternions to Gibbs' vectors, so some exposure to vectors would be optimal. Also, some exercises require knowledge of matrix algebra and calculus, but they can be passed over by students who don't feel comfortable with these things yet.

I begin the presentation in the abstract, defining such general concepts as a "vector space" and an "algebra," and become increasingly specific and concrete towards the end of the paper. The purpose of this is to give beginning students some knowledge of general mathematical structures, and also to illustrate where Quaternions lie in the grand scheme of things. Note: Please work out the exercises, they develop technique and skill, and often illustrate important points which I was too lazy to explain in the text!

I hope that these notes will motivate a few students and researchers to adopt Quaternions as the fundamental algebraic representation of vectors, rather than the conventional Gibbs' system. Since Gibbs' vectors can be derived as a limited application of Quaternions, there is nothing to loose by this change of emphasis.

I. Vector Spaces, Algebras, and Metrics

Before we can talk about Gibbs' vectors or Quaternions in particular, we must define the basic notion of a vector space. A set V is called a *vector space*[1] if there are two operations defined on the set, vector addition (denoted $\mathbf{v}+\mathbf{w}$) and scalar multiplication (denoted $c\mathbf{v}$), which satisfy the following properties:

1) Closure: If $\mathbf{v},\mathbf{w} \in V$ then $\mathbf{v}+\mathbf{w} \in V$; if $\mathbf{v} \in V, c \in \Re$ then $c\mathbf{v} \in V$.
2) Commutivity: $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$ and $c\mathbf{v} = \mathbf{v}c$.
3) Associativity: $(\mathbf{v}+\mathbf{w})+\mathbf{x} = \mathbf{v}+(\mathbf{w}+\mathbf{x})$ and $a(b\mathbf{v}) = (ab)\mathbf{v}$.
4) Distributivity: $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v}+c\mathbf{w}$ and $(a+b)\mathbf{v} = a\mathbf{v}+b\mathbf{v}$.
5) Identity: There exists an element $0 \in V$ such that $\mathbf{v}+0 = \mathbf{v}$. Also $1\mathbf{v} = \mathbf{v}$.

An element of a vector space is usually called a *vector*, but actually this term was originally invented by Sir William Hamilton and used in connection with Quaternions[2]. Also note that a "vector" in a vector space is slightly more general than the type of vectors we talk about in physical applications. Physical vectors are defined so that their components change in a specific way under coordinate transformations.

Often a vector is spatially visualized as a line segment with an arrow-head pointing at one end. This picture allows for a very useful geometric visualization of vector addition and scalar multiplication, as the reader will see though working out exercise (1.6). Naturally, however, the vector space axioms themselves don't require that we visualize vectors as arrows.

It is not difficult to see that calculus can only be defined on functions which take their values into a vector space. This is because we need to be able to add functions and and multiply by scalars in order to differentiate and integrate. However, the vector space structure is generally too rudimentary to be interesting as far as calculus is concerned. Usually we spice things up by introducing an additional algebraic structure onto the vector space—this is what is done in, for instance, complex analysis or vector calculus. Essentially, an algebraic structure on a vector space allows us to define multiplication between vectors. This multiplication generally gives us a way of comparing vectors to each other—for example, it might tell us whether two vectors are parallel or perpendicular, how their lengths compare, or how much area is enclosed between them. A vector space forms an *algebra* if there is a product between vectors (denoted $\mathbf{vw}$) satisfying:

1) Closure: If $\mathbf{v},\mathbf{w} \in V$ then $\mathbf{vw} \in V$.
2) Distributivity: $\mathbf{t}(\mathbf{v}+\mathbf{w}) = \mathbf{tv}+\mathbf{tw}$ and $(\mathbf{v}+\mathbf{w})\mathbf{t} = \mathbf{vt}+\mathbf{wt}$.
3) Associativity with Scalars: $c(\mathbf{vw}) = (c\mathbf{v})\mathbf{w} = \mathbf{v}(c\mathbf{w})$.

[1]Specifically, we define a *real* vector space because scalar multiplication is defined over the field of real numbers.

[2]Hamilton himself got the word from Latin, where *vector* means "carrier," which is related to the Latin verb *vehere*, "to carry." I suppose Hamilton's visualization was that a vector was an oriented line segment that, when added to a point, would "carry" it to another point. It's also interesting to note that "vector" has common origin with the word "vehicle."

Note that Gibbs' vector system has an algebraic structure because the cross product is a product between two 3-D vectors which returns a 3-D vector.

Another important mathematical structure that we often place on a vector space is the so-called *metric* or *inner product.* A *metric* is a function g which takes two vectors and returns a real number while satisfying the following properties:

1) Reality: If $\mathbf{v}, \mathbf{w} \in V$, then $g(\mathbf{v}, \mathbf{w}) \in \Re$.
2) Linearity: $g(a\mathbf{v} + b\mathbf{w}, \mathbf{u}) = a \cdot g(\mathbf{v}, \mathbf{u}) + b \cdot g(\mathbf{w}, \mathbf{u})$ and $g(\mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a \cdot g(\mathbf{u}, \mathbf{v}) + b \cdot g(\mathbf{u}, \mathbf{w})$.
3) Symmetry: $g(\mathbf{v}, \mathbf{w}) = g(\mathbf{w}, \mathbf{v})$.
4) Positive-Definiteness: $g(\mathbf{v}, \mathbf{v}) \geq 0$, and in this case equality will only occur when $\mathbf{v} = 0$ [3].

The metric function is very useful because it allows us to define lengths and angles between vectors. For example, two vectors $\mathbf{v}$ and $\mathbf{w}$ are *perpendicular* if and only if $g(\mathbf{v}, \mathbf{w}) = 0$. On the other hand, the length of a vector $\mathbf{v}$—often called its *norm* or *magnitude*—is defined to be $\sqrt{g(\mathbf{v}, \mathbf{v})}$. These definitions may seem a little abstract at first, but we will come to see their natural geometric significance later.

Let us remind ourselves of a few basic notions from linear algebra. I'm going to say this a little fast, so those who don't know this already should pay close attention. Consider a set S which contains k different vectors, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$. We say that S is *linearly independent* if the equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + .. + c_k\mathbf{v}_k = 0 \;, \tag{1.1}$$

implies that *all* the multiplicative constants $\{c_1, c_2, ..., c_k\}$ must vanish. If this is not implied, S is said to be *linearly dependent.* If S is linearly independent in such a way that, for every possible nonempty set $S' \subset V$, $S \cup S'$ is linearly *dependent*, then we say that S forms a *basis* for V. If S is a basis, its constituent vectors are often called *basis vectors.* Finally, if S is a basis, then the number of vectors S contains (in this case, k), is called the *dimension* of V, and V is called a k-dimensional vector space.

Exercises:

(1) Show that $0\mathbf{v} = 0$

(2) Show that the inverse of a vector and the additive identity are unique.

(3) Given a vector space with no additional structure, is it possible to determine whether two vectors are a) parallel b) linearly independent, or c) perpendicular? If so, what conditions must those the two vectors satisfy in this case?

(4) Prove or find a counterexample to the following statement: No set with a finite number of elements can form a vector space.

[3] Sometimes this restriction is relaxed. For example, the metric on vectors in four-dimensional spacetime in Einstein's theory of relativity is not positive definite.

(5) a) Does the set of real numbers form an algebra? b) Suppose we define the following map on a pair of real numbers a and b: $g(a,b) = c(ab)$ for nontrivial $c \in \Re$. Does g form a metric on the set of real numbers? c) A vector space V with both algebraic *and* metric structures is called a *normed algebra* if

$$\sqrt{g(\mathbf{vw}, \mathbf{vw})} = \sqrt{g(\mathbf{v}, \mathbf{v})}\sqrt{g(\mathbf{w}, \mathbf{w})} \ ,$$

where $\mathbf{v}, \mathbf{w} \in V$. Do real numbers form a normed algebra?

(6) a) Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a basis, then for any $\mathbf{w} \in V$, there exist real numbers $c_1, c_2, ..., c_k$ such that we can write,

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k \ .$$

In this case, we say that $\mathbf{w}$ can be expressed as a *linear combination* of the basis vectors in S. b) Prove that if S and S' are bases, they contain the same number of vectors. Thus, the notion of dimension is independent of basis, and can be thought of as an intrinsic property of the vector space.

