# Phys 233 Homework 3 Solutions

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**Problem 1:** We suppose a cloud of pure hydrogen is between us and the source described in 4) of homework 2. We also suppose that the source is energetic enough to photoionize hydrogen. Assuming the H cloud is in (ETE) and uniform in temperature T and assuming self-absorption is negligible, we want to compute the intensity ratio Lyman $\alpha$/Lyman $\beta$ as a function of the temperature $T$. Find numerical values in local thermodynamic equilibrium (LTE) for $kT =$ Ry/10 = 1.36 eV (Ry is the Rydberg constant 13.6 eV), then we will discuss the dependency on T.

From Spitzer Eq. (3-36),

$$\int I_\nu \, d\nu = h\nu_{jk}\alpha_{jk}\frac{n_p}{n_e} \cdot 2.46 \cdot 10^{17} E_m$$

where $\frac{n_p}{n_e} = 1$ (assume H only) and

$$E_m = \int n_e^2 \, ds$$

Using Spitzer Eq. (3-36) for both the Ly $\alpha$ and Ly $\beta$ transitions and dividing, most of these factors cancel leaving only

$$\frac{I(Ly\ \alpha)}{I(Ly\ \beta)} = \frac{\nu_{12}\alpha_{12}}{\nu_{13}\alpha_{13}}$$

The energy of the photons released from hydrogen atom transitions is given by

$$E = h\nu = (1\text{ Ry})\left(\frac{1}{n_f} - \frac{1}{n_i}\right)$$

so for Ly $\alpha$ and Ly $\beta$, we have

$$h\nu_{12} = Ry(1 - 1/4) = \frac{3}{4}Ry$$

$$h\nu_{13} = Ry(1 - 1/9) = \frac{8}{9}Ry$$

Since from Spitzer Eq. (3-34), we know

$$\alpha_{mn} = b_m g_m A_{mn} e^{(1\ \mathrm{Ry})/m^2 kT} f_e^{-1}$$

we therefore have

$$\frac{\alpha_{12}}{\alpha_{13}} = \frac{b_2 g_2 A_{21}}{b_3 g_3 A_{31}} e^{\frac{Ry}{kT}(\frac{1}{4}-\frac{1}{9})}$$

From Spitzer Eq. (3-13), we have

$$A_{kj} = \frac{B_{jk} g_j}{g_k} \frac{8\pi h \nu_{jk}^3}{c^3}$$

From Spitzer Eqs. (3-23) and (3-24), the coefficients $B_{jk}$ are related to the oscillator strengths via

$$B_{jk} = \frac{\pi e^2}{m_e h \nu_{jk}} f_{jk}$$

We therefore have

$$A_{21} = \frac{\pi e^2}{m_e h \nu_{12}} f_{12} \frac{g_1}{g_2} \frac{8\pi h \nu_{12}^3}{c^3}$$

$$A_{31} = \frac{\pi e^2}{m_e h \nu_{13}} f_{13} \frac{g_1}{g_3} \frac{8\pi h \nu_{13}^3}{c^3}$$

and the ratio of the Einstein coefficients is given by

$$\frac{A_{21}}{A_{31}} = \frac{f_{12}\nu_{12}^2}{f_{13}\nu_{13}^2} \frac{g_1}{g_2} \frac{g_3}{g_1} = \frac{f_{12}\nu_{12}^2}{f_{13}\nu_{13}^2} \frac{g_3}{g_2}$$

The ratio $R = I(\mathrm{Ly}\ \alpha)/I(\mathrm{Ly}\ \beta)$ is then

$$R = \left(\frac{\nu_{12}^3}{\nu_{13}^3}\right)\left(\frac{b_2}{b_3}\right)\left(\frac{f_{12}}{f_{13}}\right) e^{(\frac{5}{36}\ \mathrm{Ry})/kT} = \left(\frac{3}{4}\right)^3 \left(\frac{8}{9}\right)^3 \left(\frac{b_2}{b_3}\right)\left(\frac{f_{12}}{f_{13}}\right) e^{(\frac{5}{36}\ \mathrm{Ry})/kT}$$

The oscillator strength was given in lecture (Rybicki & Lightman Eq. (10.46)) for the Lyman series:

$$g_1 f_{1n} = \frac{2^9 n^5 (n-1)^{2n-4}}{3(n+1)^{2n+4}}$$

so for these transitions we have

$$g_1 f_{12} = \frac{2^9 \cdot 2^5}{3 \cdot 3^8} = \frac{2^{14}}{3^9}$$

$$g_1 f_{13} = \frac{2^9 3^5 2^2}{3 \cdot 4^{10}} = \frac{3^4}{2^9}$$

The ratio of oscillator strengths is therefore

$$\frac{f_{12}}{f_{13}} = \frac{2^{23}}{3^{13}}$$

Plugging this in to our relation for the ratio of intensities gives

$$R = \frac{2^{26}}{3^{16}} \left(\frac{b_2}{b_3}\right) e^{\left(\frac{5}{36}\ \mathrm{Ry}\right)/kT}$$

We solve this relation to find the kinetic temperature $T$. There is temperature dependence hidden within the $b$ factors. At large temperatures, the exponential approaches unity, so much of the temperature dependence is due to these $b$ factors at high temperatures:

$$R \approx 1.56 \left(\frac{b_2}{b_3}\right)$$

in the high temperature limit.

