Homework 7 Solutions

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Below are suggested solutions for the homework problems assigned. Other solutions may exist. If you have any question about them, feel free to consult Bill in his office hours.

(27.27) An electron has a lifetime of 1.0×10^{-8} s in a given energy state before it makes a transition to a lower state. What is the uncertainty in the energy of the photon emitted in this process?

Solution: The Heisenberg Uncertainty Principle states that the uncertainties in energy and over durations of time are related by the equation

$$\Delta E\,\Delta t \geq \frac{h}{2\pi}$$

So the minimum uncertainty in energy over an interval of time of 1.0×10^{-8} s would be given by

$$\Delta E(1.0 \times 10^{-8}\ \mathrm{s}) = \frac{h}{2\pi}$$

Solving for ΔE, we have

$$\Delta E = \frac{6.626 \times 10^{-34}\ \mathrm{J\cdot s}}{2\pi \times 10^{-8}} = 1.05 \times 10^{-26}\ \mathrm{J}$$

So the uncertainty in energy over that time interval is

$$\boxed{\Delta E = 1.05 \times 10^{-26}\ \mathrm{J}}$$

(27.31) Using the physical conditions present in the universe during the era of recombination ($T =$ 3000 K and $\rho_m = 10^{-18}$ kg/m^3), show by calculation that the Jeans length for the universe at that time was about 100 ly and that the total mass contained in a sphere with this diameter was about 4×10^5 M$_\odot$.

Solution: The Jeans length is given in the textbook as

$$L_J = \sqrt{\frac{\pi k_B t}{m G \rho_m}}$$

with k being the Boltzmann constant and G the gravitational constant, as usual. With the parameters given and assuming that the primary particle is the hydrogen atom, then, we have

$$L_J = \sqrt{\frac{\pi(1.38 \times 10^{-23}\ \mathrm{J/K})(3000\ \mathrm{K})}{(1.67 \times 10^{-27}\ \mathrm{kg})(6.67 \times 10^{-11}\ \mathrm{N\,m^2/kg^2})(10^{-18}\ \mathrm{kg/m^3})}} = 1.08 \times 10^{18}\ \mathrm{m}$$

So evidently the Jeans length during the era of recombination is

$$\boxed{L_J = 1.08 \times 10^{18} \text{ m} = 115 \text{ ly} \approx 100 \text{ ly}}$$

as expected. Now a sphere with this diameter has a radius of about 60 ly and thus a volume of

$$V_J = \frac{4}{3}\pi(60 \text{ ly})^3 = 523600 \text{ ly}^3 \times \left(\frac{9.4 \times 10^{15} \text{ m}}{1 \text{ ly}}\right)^3 \approx 7.57 \times 10^{53} \text{ m}^3$$

Noting that $\rho_m = M(V)/V$ where $M(V)$ is the mass inside a volume V, we can find that

$$M_J = \rho_m V_J = \left(10^{-18} \text{ kg/m}^3\right)\left(7.57 \times 10^{53} \text{ m}^3\right) = 7.57 \times 10^{35} \text{ kg} = 3.8 \times 10^5 \text{ M}_\odot$$

So we see that the mass enclosed in the sphere is about 40000 $M_\odot$.

(27.33) (a) If the Hubble constant is 73 km/s/Mpc, the critical density ρ_c is 1.0×10^{-26} kg/m^3. The average density of dark matter is known to be about 0.20 times the critical density. Suppose that massive neutrinos constitute this dark matter, and the average density of neutrinos throughout space is 100 neutrinos per cubic centimeter. (In fact, the density of neutrinos is far less than this.) Under these assumptions, what must be the mass of the neutrino? Give you answers in kilograms and as a fraction of the mass of the electron.

(b) Why do astronomers think that massive neutrinos are *not* the dominant type of dark matter in the universe?

Solution: If the density of the neutrinos/dark matter is $\rho_\nu = 0.20\rho_c$, we calculate it to be $\rho_\nu = 2 \times 10^{-27}$ kg/m^3. We are told that the number density of neutrinos is $n\nu = 100$ cm^{-3}. Recall now the relationship between number density and mass density:

$$\rho = mn$$

where m is the average mass of the particles in the volume. Here, all the particles are neutrinos, so $m = m_\nu$. Solving for this mass then gives

$$m_\nu = \frac{\rho_\nu}{n_\nu} = \frac{2 \times 10^{-27} \text{ kg/m}^3}{100 \text{ cm}^{-3}} \times \left(\frac{1 \text{ m}}{100 \text{ cm}}\right)^3 = 2 \times 10^{-35} \text{ kg}$$

If we note also that the mass of an electron is $m_e = 9.1 \times 10^{-31}$ kg, we can express the neutrino mass in terms of the electron mass:

$$m_\nu = 2 \times 10^{-35} \text{ kg} \times \frac{1 \; m_e}{9.1 \times 10^{-31} \text{ kg}} = 2.2 \times 10^{-5} \; m_e$$

So the mass of the neutrino in this theory would be

$$\boxed{m_\nu = 2 \times 10^{-35} \text{ kg} = 2.2 \times 10^{-5} \; m_e}$$

(c) Experiments show that an absolute maximum value for the mass for neutrinos is around 5×10^{-37} kg. This is far lighter than the number we just came up with, which also assumed a far denser neutrino gas in the universe than is actually observed. I reality, the calculated mass would need to be much higher, making it disagree with experiments even more. Thus, we conclude that neutrinos cannot possibly form the bulk of dark matter.