

Phys 233 Homework 2 Solutions

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Problem 1: (Rybicki & Lightman Problem 1.3) X-ray photons are produced in a cloud of radius R at the uniform rate Γ (photons per unit time per unit volume). The cloud is a distance d away. Neglect the absorption of these photons (optically thin medium). A detector at Earth has an angular acceptance beam of half-angle $\Delta\theta$ and it has an effective area of ΔA .

(a) Assuming that the source is completely resolved, we want to find the observed intensity (photons per unit time per unit unit area per steradian) toward the center of the cloud. Assuming it's radiating isotropically ($\Omega = 4\pi$), the emission coefficient j (photons $\text{s}^{-1} \text{cm}^{-3}$) is

$$j = \frac{\Gamma}{\Omega} = \frac{\Gamma}{4\pi}$$

The observed intensity (assuming uniform emissivity along the line of sight) is

$$I = \int j ds = j \cdot 2R = \frac{R\Gamma}{2\pi}$$

(b) Now assuming that the source is completely resolved, we want to find the observed average intensity when the source is in the beam of the detector. The luminosity of the source (photons per unit time) is

$$L = \Gamma V = \frac{4}{3}\pi R^3 \Gamma$$

The flux of the source as measured at the earth is given by

$$F = \frac{L}{4\pi d^2} = \frac{\Gamma R^3}{3d^2}$$

The angular size of the detector is just $\pi(\Delta\theta)^2$, so the observed average intensity is

$$\bar{I} = \frac{F}{\Delta\Omega} = \frac{\Gamma R^3}{3\pi d^2 (\Delta\theta)^2}$$

Since the detector will likely not have 100% throughput, it will register a smaller fraction that this based on the effective area of the detector (multiply this by

the ratio $\Delta A/[\pi(\Delta\theta)^2]$.

Problem 2: (Rybicki & Lightman Problem 1.5) A supernova remnant has an angular diameter $\theta = 4.3$ arcminutes and a flux at 100 MHz of $F_{100} = 1.6 \times 10^{-19}$ erg cm⁻² s⁻¹ Hz⁻¹ and we assume that the emission is thermal.

(a) We first want to find the brightness temperature T_b and what energy regime of the blackbody curve it corresponds to. The solid angle $\Omega = \pi(\theta/2)^2$ or $\Omega = 1.2 \times 10^{-6}$ steradians. We now find the intensity

$$I_\nu = \frac{F_{100}}{\Omega} = 1.3 \times 10^{-13} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ sr}^{-1} \text{ Hz}^{-1}$$

This frequency is in the radio part of the EM spectrum, so we will use the low frequency, long wavelength Rayleigh-Jeans approximation for a blackbody to find T_b (we will check afterwards to verify this assumption is valid). The brightness temperature is given by (R & L Eq. (1.60))

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu = 4.2 \times 10^7 \text{ K}$$

Since $h\nu = 6.6 \times 10^{-19}$ ergs while $kT_b = 5.9 \times 10^{-9}$ ergs, we see that $h\nu \ll kT_b$ and our assumption was justified.

(b) Since the emitting region is actually more compact than indicated by the observed angular diameter, we want to know what effect this has on the value of T_b . Since the brightness temperature varies linearly with intensity $T_b \propto I_\nu$ and the intensity varies inversely with the solid angle the object takes on the sky $I_\nu \propto \Omega^{-1}$, then a smaller angular size indicates a larger brightness temperature.

(c) We now want to know at what frequency this object's radiation will be a maximum if the emission is blackbody (i.e. $T = T_b$). To do this, we will use Wien's Law (R & L Eq. (1.56)):

$$\nu_{\max} = (5.88 \times 10^{10} \text{ Hz K}) T = (5.88 \times 10^{10} \text{ Hz K})(4.2 \times 10^7 \text{ K}) = 2.5 \times 10^{18} \text{ K}$$

(d) Finally, we can say the temperature of the material $T \geq T_b$ from the above results. This follows from R & L Eq. (1.62) (also given in lecture) and taking τ_ν to be small, we have that

$$T_b = T(1 - e^{-\tau_\nu})$$

We see that $T > T_b$ with $T \rightarrow T_b$ in the limit of infinite optical depth (maximum emission occurs when its optically thick).

Problem 3: (Rybicki & Lightman Problem 1.8) A certain gas emits thermally

at the rate $P(\nu)$ (power per unit volume and frequency range). A spherical cloud of this gas has radius R , temperature T , and distance d from earth ($d \gg R$).

(a) Assuming that the cloud is optically thin ($\kappa_\nu = 0$), we want to measure the brightness of the cloud as seen on earth. Let b be the distance from the center of the cloud projected onto the sky. The emission coefficient (power per unit steradian) is given by

$$j = \frac{P_\nu}{4\pi}$$

assuming the power is uniform throughout the cloud, we can find the brightness by integrating this along the line of sight

$$I(b) = \int j ds = j \int ds = j \cdot 2\sqrt{R^2 - b^2} = \frac{P_\nu \sqrt{R^2 - b^2}}{2\pi}$$

since at a distance b from the center of a spherical cloud, we are looking through a length $2\sqrt{R^2 - b^2}$.