(7) In this problem, we would like to illustrate the "parallelogram rule" for vector addition. a) Show that elements of any n-dimensional vector space can be put in one-to-one correspondence with points in a "physical" n-dimensional space—that is, some extended n-dimensional spatial region in which we have placed an orthogonal set of n coordinate axes. Thus, the elements of an abstract vector space can be diagramatically visualized as points in physical space. b) Show that adding two vectors in a vector space therefore corresponds to the following geometric construction: If we draw an arrow from the origin to the point corresponding to $\mathbf{v}_1$, and an arrow from the origin to point corresponding to $\mathbf{v}_2$, then the arrow corresponding to $\mathbf{v}_1 + \mathbf{v}_2$ points from the origin to the opposite corner of the parallelogram whose edges are defined by the arrows $\mathbf{v}_1$ and $\mathbf{v}_2$. c) Show that if $\mathbf{w} = c\mathbf{v}$, both $\mathbf{v}$ and $\mathbf{w}$ lie on a common line which intersects the origin, and $\mathbf{w}$ is c times the distance from the origin as $\mathbf{v}$.

II. Complex Numbers

Perhaps the most powerful and important algebraic structure we can place on a two dimensional vector space corresponds to algebra of complex multiplication. Complex numbers are remarkable for several reasons: they allow us to factor any polynomial; every nontrivial complex number has a unique multiplicative inverse; multiplication by complex numbers has a simple geometric interpretation; complex analysis offers profound insight into the theory of functions and differential equations. Nevertheless, it is troubling that complex numbers are only two dimensional, and so they cannot be used to represent our three dimensional world. This problem is what inspired Hamilton in the 1840's to consider generalizations of complex numbers to higher dimensions, which in turn lead him to discover the algebra of Quaternions. As we will see, the usual Gibbs' vector system which we learn in class today is basically a reduced and limited version of Quaternions.

Let us define complex numbers in a way which will suit itself best to generalization. Consider the set of ordered pairs of real numbers, (a, b). If we define addition and scalar multiplication in the usual way,

$$(a, b) + (c, d) = (a + c, b + d)$$

$$c(a, b) = (ca, cb), \tag{2.1}$$

then this set forms a vector space (verify this). If we also define the following multiplication rule,

$$(a, b)(c, d) = (ac - db, ad + cb). \tag{2.2}$$

then such ordered pairs are called *complex numbers*. The set of all complex numbers may be denoted $\mathcal{C}$.

Writing a complex number as an ordered pair is a bit unconventional, but it's really just a matter of notation. More commonly, though, people like to express complex numbers by defining:

$$1 = (1, 0), \quad i = (0, 1) \ , \tag{2.3}$$

so that $(a, b) = a + bi$. It's consistent to set the ordered pair $(1, 0)$ equal to the real number 1 since they both serve as a multiplicative identity on $\mathcal{C}$; for all practical purposes, there's no point in distinguishing them. Using eq. (2.2), we discover that

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1 \ , \tag{2.4}$$

so $i = \sqrt{-1}$, as should be no great surprise.

Here's a bit of terminology: If $z = (a, b) = a + bi \in \mathcal{C}$ then $a = \Re(z)$ is called the *real part* of z and $b = \Im(z)$ is called the *imaginary part*. The *complex conjugate* $\overline{z}$ is the reflection of z about the real axis, $\overline{z} = (a, -b) = a - bi$. The *magnitude* of z is defined $|z| = \sqrt{a^2 + b^2}$.

Exercises:

(1) Show that any complex number has a multiplicative inverse, and give a formula for it.

(2) a) Prove Euler's formula, $e^{i\theta} = \cos\theta + i \sin\theta$ using Taylor expansions. b) Use this result to verify that the product of two complex numbers z_1 and z_2 gives a complex number whose magnitude is the product of the magnitudes of z_1 and z_2, and whose angle is the sum of the angles of z_1 and z_2.

(3) Suppose that a vector in the Cartesian coordinate plane is rotated counterclockwise by ninety degrees. How could this transformation be represented by complex numbers?

(4) Prove that

$$\sum_{n=1}^{N} e^{2\pi i n/N} = 0.$$

provided that $N > 1$.

(5) Suppose we define an alternative multiplication rule between ordered pairs (a,b) so that,

$$(a,b)(c,d) = (ac + db, ad + cb)$$

(note that the minus sign is missing). Show that there are nontrivial ordered pairs which have no multiplicative inverse. In this example, such ordered pairs are *nilpotents*. This algebraic structure is closely tied with the geometric structure of spacetime in special relativity.

(6) a) Show that the the matrix,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

squares to minus one, and therefore, this matrix taken together with the identity matrix forms a *representation* for the complex algebra. b) Show that taking the *transpose* of a linear combination of this matrix with the identity is equivalent to taking the *complex conjugate* of the same linear combination of "1" and i.

(7) a) Show that the function,

$$g(z_1, z_2) = \Re(z_1 \overline{z}_2) \quad z_1, z_2 \in \mathcal{C} \ ,$$

defines a metric on complex numbers. b) Prove also that the the magnitude of a complex number can be computed as $\sqrt{g(z,z)}$, precisely as we defined more abstractly in section 1. c) Prove that complex numbers are a normed algebra, as defined in excercise 1.5.

III. Definition of Quaternions

As we have stated it, the multiplication rule for complex numbers can be extended in an obvious way. Instead of considering ordered pairs of real numbers, why don't we consider ordered pairs of *complex* numbers, (a,b) with $a,b \in \mathcal{C}$? We can define vector addition and scalar multiplication as in eq. (2.1), and also define a product between pairs in analogy with eq. (2.2):

$$(a,b)(c,d) = (ac - d\bar{b}, \bar{a}d + cb) \quad . \tag{3.1}$$

Presumably, this defines a new algebraic system whose relationship with complex numbers is analogous to the relationship complex numbers have with real numbers. For example, this system is clearly *four dimensional*—twice the dimension of complex numbers—just as complex numbers have twice the dimension of real numbers. The algebra defined by eq. (3.1) is exactly the algebra of Quaternions, discovered by Hamilton in 1843. An ordered pair of complex numbers subject to the multiplication rule eq. (3.1) is called a *quaternion*, and the set of all quaternions is denoted $\mathcal{H}$ ("H" is for Hamilton).

Note that eq. (3.1) involves taking the complex conjugate, whereas eq. (2.2) does not. Of course, writing complex conjugates into eq. (2.2) wouldn't change anything, since a real number is its own complex conjugate—so eq. (3.1) is a consistent generalization of eq. (2.2). Still, that doesn't explain why I put the conjugates there. It turns out that this expression gives the unique generalization which preserves the following crucial property: that multiplication between any two nonzero elements is *invertable.* We can see this by explicitly constructing an inverse. Note:

$$(a,b)(\bar{a},-b)=(a\bar{a}+b\bar{b},-\bar{a}b+\bar{a}b)=(|a|^2+|b|^2,0) \quad . \tag{3.2}$$

Therefore,

$$(a,b)^{-1}=\frac{(\bar{a},-b)}{|a|^2+|b|^2} \quad . \tag{3.3}$$

A quick calculation on the reader's part would reveal that other variations on eq. (3.1) would create algebras containing nontrivial elements without an inverse. (This would become evident, for instance, when we tried to create an expression analogous to eq. (3.3); the denominator would sometimes be zero.)

