Phys 233 Homework 1 Solutions

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Problem a: (You only had to give a qualitative answer for this problem) We are given a system where $N_e$ electrons and $N_p$ protons ($N_e = N_p$) are in kinetic equilibrium at temperatures $T_1$ and $T_2$ ($>> 10^6$ K) respectively are mixed together in an adiabatic container of volume $V$. We will assume here the protons are the field particles and the test particles are the electrons with a bulk velocity $w$ (if present).

(1) If there is any bulk relative motion present (such as when the coffee cup was discussed in lecture), then the timescale to establish zero net momentum for the system will be the shortest (Equation 2.7 in Spitzer) and for heavy ions moving through lighter, faster-moving field particles (electrons) is given by

$$t_s = \frac{3m_p(2\pi)^{1/2}(kT_1)^{3/2}}{8\pi m_e^{1/2} n_e Z_p^2 e^4 \ln \Lambda} = \frac{503A_p T_1^{3/2}}{n_e Z_p^2 \ln \Lambda} = \frac{503A_p T_1^{3/2} V}{N_e \ln \Lambda}$$

where $n_e = n_p = N_e/V = N_p/V$. This timescale appears much longer than the timescale for self-collisions to establish an equilibrium. However, while self-collisions are trying to establish a kinetic equilibrium, this effort will be undone by the collisions acting to remove bulk motion. No kinetic equilibrium will be established until the net momentum of the system is zero.

(2) Once collisions have stopped the bulk motions, the electrons will rebalance themselves first via self-collisions over the self-collision timescale ($5\cdot26$ in reader):

$$t_c = \frac{m_e^{1/2}(3kT_1)^{3/2}}{8 \times 0.714\pi n_e e^4 Z_e^2 \ln \Lambda} = \frac{11.4A_e T_1^{3/2}}{n_e Z_e^2 \ln \Lambda} = 0.267 \frac{T_1^{3/2} V}{N_e \ln \Lambda}$$

(3) Protons will do the same thing, over a generally longer timescale unless the mean kinetic energies are very different:

$$t_c = \frac{11.4T_2^{3/2} V}{N_p \ln \Lambda}$$
For simplicity, we can assume the mean kinetic energies of protons and electrons are similar, in which case the proton self-collisions take $1823^{1/2} = 43$ times longer to come into kinetic equilibrium with themselves (as discussed on page 81 of the reader).

(4) Finally, the protons and electrons will reach a common temperature

$$T = \frac{N_eT_1 + N_pT_2}{N_e + N_p} = \frac{T_1 + T_2}{2}$$

over an equipartition timescale (Equation 5-31 in the reader) of

$$t_{eq} = 5.87 \text{ sec} \frac{A_eA_pV}{N_eZ_e^2Z_p^2 \ln \Lambda} \left( \frac{T_1}{A_e} + \frac{T_2}{A_p} \right)^{3/2}$$

Neglecting small numerical factors between the self-collision timescale and equipartition timescale, the equipartition timescale is roughly $A_e^{-1/2} = 43$ times longer than the proton timescale and $A_e^{-1} = 1823$ times longer than the electron self-collision timescale. If we take $T \approx T_1 \approx T_2$, we get

$$t_{eq} = (251 \text{ sec}) \left( \frac{T^{3/2}V}{N_e \ln \Lambda} \right)$$

**Problem b:** Assuming now that the temperature of the system has cooled to $100 < T_f < 1000$ K and neglecting molecular hydrogen, we wish to calculate the ratio of proton density to atomic hydrogen density as a function of $T_f$ and total mass ($\text{H} \, \text{I, H} \, \text{II, and e}^{-}$). Neutral H has two states $g_H = 2$. Ionized H (i.e. protons) has no states and therefore $g_p = 1$. From the Saha equation, we have

$$\frac{n_p n_e}{n_H} = \frac{2g_p}{g_H} \left( \frac{2\pi m_e kT_f}{\hbar^2} \right)^{3/2} e^{-\chi_I/kT_f}$$

where $\chi_I = 13.6$ eV is the ionization energy of hydrogen. Since the gas is neutral (and from Problem a), we know $n_p = n_e$. The total mass density $\rho$ is given by

$$\rho = (n_H + n_p)m_H = (n_H + n_e)m_H$$

such that

$$n_H = \frac{\rho}{m_H} - n_e$$

Plugging this into the Saha equation gives

$$\frac{n_e^2}{n_H} = \frac{n_e^2}{m_H} - n_e = \alpha(T_f) = \left( \frac{2\pi m_e kT_f}{\hbar^2} \right)^{3/2} e^{-\chi_I/kT_f}$$