**Problem 2:** (Rybicki & Lightman Problem 1.10) We are asked to consider a semi-infinite half-space in which both the scattering ($\sigma$) and absorption and emission ($\alpha_\nu$) occur. We can assume the medium is homogeneous and isothermal (such that the coefficients do not vary with depth) and that the scattering is isotropic.

(a) We first want to apply the radiative diffusion equation with two-stream boundary conditions (R & L pp. 42-45) to find the mean intensity $J_\nu(\tau)$ and the emergent flux $F_\nu(0)$. The probability that a free path will end in a true absorption event is given by

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$$

The radiative diffusion equation (which in this form makes use of this parameter) is from R & L Eq. (1.119)

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B)$$

From the assumption that the coefficients do not vary with depth, we see that $\epsilon_\nu$ is also constant with depth and for the effective optical depth (R & L Eq. (1.120))

$$\tau_* \equiv \sqrt{3\epsilon}\tau = \sqrt{3\tau_a(\tau_a + \tau_s)}$$

The transfer equation becomes (R & L Eq. (1.121))

$$\frac{\partial^2 J_\nu}{\partial \tau_*^2} = J_\nu - B_\nu$$

This is a second-order partial differential equation with the solution

$$J_\nu(\tau_*) = C_1 e^{-\tau_*} + C_2 e^{\tau_*} + B_\nu$$

We see that the second term blows up as the effective optical depth $\tau_* \to 0$, so $C_2 = 0$ from boundary conditions. This gives us

$$J_\nu(\tau_*) = C_1 e^{-\tau_*} + B_\nu$$

For the two stream approximation, we assume the entire radiation field can be represented by radiation traveling at just two angles ($\mu = \pm 1/\sqrt{3}$) and has a boundary condition given by (R & L Eq. (1.124))

$$\frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau}(\tau = 0) = J(\tau = 0)$$

or since $\tau = \frac{1}{\sqrt{3\epsilon}}\tau_*$, this boundary condition becomes

$$\sqrt{\epsilon}\frac{\partial J}{\partial \tau_*}(\tau_* = 0) = J(\tau_* = 0)$$

Applying this to our previous relation gives

$$C_1 = \frac{-B_\nu}{1+\sqrt{\epsilon}}$$

Plugging in gives

$$J_\nu(\tau_*) = \frac{-B_\nu e^{-\tau_*}}{1+\sqrt{\epsilon}} + B_\nu = B_\nu\left(1 - \frac{e^{-\tau_*}}{1+\sqrt{\epsilon}}\right)$$

In the Eddington approximation, $H$ is proportional to flux while $K$ is proportional to radiation pressure. Using R & L Eqs. (1.113b) and (1.118), we have that the flux is

$$F_\nu(\tau) = 4\pi H = 2\pi\int_{-1}^{+1}\mu I\, d\mu = \frac{4\pi}{3}\frac{\partial J}{\partial \tau} = 4\pi\sqrt{\frac{\epsilon}{3}}\frac{\partial J}{\partial \tau_*}$$

or

$$F_\nu(\tau) = 4\pi\sqrt{\frac{\epsilon}{3}}\left(\frac{e^{-\tau_*}}{1+\sqrt{\epsilon}}\right)B_\nu$$

and the emergent flux is

$$F_\nu(0) = \frac{4\pi\sqrt{\epsilon}B_\nu}{\sqrt{3}(1+\sqrt{\epsilon})}$$

(b) We now wish to show that $J_\nu(\tau) \to B_\nu$ at an effective optical depth $\tau_* = \sqrt{3\tau_a(\tau_a+\tau_s)}$. We see from before that

$$J_\nu(\tau_*) = C_1 e^{-\tau_*} + B_\nu$$

Taking that $\tau >> 1$ (as we did in the last homework), we see that $J_\nu(\tau_*) \to B_\nu$ in the optically thick limit ($\tau_* \to \infty$).

**Problem 3:** We now consider a neutral hydrogen cloud at a kinetic temperature $T$ and column density $N(HI)$. We wish to compute the specific intensity of the radiation coming out of the cloud at wavelengths around 21 cm. What physical properties of the system can we infer if we measure the brightness temperature?