Just as in complex numbers, it is somewhat unusual to express a quaternion as an ordered pair. Rather, people usually write:

$$1=(1,0), \quad \mathbf{i}=(i,0), \quad \mathbf{j}=(0,i), \quad \mathbf{k}=(0,1) \quad . \tag{3.4}$$

so that

$$(a+bi,d+ci)=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k} \quad . \tag{3.5}$$

This was Hamilton's original notation. Obviously, his idea was that there are now two new imaginary units, **j** and **k** to accompany the usual imaginary unit, i. Interestingly, this is exactly why we still use the letters **i**, **j**, and **k** to denote unit vectors in 3-D space in ordinary vector arithmetic. We can use eq. (3.1) to generate a full multiplication table for products between **i**, **j**, and **k**; multiplication between arbitrary quaternions can then be easily inferred from the vector space axioms and the distributive law. We have:

$$\mathbf{i}^2=(i,0)(i,0)=(-1,0)=-1; \quad \mathbf{j}^2=(0,i)(0,i)=(-i\bar{i},0)=-1;$$

$$\mathbf{k}^2=(0,1)(0,1)=(-1,0)=-1.$$

And,

$$\begin{aligned}\mathbf{ij}&=(i,0)(0,i)=(0,\bar{i}i)=(0,1)=\mathbf{k}\\ \mathbf{jk}&=(0,i)(0,1)=(-\bar{i},0)=\mathbf{i}\\ \mathbf{ki}&=(0,1)(i,0)=(0,i)=\mathbf{j} \quad .\end{aligned} \tag{3.6}$$

Note that with quaternions, we have *three* different quantities which square to minus one! But there's a still more surprising fact: one might think that, since we've calculated the product of every pair of quaternions, we've calculated the whole multiplication table. This is not actually so, since there are two ways you can multiply a pair of quantities—either the first times the second, or the second times the first. Note that

$$\mathbf{ji}=\mathbf{j}(\mathbf{jk})=\mathbf{j}^2\mathbf{k}=-\mathbf{k}=-\mathbf{ij} \quad . \tag{3.7}$$

Similarly, $\mathbf{jk} = -\mathbf{kj}$ and $\mathbf{ki} = -\mathbf{ik}$.[4] This means that multiplication of quaternions is not commutative! $ab \neq ba, a, b \in \mathcal{H}$! Hamilton's discovery of the Quaternions was in fact the first time non-commutative multiplication had been considered in the history of mathematics. Since then, however, with the study of groups, matrices, and covariant differentiation, among other things, noncommutivity has become commonplace in advanced mathematics. In fact, Sir Arthur Eddington said, "Anticommutivity is perpendicularity," foreshadowing a deep connection between spatial geometry and algebraic structure, as we're about to see. The results of our multiplication table can be summarized in a nicely compact equation:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \ . \tag{3.6}$$

This expression is exactly as Hamilton himself carved into the Brougham Bridge in Edinburgh, 1843, after the multiplication rule had come to him in a sudden flash of inspiration. Apparently, his etching can still be seen on the bridge to this day.

One way to memorize quaternionic multiplication is to draw a picture of a circle with three dots equally spaced counterclockwise around a circle, labeled **i**, **j**, and **k** respectively. To multiply any two of the units together, start from the dot labeled by the first unit and move to the dot labeled by the second. If the movement was counterclockwise, the answer is the label of the third remaining dot. If the movement was clockwise, the answer is minus the label of the third remaining dot.

The reader might wonder, at this point, why we don't try generalizing the complex numbers yet again! That is, why not consider, now, *ordered pairs of quaternions* (a, b) with $a, b \in \mathcal{H}$, subject to something like a multiplication rule eq. (3.1)? We'd have to be a little careful here, since quaternions don't commute, and we'd also need to define a "quaternionic" conjugation (see next section). But if we do this right, we can get an eight dimensional algebra which goes by the name of *Octonions*, denoted $\mathcal{O}$. As you might expect, the Octonions have seven units squaring to -1 and are non-commutative. But the really baffling new feature is that they turn out to be *non-associative*! This makes them extremely tricky and confusing to work with, and there is currently only a handful of researchers worldwide who have enough grit to study them actively. Now we might ask, what would happen if we considered ordered pairs of *octonions*? Or ordered pairs of pairs of octonions, and so on *ad infinitum*? This unending process of generating new algebras actually has a name: the *Cayley-Dickson construction.* However, it turns out that, beyond Octonions, none of the bigger algebras are of much particular interest. The reason is that multiplication in these algebras is not generally invertable. This is because, roughly speaking, every time we apply the Cayley-Dickson procedure we generate a stranger algebra—the first time, we get a number which squares to -1; the second time, we get numbers which don't commute; the third time, we get numbers which

[4]In this equation, we've assumed that quaternionic multiplication is associative (or at least alternative). An explicit calculation from eq. (3.1) will quickly verify this is the case. See exercise 4.2.

don't associate. Finally, the fourth time, we get multiplication which is not invertable. This nasty property then persists through all of the higher algebraic generalizations. One might think that we could modify our multiplication rule eq. (3.1) in some clever way so that we might continue to generate larger algebras with completely invertable multiplication. But a theorem constructed by Hurwitz in 1898 proves that all *division algebras*—algebras with completely invertable multiplication—are isomorphic to the reals, the complex numbers, the Quaternions, or the Octonions. Therefore, there can be no division algebra with dimension greater than eight. In fact, $\Re$, $\mathcal{C}$, $\mathcal{H}$, and $\mathcal{O}$ are the *only* division algebras with a norm satisfying the intuitive property $|a||b| = |ab|$—thus they are the *normed* division algebras (see exercises 1.5, 4.6).

Exercises:

(1) Draw a picture of the multiplication diagram described at the end of this section. Use it to compute the product $(3 + \mathbf{i} - 4\mathbf{j})(7\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ for practice.

(2) Consider the following matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the so-called *Pauli spin matrices*. Show that if we set

$$\mathbf{i} = i\sigma_1, \quad \mathbf{j} = i\sigma_2, \quad \mathbf{k} = i\sigma_3,$$

the resulting matrices satisfy the multiplication rules for Quaternions, and so form a *representation* for the algebra. (Note that the spin matrices themselves form a representation for a 3-dimensional Clifford Algebra).

(3) Show that Quaternions form an algebra, that is, verify that eq. (3.1) is consistent with the definition of an algebra as given in section 1.

(4) Defining the conjugate of a quaternion to be $\overline{(a, b)} = (\bar{a}, -b)$, use eq. (3.1) to generate the algebra of Octonions. The multiplication should be explicitly ordered as written in eq. (3.1). Show that octonionic multiplication is non-associative.

IV. Relation Between Quaternions and Gibbs' Vectors

So now that we have finally defined the quaternionic algebra, let's introduce a little terminology. Suppose we have $\mathbf{Z} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathcal{H}$, then $a = \mathrm{SP}(\mathbf{Z})$ is called the *scalar part* and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \mathrm{VP}(\mathbf{Z})$ is called the *vector part* of $\mathbf{Z}$. Notice that the vector part of a quaternion is three dimensional, and so is exactly what is needed to describe an oriented line segment in three dimensional space. A quaternion which has a vanishing scalar part is called a *vector*[5], but sometimes people also call it a *pure quaternion*.

[5] Remember, this "vector" should not be confused with the *abstract* vector which is an element of a vector space.

The *quaternionic conjugate* $\overline{\mathbf{Z}}$ is the reflection (in four dimensions!) of the vector part about the real axis:

$$\overline{\mathbf{Z}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \ . \tag{4.1}$$

The *magnitude* or *norm* of a quaternion is given by,

$$|\mathbf{Z}| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{\mathbf{Z}\overline{\mathbf{Z}}} \ . \tag{4.2}$$

If $|\mathbf{Z}| = 1$, we say that $\mathbf{Z}$ is a *unit quaternion* or that it is *normalized.* We can express the scalar and vector parts of a quaternion in terms of the quaternionic conjugate:

$$\begin{aligned} \mathrm{SP}(\mathbf{Z}) &= \frac{\mathbf{Z} + \overline{\mathbf{Z}}}{2} \\ \mathrm{VP}(\mathbf{Z}) &= \frac{\mathbf{Z} - \overline{\mathbf{Z}}}{2} \ . \end{aligned} \tag{4.3}$$

Quaternionic conjugation satisfies the following useful properties, which should be familiar from the complex case:

$$\begin{aligned} \overline{\overline{\mathbf{Z}}} &= \mathbf{Z} \\ \overline{\mathbf{Z} + \mathbf{W}} &= \overline{\mathbf{Z}} + \overline{\mathbf{W}} \\ \overline{\mathbf{Z}\mathbf{W}} &= \overline{\mathbf{W}}\ \overline{\mathbf{Z}} \ . \end{aligned} \tag{4.4}$$

Note that the conjugation reverses the order of the product—in this sense, it is analogous to transposing a matrix. As a shorthand, we sometimes write $\mathbf{Z} = a + \mathbf{v}$, where a is understood to be the scalar part of $\mathbf{Z}$ and $\mathbf{v}$ the vector part.