This gives the quadratic equation

$$n_e^2 + \alpha n_e - \frac{\alpha \rho}{m_H} = 0$$
Using the quadratic formula gives
\[ n_e = \frac{\alpha}{2} + \frac{1}{2} \sqrt{\alpha^2 + \frac{4\alpha \rho}{m_H}} = \frac{\alpha}{2} \left( \sqrt{1 + \frac{4\rho}{\alpha m_H}} - 1 \right) \]

The ratio of the number density of protons to atomic hydrogen is therefore
\[ \frac{n_p}{n_H} = \frac{2}{\alpha} \frac{1}{\sqrt{1 + \frac{4\rho}{\alpha m_H}} - 1} \sqrt{1 + \frac{4\rho}{m_H} \left( \frac{h^2}{2\pi m_e k T_f} \right)^{3/2} e^{\lambda l/k T_f} - 1} \]

Since \( \alpha \) is typically small (and hence the second term in the square root is large), we can ignore the other terms in the denominator such that
\[ \frac{n_p}{n_H} \approx \frac{2}{\alpha} \frac{1}{\sqrt{1 + \frac{4\rho}{m_H} \left( \frac{h^2}{2\pi m_e k T_f} \right)^{3/2}}} \frac{e^{\lambda l/k T_f}}{e^{\lambda l/2k T_f}} \]

**Problem c:** (You can ignore cooling effects in this problem) If we add carbon atoms to the mix from Problem a (which is at a very hot temperature exceeding \( 10^6 \) K), the carbon atoms will become multiply ionized. For temperatures around \( 6 \times 10^6 \) K or higher, carbon will be fully ionized. The time scales that follow will be affected by what we assume for the ionization of carbon:

**Electron Density:** The ionization of the carbon atoms will free up several electrons per atom which will increase the number density \( n_e \) of electrons within the gas. Let’s assume the mixture is hot enough to fully ionize carbon and that the number density of carbon is roughly the same ratio for metals in the ISM (~ 2%). This gives the number density of the freed electrons to be about 12% of the total number of electrons present in the mixture since each carbon atom releases 6 electrons. This is enough such that the electrons will again have to establish a Maxwell-Boltzmann distribution with themselves again. This happens very quickly do to the low mass of the electrons (\( A_e = 1/1823 \)).

**Collision Time Scale:** Assuming the carbon atoms become fully ionized (\( Z_C = 6 \)) and the mass of the carbon ions are a factor of 12 larger than the protons, this implies the timescale for carbon to get rid of its bulk motions will be
\[ t_s = \frac{503 A_C T_f^{3/2}}{n_e Z_C^2 \ln \Lambda} = \frac{168 T_f^{3/2} V}{N_e \ln \Lambda} \]

so carbon ions will slow down and lose their bulk motion in a third of the time as the protons will moving through the electrons (due to the stronger Coulomb interactions).

**Self-Collision Time Scale:** We will assume that we introduce carbon such that \( n_C = 0.02 n_p \) (a reasonable concentration for the ISM assuming roughly solar metallicity), that the kinetic energy of the carbon is similar to the protons and electrons, and that carbon is fully ionized (i.e. \( Z_C = 6 \)). The self-collision timescale for carbon is then given by
\[ t_c = \frac{11.4 A_C T_f^{3/2}}{n_C Z_C^2 \ln \Lambda} = \frac{5.3 T_f^{3/2}}{n_p \ln \Lambda} \]
where we have taken $A_C = 12$. We see that the carbon will actually come into kinetic equilibrium with themselves faster than the protons will in about half the time.

**Equipartition Time Scale:** Assuming the same conditions, the equipartition timescale for carbon to come into thermal equilibrium with the electrons is

$$t_{eq} = 5.87 \text{ sec} \frac{A_e A_C V}{N_e Z_e^2 Z_C^2 \ln \Lambda} \left( \frac{T_1}{A_e} + \frac{T_C}{A_C} \right)^{3/2}$$

or

$$t_{eq} = (84.4 \text{ sec}) \frac{T^{3/2} V}{N_e \ln \Lambda}$$

(assuming electrons are the field particles). So carbon reaches equipartition with the electrons in a third of the time as that for the protons.