(b) The effective temperature of the cloud from the Stefan-Boltzmann Law is

$$T_{\text{eff}} = \left(\frac{L}{4\pi\sigma R^2} \right)^{1/4}$$

where L is given by

$$L = V \int P_\nu d\nu = \frac{4}{3}\pi R^3 \int P_\nu d\nu$$

such that

$$T_{\text{eff}} = \left(\frac{R}{3\sigma} \int P_\nu d\nu \right)^{1/4}$$

(c) The flux F_ν that is measured at the earth coming from the entire cloud is

$$F_\nu = \frac{L_\nu}{4\pi d^2} = \frac{P_\nu V}{4\pi d^2} = \frac{P_\nu R^3}{3d^2}$$

(d) The cloud's temperature is related to intensity by (taking a constant source function $S_\nu = B_\nu(T)$ and $I(0) = 0$ in R & L Eq. (1.30))

$$I_\nu(\tau_\nu) = I_\nu(\tau_\nu)e^{-\tau_\nu} + \int_0^{\tau_\nu} B_\nu(T)e^{-\tau_\nu} d\tau_\nu$$

Since brightness temperature is a measured of the intensity of the cloud (i.e. $I_\nu = B_\nu(T_b)$), then

$$B_\nu(T_b) = B_\nu(T)(1 - e^{-\tau_\nu})$$

So $B_\nu(T_b) < B_\nu(T)$ which implies that $T_b < T$. If we assume that the cloud is very optically thin (i.e. $\tau_\nu \ll 1$), then we can take

$$e^{-\tau_\nu} \approx 1 - \tau_\nu$$

and therefore

$$B_\nu(T_b) = \tau_\nu B_\nu(T)$$

Since

$$B_\nu(T)\tau_\nu \ll B_\nu(T)$$

then

$$B_\nu(T_b) \ll B_\nu(T)$$

and $T_b \ll T$.

(e) We now examine each of these for the optically thick case:

(i) Since the cloud is now optically thick, its properties should be that of a blackbody, therefore the brightness of the cloud should have no dependence on b as it is radiating uniformly from every point on its "surface" with an intensity given by $I_\nu = S_\nu = B_\nu(T)$.

(ii) Since the cloud is now a blackbody, its effective temperature should be the same as its temperature (i.e. $T_{\text{eff}} = T = T_b$)

(iii) The flux F_ν that is measured at the "surface" of the entire cloud is simply

$$F_{\nu,\text{surf}} = \pi B_\nu(T)$$

From the inverse square law, the ratio of the flux at the earth F_ν to that at the cloud's surface is

$$\frac{F_\nu}{F_{\nu,\text{surf}}} = \frac{R^2}{d^2}$$

We therefore have

$$F_\nu = \frac{\pi R^2 B_\nu(T)}{d^2}$$

(iv) The measured brightness temperatures are now the same as the cloud's temperature (as we stated in (ii), $T = T_b$).

Problem 4: We are asked to consider an opaque ISM cloud where the source function varies linearly with optical depth. We want to show that the emergent intensity is characteristic of the source function at the last scattering surface. Beginning from the transfer equation (R & L Eq. 1.91), we have

$$\frac{dI_\nu}{ds} = -(\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

where κ_ν is the absorption coefficient and σ_ν is the scattering coefficient. The effective optical depth is therefore given by

$$d\tau_\nu = (\kappa_\nu + \sigma_\nu) ds$$

such that we get back the transfer equation in the form

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

If we assume the source function varies linearly with optical depth, we have that

$$S_\nu = S_1\tau_\nu + S_0$$

for some constant of proportionality k . We therefore get the following differential equation

$$I' + I = S_\nu = S_1\tau_\nu + S_0$$

with a solution given by

$$I_{\nu\infty}(\tau_{\nu\infty}) = I_\nu(\infty) e^{-\tau_{\nu\infty}} + \int_0^{\tau_{\nu\infty}} e^{-\tau_\nu} S_\nu(\tau_\nu) d\tau_\nu$$

which for $\tau_\nu \gg 1$ gives

$$I_{\nu\infty}(\tau_{\nu\infty}) = \int_0^{\tau_{\nu\infty}} e^{-\tau_\nu} (S_1\tau_\nu + S_0) d\tau_\nu$$

Integrating by parts gives

$$I_{\nu\infty}(\tau_{\nu\infty}) = -(S_1\tau_\nu + S_0)e^{-\tau_\nu} \Big|_0^{\tau_{\nu\infty}} - S_1 e^{-\tau_\nu} \Big|_0^{\tau_{\nu\infty}}$$

or for $\tau_{\nu\infty} \gg 1$,

$$I_{\nu\infty}(\tau_{\nu\infty}) \approx S_0 + S_1$$

Since $\tau_\nu \approx 1$ at the last scattering surface where $S_\nu(\tau_\nu = 1) = S_0 + S_1$, the emergent intensity is about the same as the source function at the last scattering surface.