Now consider what happens when we multiply two quaternions together:

$$\mathbf{Z}_1\mathbf{Z}_2 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_2\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \ .$$

Multiplying everything out,

$$\begin{aligned} \mathbf{Z}_1\mathbf{Z}_2 = {} & a_1a_2 + a_1(b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) + b_1a_2\mathbf{i} - b_1b_2 + b_1c_2\mathbf{ij} + b_1d_2\mathbf{ik} \\ & + c_1a_2\mathbf{j} + c_1b_2\mathbf{ji} - c_1c_2 + c_1d_2\mathbf{jk} + d_1a_2\mathbf{k} + d_1b_2\mathbf{ki} + d_1c_2\mathbf{kj} - d_1d_2 \ , \end{aligned}$$

and sorting terms:

$$\begin{aligned} \mathbf{Z}_1\mathbf{Z}_2 = {} & a_1a_2 + \\ & a_1(b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) + a_2(b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \\ & -b_1b_2 - c_1c_2 - d_1d_2 \\ & (c_1d_2 - c_2d_1)\mathbf{i} + (b_1c_2 - b_2c_1)\mathbf{k} + (b_2d_1 - b_1d_2)\mathbf{j} \end{aligned} \ . \tag{4.5}$$

Now let us write $\mathbf{Z}_1 = a_1 + \mathbf{v}_1$ and $\mathbf{Z}_2 = a_2 + \mathbf{v}_2$, and define:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = b_1b_2 + c_1c_2 + d_1d_2$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = (c_1 d_2 - c_2 d_1)\mathbf{i} + (b_1 c_2 - b_2 c_1)\mathbf{k} + (b_2 d_1 - b_1 d_2)\mathbf{j} \ . \tag{4.6}$$

We can then write eq. (4.5) in the form,

$$\mathbf{Z}_1 \mathbf{Z}_2 = a_1 a_2 + a_1 \mathbf{v}_2 + a_2 \mathbf{v}_1 - \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_2 \ . \tag{4.7}$$

Now we have finally reached the punch line! Of course, the products defined in eq. (4.6) are precisely the dot and cross products we know from Gibbs' vector algebra. This is, in fact, how these two products were first discovered by Hamilton in the mid nineteenth century. J. Willard Gibbs borrowed these products when he concocted his own system around 1880, but in doing so he obscured their algebraic origin to an extreme. I think that it is fantastic that the dot and cross products are actually an inevitable consequence of generalizing complex numbers, and by extension, real numbers and even integers!

Take a moment to appreciate eq. (4.7). Note that every conceivable type of product is lumped together into one equation: multiplication between real numbers, scalar multiplication of a vector, and even the dot and cross products. Note also that the square of a vector is negative:

$$\mathbf{v}^2 = -\mathbf{v} \cdot \mathbf{v} = -|\mathbf{v}|^2 \ . \tag{4.8}$$

This peculiar fact created a lot of resistance to the quaternionic system at the time. There was a strong philosophical prejudice, which is still present to a extent even today, which says that anything which squares to a negative number cannot be physically interpreted or intuitively imagined. Certainly, they said, something as concrete as a pointed arrow in space could not square to a negative number! A heated debate ensued, and those who advocated Quaternions were given such derogatory names as "the minus men." For this reason and a few others, Hamilton's Quaternions eventually fell out of favor and Gibb's system was adopted. This quibbling seems a little bit adolescent these days—a modern physicist deals with many mathematical objects which square to a negative number. For example, the unit vector in the time direction in special relativity[6], the Hodge star operator, a complex structure on manifolds, etc. Besides, who is to say that the square of an arrow needs to be positive?

Note that the dot product of two vectors is a scalar, while the cross product is a vector. Thus we have:

$$\mathbf{a} \cdot \mathbf{b} = -\mathrm{SP}(\mathbf{ab}) = -\tfrac{1}{2}(\mathbf{ab} + \overline{\mathbf{ab}})$$
$$\mathbf{a} \times \mathbf{b} = \mathrm{VP}(\mathbf{ab}) = \tfrac{1}{2}(\mathbf{ab} - \overline{\mathbf{ab}}) \ .$$

The quaternionic conjugate of a product is the product of the conjugates in the reverse order, and since the conjugate of a pure vector gives an overall sign, we derive the following extremely useful formulas:

$$\mathbf{a} \cdot \mathbf{b} = -\tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \ = -\tfrac{1}{2}\{\mathbf{a}, \mathbf{b}\}$$

[6]In fact, another equally prevalent convention has the three spatial directions square to minus one, rather than the time direction. But theis is just as the quaternionic system would have it!

$$\mathbf{a}\times\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba}) \ = \tfrac{1}{2}[\mathbf{a},\mathbf{b}] \ . \tag{4.9}$$

From this we see that the dot product is *symmetric* (i.e. commutative) and the cross product is *antisymmetric* (i.e. switching the order of the product changes the answer by a minus sign). Note also that the cross product forms an algebra on three dimensional vectors, while the dot product forms a metric.

The formulas of eq. (4.9) are especially useful for deriving the notorious vector "identities," which otherwise have to be derived by brute force. To see an example, let's derive the famous "BAC-CAB" rule for the triple cross product.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\left(\frac{1}{2}\mathbf{a}(\mathbf{bc}-\mathbf{cb}) - \frac{1}{2}(\mathbf{bc}-\mathbf{cb})\mathbf{a}\right)$$

$$= \frac{1}{4}(\mathbf{abc}-\mathbf{acb}-\mathbf{bca}+\mathbf{cba}) \ .$$

Adding and subtracting **bac** and **cab** from the expression, we can write,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{abc}+\mathbf{bac}) - \frac{1}{2}(\mathbf{acb}+\mathbf{cab}) - \frac{1}{2}(\mathbf{bca}+\mathbf{bac}) + \frac{1}{2}(\mathbf{cba}+\mathbf{cab})\right) \ .$$

Substituting back in using the formulas eq. (4.9),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -\frac{1}{2}((\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) + \mathbf{c}(\mathbf{a}\cdot\mathbf{b}))$$

Or,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \ . \tag{4.10}$$

This is the "BAC-CAB" rule, perhaps the world's easiest derivation[7]!

Let us pause a moment to think about what we have done. Originally Hamilton's idea was that, by generalizing complex numbers, we could find an algebraic representation of three dimensional space analogous to the complex representation of the 2-D plane. However, this is not quite what has happened. We actually found a *four dimensional* generalization of the complex numbers, but it turned out that the "vector part" of this was three dimensional, and hence could be used to describe oriented line segments in 3-D. Apparently no purely three dimensional analogy to complex numbers exists. However, it is clear that complex numbers form an extremely compelling algebraic representation of the 2-D plane. So it is natural to wonder whether, in spite of our failure to find a 3-D generalization of complex numbers, the Quaternions form an adequate representation of three dimensional geometry. The answer is, resoundingly, "Yes!" In fact, Quaternions probably are a *better* representation of 3-D then complex numbers are of 2-D. The reason for this is that the vector part of a Quaternion encodes the fundamental physical principle of *isotropy.* Isotropy is the observed fact that there are no physically preferred directions in the

[7]Note for instance, that in Colley's book *Vector Calculus* she suggests (p.43) that the reader prove this formula using a computer. Clearly the quaternionic system makes this an overkill!

universe. In particular, if you set up one coordinate system of axes to describe the universe, and I set up another which is rotated or skewed with respect to yours, the same laws of physics will apply in either coordinate system. The complex representation of the two dimensional plane violates isotropy for the simple reason that the real axis is treated very differently from the imaginary axis—1 squares to itself, while i squares to -1. However, this is not the case for the vector part of a quaternion. All three basis elements are algebraically equivalent—that is, each squares to "-1," and the product of any two is the third. The mathematical equivalence of the three imaginary units translates directly into the physical equivalence of the three directions of space.

Exercises:

(1) Verify eq. (4.4).

(2) Use eq. (3.1) to prove that quaternionic multiplication is associative, i.e. $\mathbf{Z}_1(\mathbf{Z}_2\mathbf{Z}_3) = (\mathbf{Z}_1\mathbf{Z}_2)\mathbf{Z}_3 = \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3$. Also show that the vector cross product is non-associative.

(3) Show that the dot product follows these basic rules:

i. $\mathbf{a}\cdot\mathbf{a} \geq 0$, and $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = 0$.

ii. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

iii. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.

iv. For $c \in \Re$, $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$.

(4) Show that the cross product follows these basic rules:

i. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

ii. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$.

iii. For $c \in \Re$, $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$.

(5) Prove the following identities:

i. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

ii. $\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

iii. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \det\begin{pmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{pmatrix}$.

iv. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$. This important result is known as the *Jacobi identity*.

(6) a) Convince yourself that,

$$g(\mathbf{Z_1},\mathbf{Z_2}) = \mathrm{SP}(\mathbf{Z_1},\overline{\mathbf{Z_2}}),\ \mathbf{Z_1},\mathbf{Z_2} \in \mathcal{H}\ ,$$

defines a metric on quaternions. b) Prove that the magnitude of a quaternion $\mathbf{Z}$ can be computed as $\sqrt{g(\mathbf{Z},\mathbf{Z})}$ as in section 1. c) Prove that quaternions are a *normed* division algebra.

V. Geometric Interpretation of the Dot and Cross Products

So far, basically all we have done is algebra. We derived the dot and cross products from the premise of the quaternionic multiplication rule, but we have not shown that these products have any intrinsic geometric significance (though, the reader probably knows from previous experience that they do).

So, let us make the connection between the algebra and geometry explicit. Visualize a three dimensional space where we have set up an *righthanded* and orthogonal Cartesian coordinate system, (x, y, z). What we mean by "righthanded" is that if you point your right hand along positive x, curl your fingers towards positive y, and sticks your thumb up, it should point in the direction of positive z.

Imagine three arrows of unit length, which we denote $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, pointing along the positive x, y, and z axes, respectively. Suppose we geometrically define addition and scalar multiplication between $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ according to the parallelogram rule (see exercise 1.7). We can then construct arrows of arbitrary length and orientation by simply taking linear combinations of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Let's further suppose that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ can be *multiplied* together in correspondence to the quaternionic multiplication rule, eq. (3.6). Then, a pointed arrow in 3-D space can be thought of as a visual representation of the vector part of a quaternion. Therefore we frequently refer to oriented arrows simply as *vectors* because we don't differentiate between the visual image of a vector and the vector itself as an abstract mathematical entity. This association is what imbues Quaternions with geometric significance.

So, imagine a triangle floating in space with no particular shape or orientation. We can completely describe the triangle by specifying two vectors which lie along two of its sides, say $\mathbf{a}$ and $\mathbf{b}$, and the third side will, naturally, coincide with the vector $\mathbf{a} - \mathbf{b}$. According to the Law of Cosines, the lengths of the triangle's sides, and hence the magnitudes of the corresponding vectors[8], must satisfy the equation,

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta_{\mathbf{a},\mathbf{b}} \ , \tag{5.1}$$

where $\theta_{\mathbf{a},\mathbf{b}} \in [0, 2\pi]$ is the angle in between $\mathbf{a}$ and $\mathbf{b}$. But, as we saw in eq. (4.8) the magnitude squared of a vector is simply its dot product with itself. So,

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} \ . \tag{5.2}$$

Combining eq. (5.1) and eq. (5.2) we discover that:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta_{\mathbf{a},\mathbf{b}} \ . \tag{5.3}$$

This equation tells us how we can compute the dot product explicitly in terms of geometric quantities intrinsic to the vectors themselves—their magnitudes and respective angle. So, as it turns out, the dot product doesn't actually depend on a vector's orientation with respect to a coordinate system, even though the definition eq. (4.6) involves vector components explicitly. The coordinate independence of the dot product is basically a manifestation of the "isotropic" nature of Quaternionic multiplication, as we were just talking about.

Can we find a similar formula to describe the cross product? Sure, but we have to do a little more work because the cross product is a *vector* and so we

[8] Note that because the arrows $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ have a spatial length of exactly one, the length of the any arrow in 3-D is identical with the magnitude of the corresponding quaternion.

have to find its magnitude *and* direction. To find its magnitude, recall:

$$\mathbf{a} \times \mathbf{b} = \mathbf{ab} + \mathbf{a} \cdot \mathbf{b} \ .$$

Take the square of this,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b})^2 = -|\mathbf{a} \times \mathbf{b}|^2 &= (\mathbf{ab} + \mathbf{a} \cdot \mathbf{b})^2 \\ &= \mathbf{abab} + 2\mathbf{ab}(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \mathbf{abab} - \mathbf{ab}(\mathbf{ab} + \mathbf{ba}) + (\mathbf{a} \cdot \mathbf{b})^2 \\ &= -\mathbf{a}^2\mathbf{b}^2 + (\mathbf{a} \cdot \mathbf{b})^2 \ .\end{aligned}$$

Substituting eq. (5.3) into this, we get,

$$\begin{aligned}-|\mathbf{a} \times \mathbf{b}|^2 &= -|\mathbf{a}|^2|\mathbf{b}|^2 + |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta_{\mathbf{a},\mathbf{b}} \\ &= -|\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta_{\mathbf{a},\mathbf{b}} \ . \end{aligned} \tag{5.4}$$

I'm going to leave it to the reader as an exercise to prove that the direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in such a way that if you point your fingers along $\mathbf{a}$, curl them towards $\mathbf{b}$ and lift your thumb it should point along $\mathbf{a} \times \mathbf{b}$. This mnemonic is called the "right hand rule." There can only be one such direction in three dimensions, so let us denote it with the unit vector $\mathbf{n}_{\mathbf{a},\mathbf{b}}$. Then we can write a coordinate invariant formula for the cross product, making use of eq. (5.4),

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}||\sin\theta_{\mathbf{a},\mathbf{b}}|\mathbf{n}_{\mathbf{a},\mathbf{b}} \ . \tag{5.5}$$

Note that if we visualize $\mathbf{a}$ and $\mathbf{b}$ as defining the edges of a parallelogram then the magnitude of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram, and the direction can be visualized as specifying the orientation of the parallelogram. Therefore the cross product essentially gives us a mathematical description of an oriented area.

As it turns out, it's also possible to form a description of an oriented volume by composing the dot and cross products together into the "triple product," $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Taking eq. (5.4) and eq. (5.5) together, we find,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin\theta_{\mathbf{a},\mathbf{b}} \cos\theta_{\mathbf{a}\times\mathbf{b},\mathbf{c}} \ . \tag{5.6}$$

It's not too difficult to visualize that if $|\mathbf{a}|$ is the length of one leg of a parallelepiped, $|\mathbf{b}| \sin\theta_{\mathbf{a},\mathbf{b}}$ is the depth and $|\mathbf{c}| \cos\theta_{\mathbf{a}\times\mathbf{b},\mathbf{c}}$ is the height, so the product is the parallelepiped's volume.

Let me introduce a bit of terminology that comes up when discussing cross and triple products. A *pseudovector* is any vector which is constructed by taking the cross product of two other vectors. Some examples of this include oriented areas, angular momentum, magnetic field, etc. A *pseudoscalar* is a real number which is constructed by taking a triple product between three vectors. Some examples include volumes, densities, electric flux. I should warn the reader that this terminology is a little bit artificial, though. Given any vector or scalar out of context, there is no unambiguous way of deciding whether it was constructed using a cross or triple product[9].

Another geometric application of the dot and cross products is in calculating projections. Frequently we find it convenient to "decompose" quantities and express them in terms of the basis vectors of a coordinate system (in particular, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$). However, the algebraic behavior of a vector is independent of the coordinate system we chose, and so we often find it useful to "decompose" it into components which are not strictly coordinate basis vectors. We could, for instance, break up a vector, $\mathbf{a}$, according to how much it points in the same direction, and points away from, another vector, $\mathbf{b}$. Specifically, $\mathbf{a}$ can be broken up into two parts: a part *parallel* to $\mathbf{b}$, and a part *perpendicular* to $\mathbf{b}$. The parallel part is called the *orthogonal projection* (or simply "projection") of $\mathbf{a}$ onto $\mathbf{b}$, given by the equation,

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}| \cos\theta_{\mathbf{a},\mathbf{b}} \frac{\mathbf{b}}{|\mathbf{b}|} \ . \tag{5.7}$$

The perpendicular part is called the *antiprojection* of $\mathbf{a}$ onto $\mathbf{b}$, denoted $\mathrm{aproj}_{\mathbf{b}}\mathbf{a}$, defined so that,

$$\mathbf{a} = \mathrm{proj}_{\mathbf{b}}\mathbf{a} + \mathrm{aproj}_{\mathbf{b}}\mathbf{a} \ . \tag{5.8}$$

The reader might notice that eq. (5.7) bears some resemblance to our formula for the dot product, eq. (5.3). So if we take the hint, we should try expressing the projection in terms of the dot product:

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}||\mathbf{b}| \cos\theta_{\mathbf{a},\mathbf{b}} \frac{\mathbf{b}}{|\mathbf{b}|^2} = -\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}^2}\mathbf{b} \ .$$

In other words,

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = -\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}} \ . \tag{5.9}$$

This is a pretty neat formula—notice that we are dividing by a vector, something that is unheard of in the Gibbs' formalism! Using eq. (5.8), we can also find a nifty formula for the antiprojection:

$$\mathrm{aproj}_{\mathbf{b}}\mathbf{a} = \mathbf{a} + \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}} = \frac{\mathbf{ab} + \mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}} \ .$$

The bar on the left of $\mathbf{b}$ in the denominator indicates that division is performed from the left, using the notation,

$$\frac{\mathbf{Z}}{|\mathbf{W}} = \mathbf{Z}(\mathbf{W}^{-1}) \qquad \frac{\mathbf{Z}}{\mathbf{W}|} = (\mathbf{W}^{-1})\mathbf{Z} \ . \tag{5.10}$$

[9] Actually, the terminology is not completely arbitrary. It tries to highlight the fact that, in tensor algebra, areas and volumes are represented by antisymmetric two and three index tensors, and are *not* generally represented by vectors and scalars as above. So, in essence, people have tried to maintain a distinction in terminology that is not reflected by the vector algebra.

Therefore, we find

$$\begin{aligned}\mathrm{aproj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{ab} - \frac{1}{2}(\mathbf{ab} + \mathbf{ba})}{\big|\mathbf{b}} \\ &= \frac{\frac{1}{2}(\mathbf{ab} - \mathbf{ba})}{\big|\mathbf{b}} \ .\end{aligned}$$

Substituting the now famous eq. (4.9),

$$\mathrm{aproj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \times \mathbf{b}}{\big|\mathbf{b}} \ . \tag{5.11}$$

Therefore, the dot product is to the projection exactly as the cross product is to the *anti*projection.

Note that when two vectors are parallel—and, in a sense, most "similar" to one another—the projection is maximum, but when they are perpendicular—and hence most "different"—the projection is zero. For the antiprojection it works the other way around. This suggests, in light of eq. (5.9) and eq. (5.11), that the dot product can be interpreted as a positive measure of "similarity" between vectors, while the cross product can be imagined as a measure of "discrepancy." This is a useful interpretation to keep in mind. For example, in vector calculus the divergence and curl of a vector field $\mathbf{v}(x, y, z)$ are the dot and cross products of the gradient ∇ with $\mathbf{v}$, respectively. Hence, in a sense, the divergence is a measure of how similar the *change* in $\mathbf{v}$ is to $\mathbf{v}$ itself. We can roughly visualize how a vector field which changes in the direction of itself will "diverge." On the other hand, the curl measures how different the change in $\mathbf{v}$ is from $\mathbf{v}$. A vector field which changes in an orthogonal direction from itself will tend to rotate, hence the name "curl."

Exercises:

(1) Suppose that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Consider the product $\mathbf{a} \odot \mathbf{b}$ defined,

$$\mathbf{a} \odot \mathbf{b} = a_1b_1\mathbf{i} + a_2b_2\mathbf{j} + a_3b_3\mathbf{k} \ .$$

a) Show that the product defines an associative and commutative algebra on 3-D vectors. b) Show also that the product is *not* isotropic—i.e. it cannot be expressed in terms of quantities intrinsic to the vectors themselves, their lengths and respective angle. Therefore, such a product could never play a fundamental role in physical applications.

(2) Show that if two vectors are perpendicular, their dot product vanishes while their cross product has its greatest magnitude. Likewise, show that if the two vectors are parallel, their cross product vanishes while the dot product has its greatest magnitude.

(3) Show that if $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in accordance with the right hand rule.

(4) Suppose that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Show, then, that the triple product can be computed in terms of the determinant,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} .$$

(5) Consider a cart which is constrained to roll along a track which lies along the vector $\mathbf{v}$. Suppose that we pull the cart with a force $\mathbf{F}$. What is the cart's acceleration?

(6) Show that the *reflection* of a vector $\mathbf{a}$ onto the opposite side of a vector $\mathbf{b}$ can be computed with the formula,

$$\text{refl}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{ab}}{\mathbf{b}|} .$$

VI. Rotations

The crowning glory of the Quaternionic system is the ease with which it allows us to rotate vectors in three dimensions. In this respect, Quaternions are so vastly superior to other approaches that even hard-core Gibbs' vector devotees often find themselves reverting to some form of this method whenever they have a really tough rotating problem[10].

What does it mean to rotate a vector? Imagine sitting on a turntable situated in three dimensions so that its axis is aligned with the vector $\mathbf{n}$. Now, if you keep your eye on a stationary vector $\mathbf{a}$ while the turntable is rotating, the vector will appear, according to you, to be moving around. This motion corresponds to a rotation of the vector $\mathbf{a}$ about the axis $\mathbf{n}$.

But let's be a little more specific. To rotate a vector $\mathbf{a}$, we must first choose both an axis which defines the bearing of the rotation, which we denote by a unit vector $\mathbf{n}$, and an angle of rotation, ϕ. Therefore, we can specify any rotation by giving the vector,

$$\mathbf{\Phi} = \phi\mathbf{n} . \tag{6.1}$$

Now, imagine splitting $\mathbf{a}$ into two components: the part which lies in the plane of rotation, and the part which points out. Since the plane of rotation is defined by the direction of $\mathbf{n}$, this is equivalent to breaking up $\mathbf{a}$ into the sum of $\text{aproj}_{\mathbf{n}}\mathbf{a}$ and $\text{proj}_{\mathbf{n}}\mathbf{a}$, respectively. We now rotate $\text{aproj}_{\mathbf{n}}\mathbf{a}$ within the plane of rotation by the angle ϕ, while $\text{proj}_{\mathbf{n}}\mathbf{a}$ is left alone. The two pieces are then added back together to get the rotated vector, which we denote $\text{rot}_{\mathbf{\Phi}}\mathbf{a}$.

This is what we mean precisely when we talk about rotating a vector. It's a bit easier to visualize than define in words!

[10]In standard theory, the quaternionic rotation rule appears, in disguised form, in the method of *Cayley-Klein parameters*. See Herbert Goldstein's classic text *Classical Mechanics* 2nd ed. p.148 for a discussion of this method. Goldstein's presentation is more difficult to understand than our presentation here because he is forced to get bogged down with matrix representations and matrix algebra, and the underlying geometric process is obscured. See exercise (3.1) to translate notations.

Let's now translate this procedure into mathematical notation. To make it easier, we define a special set of basis vectors:

$$\left\{ \mathbf{n}, \quad \tfrac{\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|}, \quad \tfrac{\mathbf{n}\times\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|} \right\} . \tag{6.2}$$

This basis is called an *orthonormal basis* because the three vectors are mutually orthogonal and have unit magnitude. Furthermore, since $\mathbf{n}$ is orthogonal to the plane of rotation, $\frac{\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|}$ and $\frac{\mathbf{n}\times\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|}$ must together form an orthonormal basis in the plane of rotation. Therefore, if we start with $\mathrm{aproj}_{\mathbf{n}}\mathbf{a} = |\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|\frac{\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|}$ and rotate in the plane by ϕ, we end up with,

$$|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|\cos\phi\frac{\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|} + |\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|\sin\phi\frac{\mathbf{n}\times\mathrm{aproj}_{\mathbf{n}}\mathbf{a}}{|\mathrm{aproj}_{\mathbf{n}}\mathbf{a}|} \ ,$$

as can be seen by simple trigonometry. Adding back in the normal component of $\mathbf{a}$, we find the following formula for rotations:

$$\mathrm{rot}_{\mathbf{\Phi}}\mathbf{a} = \mathrm{proj}_{\mathbf{n}}\mathbf{a} + \cos\phi(\mathrm{aproj}_{\mathbf{n}}\mathbf{a}) + \sin\phi(\mathbf{n}\times\mathrm{aproj}_{\mathbf{n}}\mathbf{a}) \ . \tag{6.3}$$

This is hardly a good looking formula, and not particularly easy to calculate with either (especially when compared with the analogous formula for rotations in the complex plane, where all we have to do is multiply by $e^{i\phi}$). But with the help of Quaternions, we can cast this equation into a particularly nifty form. Substituting equations (5.9) and (5.11), we can rewrite,

$$\begin{aligned}
\mathrm{rot}_{\mathbf{\Phi}}\mathbf{a} &= -\frac{\mathbf{n}\cdot\mathbf{a}}{\mathbf{n}} + \cos\phi\frac{\mathbf{n}\times\mathbf{a}}{\mathbf{n}|} + \sin\phi(\mathbf{n}\times(\mathrm{aproj}_{\mathbf{n}}\mathbf{a} + \mathrm{prog}_{\mathbf{n}}\mathbf{a})) \\
&= \tfrac{1}{2}\mathbf{n}^{-1}[\mathbf{na}+\mathbf{an}+\cos\phi(\mathbf{na}-\mathbf{an})] + \sin\phi\ \mathbf{n}\times\mathbf{a} \ .
\end{aligned}$$

Note that $\mathbf{n}^{-1} = -\mathbf{n}$ because $\mathbf{n}$ is a unit vector. After rearranging some terms,

$$\mathrm{rot}_{\mathbf{\Phi}}\mathbf{a} = \frac{1+\cos\phi}{2}\mathbf{a} - \frac{1-\cos\phi}{2}\mathbf{nan} + \sin\phi\ \mathbf{n}\times\mathbf{a} \ .$$

Recall the common "trig" identities:

$$\begin{aligned}
\cos^2\theta &= \frac{1+\cos 2\theta}{2} \\
\sin^2\theta &= \frac{1-\cos 2\theta}{2} \\
\sin 2\theta &= 2\cos\theta\sin\theta \ ,
\end{aligned}$$

plugging these in,

$$\begin{aligned}
\mathrm{rot}_{\mathbf{\Phi}}\mathbf{a} &= \cos^2\tfrac{1}{2}\phi\,\mathbf{a} - \sin^2\tfrac{1}{2}\phi\,\mathbf{nan} + (\mathbf{na}-\mathbf{an})\sin\tfrac{1}{2}\phi\ \cos\tfrac{1}{2}\phi \\
&= (\cos\tfrac{1}{2}\phi\ + \mathbf{n}\sin\tfrac{1}{2}\phi\,)\mathbf{a}(\cos\tfrac{1}{2}\phi\ - \mathbf{n}\sin\tfrac{1}{2}\phi\,) \ . \qquad (6.4)
\end{aligned}$$

The factorization in the last step takes a moment to see, but if the reader multiplies it out, he'll see that it works. We're almost there—all we have to do use the generalized Euler's formula (see Exercise 6.1) and we find the remarkable result:

$$\mathrm{rot}_{\mathbf{\Phi}}\mathbf{a} = e^{\frac{1}{2}\mathbf{\Phi}}\mathbf{a}e^{-\frac{1}{2}\mathbf{\Phi}} \; . \tag{6.5}$$

We find ourselves in the presence of some extremely profound mathematics. Not only is this expression extremely elegant, but its full repercussions are even still being felt at the forefront of mathematics and theoretical physics today—for example, in spin geometry, string theory, etc[11]. Not that I'd expect the reader to know a lot about these things. But, suffice it to say, this equation is extremely fundamental.

But, on a more grounded level, eq. (6.5) is much easier to apply and calculate, step by step, than the standard alternative using rotation matrices. This is why so many computer programmers prefer Quaternions when trying to animate complicated rotating graphical figures. Simply, Quaternions are the easiest way to do rotations.

Let me mention a bit of terminology. The type of rotation that we calculated in eq. (6.5) is called an *active transformation* because we, so to speak, *actively* grabbed the vector and *moved* it. This should be distinguished from a *passive transformation*, in which we don't actually move the vector at all, but we rotate our coordinate system. Thus, the vector will *appear* to move from our frame of reference, but really, it is we who are moving. The example I gave earlier of an observer watching a stationary vector "move" as he rotates on a turntable is an example of a passive transformation. The reader might wonder what the difference is—in either case, we see the vector rotate, right? True, but the direction of rotation will be *opposite* from one case to the other. This is not too difficult to see. For, if we actively rotate a vector forward by ϕ, we will obviously see it move forward, whereas if we rotate *ourselves* forward by ϕ we will see the vector move in the opposite direction because it lags behind. Therefore, if we want to fix up eq. (6.5) so that it describes passive rotations, we have to replace $\mathbf{\Phi}$ with $-\mathbf{\Phi}$. This is a constant source of confusion and sign errors, so be aware.

Let me make one more important point before we finish. Note that since $e^{\frac{1}{2}\mathbf{\Phi_1}}$ and $e^{\frac{1}{2}\mathbf{\Phi_2}}$ are quaternions, we don't generally expect them to commute[12]. But this, then, must mean that consecutive rotations can't commute either, because:

$$\begin{aligned} \mathrm{rot}_{\mathbf{\Phi_1}}(\mathrm{rot}_{\mathbf{\Phi_2}}\mathbf{a}) &= e^{\frac{1}{2}\mathbf{\Phi_1}}e^{\frac{1}{2}\mathbf{\Phi_2}}\mathbf{a}e^{-\frac{1}{2}\mathbf{\Phi_2}}e^{-\frac{1}{2}\mathbf{\Phi_1}} \\ &\neq e^{\frac{1}{2}\mathbf{\Phi_2}}e^{\frac{1}{2}\mathbf{\Phi_1}}\mathbf{a}e^{-\frac{1}{2}\mathbf{\Phi_1}}e^{-\frac{1}{2}\mathbf{\Phi_2}} = \mathrm{rot}_{\mathbf{\Phi_2}}(\mathrm{rot}_{\mathbf{\Phi_1}}\mathbf{a}) \; . \end{aligned} \tag{6.6}$$

It's not difficult to visualize that rotations, in fact, do not commute. Suppose, for example, that we take a pencil and align it so that its center lies at the origin

[11] Roughly speaking, the reason why this is so comes down to the curious factor of $\frac{1}{2}$ in the exponential, which makes it so that the rotational periodicity of the exponential is 4π, rather than 2π. This suggests a deep connection between spatial geometry and quantum mechanical *spinors*, which also have period 4π and represent the wavefunction of fermionic matter.

[12] Note that with quaternions, the power law for exponentials must fail, because the equation $e^{\mathbf{a}}e^{\mathbf{b}} = e^{\mathbf{a}+\mathbf{b}}$ is inconsistent with the exponentials not commuting. See Exercise 6.1.

while its tip points along the positive z-axis. If we rotate it 90 degrees about the x-axis, and follow that by a 90 degree rotation about the z-axis, the pencil tip will first point along negative y, and then along positive x. Now, on the other hand, if we performed the rotations in the *opposite* order, the 90 degree rotation about z would do nothing, and the 90 degree rotation about x would make the pencil tip point along negative y—which is in fact *orthogonal* to positive x! So, eq. (6.6) is no accident, rotations clearly do not commute. If we think about it, this is sort of neat because the abstract noncommutivity of Quaternions can now be seen as a direct consequence of the geometric fact that rotations don't commute in 3-D. This is about the most concrete justification for quaternionic noncommutivity I can imagine.

Exercises:

(1) Prove via Taylor expansion the generalized Euler's formula, $e^{\mathbf{n}\phi} = \cos\phi + \mathbf{n}\sin\phi$, where $\mathbf{n}$ is a unit vector. b) Prove that the power law fails in Quaternions, and in particular, $e^{\mathbf{a}}e^{\mathbf{b}} = e^{\mathbf{a}+\mathbf{b}}e^{\mathbf{a}\times\mathbf{b}}$.

(2) a) Convince yourself *geometrically* that a full rotation of ϕ about $\mathbf{n}$ can be thought of as a composition of many smaller rotations about $\mathbf{n}$. b) Use this fact to prove eq. (6.5) from eq. (6.4). You will need to make use of the limit,

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n .$$

(3) Show that any quaternion can be cast into *polar form*

$$\mathbf{Z} = |\mathbf{Z}|e^{\mathbf{n}\phi} ,$$

where $\mathbf{n}$ is a unit vector pointing in the direction of the vector part of $\mathbf{Z}$, and ϕ is the angle between the real axis and the vector part of $\mathbf{Z}$. With complex numbers, the polar form is important mostly because it allows us to multiply so easily—all we have to do is multiply magnitudes and add angles, as we saw in exercise 2.2. The polar form for a quaternion is not quite as useful, though, because of the failure of the power law. However, it can be readily seen that when the vector parts of two quaternions are parallel, the multiplication rule for complex numbers directly generalizes.

(4) Using the results of exercises 5.6 and 6.2, show that any rotation can be realized as the composition of two consecutive reflections.

(5) Show that any rotation is a linear transformation which preserves lengths and angles between vectors. This property is, in fact, sometimes used to *define* rotations.

(6) Using eq. (6.5), calculate the rotation of a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ about the z-axis. How would we describe this transformation with a matrix?

VII. Quaternions: Pro and Con

Any reader who is familiar with Gibbs' vectors will probably be impressed with the much greater algebraic fluency and notational economy afforded by Quaternions. We are only left to wonder why scientists such as Gibbs and Heaviside felt so dissatisfied with Hamilton's formalism. There are two significant objections which were frequently voiced. One is the negative square of the vector, which we have already discussed. The other is that in physical applications the dot and cross products were frequently found to be useful, whereas the full quaternion product seemed to be less important. Accordingly, Gibbs and Heaviside proposed a new system where the dot and cross products were postulated as fundamental from the beginning, but Hamilton's quaternionic product was disallowed.

There are several arguments which I could mount against the second objection. First, the dot and cross products are really just two faces of the quaternionic product, and any place we find application for the dot and cross products, we also find application for the quaternionic product. It does not make sense to say that the dot and cross products are useful, while the quaternionic product is not, because the quaternionic product is composed of the dot and cross products. Secondly, all of our useful formulas for projections, anti-projections, reflections, and rotations involve quaternionic multiplication explicitly. The corresponding formulas in Gibbs' formalism are ugly and difficult to calculate. Thirdly, even formulas which don't explicitly involve quaternionic products, such as the BAC-CAB rule for example, are often most easily derived and manipulated using quaternionic multiplication. Fourthly, the quaternionic product frequently *does* appear in physical applications. Take, for example, Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \qquad & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \qquad & \nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ . \end{aligned} \tag{7.1}$$

Since the divergence and curl appear in pairs, we can simplify by invoking quaternionic multiplication,

$$\begin{aligned} \nabla \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \rho \\ \nabla \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{J} \ . \end{aligned} \tag{7.2}$$

Not only is this notation more economical, but the fundamental contrast between electric and magnetic fields is more clearly exposed.

Another objection against Quaternions might be that while a scalar and vector each *individually* has an intuitive interpretation, a full quaternion itself is not quite as easy to visualize, except in as far as it is an aggregate of a scalar and vector. This point is well taken, but those who are bothered by it should consider the quaternion to be a necessary abstraction. If the abstraction is not allowed, then we are left with Gibbs' system, where the dot and cross products become separate and unrelated, and all computational fluency and algebraic structure is lost. It is difficult to overemphasize that without algebraic structure, there is no way to manipulate vectorial expressions without reverting to magical "identities" or primitive coordinate methods. To drive this point home, I would like to point out that a similar issue arises when we use complex arithmetic where, in the end, we always take the real and imaginary parts of our expressions to obtain physical results. Certainly, quaternions are no more abstract than complex numbers, and they have the potential to be at least as useful in applications.

So, having accepted this argument, we might reflect and wonder whether the Quaternionic algebra forms the ultimate mathematical description of pointed arrows in space. The answer, in my opinion, is unfortunately "no." The shortcomings of the Quaternionic algebra are twofold: First, Quaternions only effectively describe arrows in three dimensions. Since our universe is known to be *at least* four dimensional, this presents a problem to those of us who might be interested in relativistic physics or string theory[13]. The second problem, which is related to the first, is that quaternions don't offer any independent mathematical description for oriented planes, volumes, or any higher dimensional objects. Clearly, an oriented line segment is not the only type of extended object we could imagine in space! This shortcoming becomes particularly problematic starting in four dimensions, where there are no less than *six* independent planes, so that any attempt to represent plane-like quantities with four independent vectors is doomed from the start. Note that in three dimensions there happen to be as many planes as vectors, so this difficulty does not arise.

While these problems are unfortunate, they are unlikely to trouble anybody accept mathematicians and theoretical physicists, because in practical physical regimes, the universe is three dimensional. But for those who are interested, I should mention that a system called *Clifford Algebra* resolves these difficulties, while at the same time preserving the algebraic elegance of Hamilton's Quaternions. In fact, Quaternions are a special case of Clifford Algebra. However, I have not presented Clifford Algebra here because it is both more complex and requires some "unlearning" of the conventional Gibbs' vector approach. Hamilton's Quaternions, on the other hand, supplement and unify the Gibbs' system very nicely, and are more than adequate for anyone trying to solve practical problems with vectors in three dimensions.

Appendix: List of Key words and Concepts

Here's an alphabetical list of important words and ideas that the student should hopefully understand after reading through this paper.

[13]The attentive reader might be quick to wonder why we don't use the scalar component of a quaternion to describe the time dimension. The answer is that the squared norm of a quaternion is positive definite: $a^2 + b^2 + c^2 + d^2$ following eq. (4.2), whereas the squared norm of a spacetime 4-vector has minus signs: $a^2 - b^2 - c^2 - d^2$. These minus signs are vital to the mathematical structure of special relativity. The reader might imagine that we could fix this up by considering complexified quaternions, multiplying the vector components by i, but this basically leads to Clifford Algebra.

Algebra
Anticommutivity
Antisymmetric
BAC − CAB Rule
Basis
Basis Vector
Complex Conjugation
Complex Numbers
Cross product
Dimension
Division Algebras
Dot Product
Gibbs′ Vectors
Imaginary Part (of a complex number)
Isotropy
Jacobi Identity
Linear Combination (of vectors)
Linearly Dependent (vectors)
Linearly Independent (vectors)
Magnitude (of a quaternion)
Nilpotent
Norm (of a quaternion)
Normalized Quaternion

Orthonormal Basis
Parallelogram Rule
Passive Transformation
Polar Form
Projection
Pseudoscalar
Pseudovector
Pure Quaternion
Quaternion
Quaternionic Conjugation
Real Part (of a complex number)
Reflection
Right Hand Rule
Right Handed Coordinate System
Rotation
Scalar
Scalar Part (of a quaternion)
Symmetric
Unit Quaternion
Vector
Vector Part (of a quaternion)
Vector Space

(1)

References

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[2] M. Crowe, *A History of Vector Analysis* (Dover, 1994). A thorough account of the history of vector algebra. Discusses the development of Hamilton's work, Grassman's work, and the historical reasons why Gibbs' system was eventually adopted over both systems.

[3] H. Goldstein, *Classical Mechanics* 2nd ed., (Addison-Wesley, 1980). A good, standard source on classical mechanics that makes advanced use of Gibbs' vectorial system.

[4] B. Hoffmann, *About Vectors*, (Dover, 1975). An unconventional, humorous, but at times pedantic, presentation of vectors from the Gibbs' standpoint. The book does an excellent job at highlighting the numerous irritating shortcomings of the Gibbs' system, but Hoffmann doesn't quite grasp that Quaternions could be an elegant and unified alternative, even though he does discuss them briefly.

[5] S. Hoggar, *Mathematics for Computer Graphics*, (Cambridge, 1992). Includes discussion of the application of Quaternions and the rotation rule in computer graphics.

[6] T. Needham, *Visual Complex Analysis*, (Oxford, 1997). Simply a beautiful introduction to the calculus of complex numbers. Includes some discussion of Quaternions too.

Beyond Quaternions:

[1] D. Hestenes, *New Foundations for Classical Mechanics* 2nd ed., (Kluwer, 1999). An excellent introduction to Clifford Algebra as a unified algebraic basis for Newtonian Mechanics in three dimensions. Very pedagogical, and contains many nonstandard geometric approaches.

[2] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, (Cambridge, 1995). An advanced text, but there are no elementary texts which discuss applications of Octonions, the final and most complex division algebra.