Beginning with Spitzer Section 3.3b, we see that this corresponds to a frequency $\nu_{jk} = 1420.406$ MHz and transition probability $A_{kj} = 2.869 \times 10^{-15}$ s$^{-1}$. From Spitzer Eq. (3-20), the line profile is

$$\phi_a(\Delta\nu) = \frac{c}{\nu_{jk}} P(w) = \frac{c}{\nu_{jk}} \frac{e^{-(w/b)^2}}{\pi^{1/2} b}$$

where $w = c(\Delta\nu/\nu_{jk})$ and where for neutral hydrogen $(A = 1)$

$$b = \left(\frac{2kT}{m_H}\right)^{1/2} = 1.290 \times 10^4 T^{1/2}$$

such that

$$\phi_a(\Delta\nu) = 9.24 \times 10^{-4} T^{-1/2}\, e^{-6.009 \times 10^{-9} T^{-1}}$$

Fitting this relation to the line profile will enable us to find the kinetic temperature $T$.

We now find a relation for optical depth. The optical depth is given by

$$\tau_{\nu r} = \int \kappa_\nu \, ds$$

Expressing the absorption coefficient $\kappa_\nu$ in terms of number density $n$ (Spitzer Eq. 3-16), we have

$$\kappa_\nu = n s_\nu = n s \phi(\Delta\nu)$$

where $s$ for $h\nu << kT$ and $b_k/b_j = 1$ is given by (Spitzer Eq. 3-29)

$$s = s_u \left(1 - \frac{b_k}{b_j} e^{-h\nu/kT}\right) = s_u \frac{h\nu}{kT}$$

The oscillator strength $f_{jk}$ is related to $s_u$ via (Spitzer Eq. 3-25)

$$s_u = \frac{\pi e^2}{m_e c} f_{jk} = 2.654 \times 10^{-2} f_{jk}$$

We can put this in terms of the Einstein coefficient since we know from Problem 1 that

$$A_{kj} = \frac{\pi e^2}{m_e h \nu_{jk}} \frac{g_j}{g_k} \frac{8\pi h \nu_{jk}^3}{c^3} f_{jk} = \frac{8\pi \nu_{jk}^2}{c^2} \frac{g_j}{g_k} s_u$$

Therefore, $s$ can be written

$$s = \frac{h\nu}{kT} \frac{c^2}{8\pi \nu_{jk}^2} \frac{g_k}{g_j} A_{kj}$$

The optical depth can therefore be written as

$$\tau_{\nu r} = \left( \frac{h\nu}{k} \frac{c^2}{8\pi\nu_{jk}^2} \frac{g_k}{g_j} A_{kj} \right) N(HI) \frac{c}{\nu_{jk}} \frac{e^{-(w/b)^2}}{\pi^{1/2} bT}$$

where we know $A_{kj}$ and $\nu_{jk}$ (stated above), and $g_k/g_j = 3.$ or equivalently from Spitzer Eq. (3-37),

$$\tau_{\nu r} = 5.49 \times 10^{-14} \frac{N(HI)P(w)}{T}$$

We can therefore calculate the brightness temperature using Spitzer Eq. (3-8) (and assuming no incident radiation on the far side of the cloud such that $T_{b0} = 0$), then

$$T_b = T(1 - e^{-\tau_{\nu r}})$$

Since we are in the Rayleigh-Jeans limit (radio region of spectrum), we can calculate the specific intensity using R & L Eq. (1.53):

$$I_\nu = \frac{2k\nu^2}{c^2} T_b = \frac{2kT\nu^2}{c^2} (1 - e^{-\tau_{\nu r}})$$

If we measure the brightness temperature $T_b$ (i.e. we can find using the Rayleigh-Jeans approximation (R & L Eq. (1.53),

$$T_b = \frac{c^2}{2k\nu^2} I_\nu$$

then we can find the optical depth $\tau_{\nu r}$ from Spitzer Eq. 3-8 if we know the kinetic temperature from fitting the line profile (in general we cannot assume isotropy) since

$$\tau_{\nu r} = \ln \frac{T}{T - T_b}$$

We found previously there is a relation between optical depth and column density. From above, we have

$$\tau_{\nu r} T = \frac{\tau_{\nu r} T_b}{1 - e^{-\tau_{\nu r}}} = \left( \frac{h\nu}{k} \frac{c^2}{8\pi\nu_{jk}^2} \frac{g_k}{g_j} A_{kj} \frac{c}{\nu_{jk}} \right) N(HI) \frac{e^{-(w/b)^2}}{\pi^{1/2} b}$$

Integrating both sides over velocity space, we have

$$\int T_b(w) \left[ \frac{\tau_{\nu r}}{1 - e^{-\tau_{\nu r}}} \right] dw = \left( \frac{h\nu}{k} \frac{c^2}{8\pi\nu_{jk}^2} \frac{g_k}{g_j} A_{kj} \frac{c}{\nu_{jk}} \right) N(HI) \int \frac{e^{-(w/b)^2}}{\pi^{1/2} b} dw$$

The right side integral evaluates to $1/2$ if we take the lower limit on $w$ to be zero instead of $-\infty$. We therefore get for the column density $N(HI)$ using this optical depth and the brightness temperature is

$$N(HI) = \left( \frac{2}{A_{kj}} \frac{8\pi\nu_{jk}^2}{c^2} \frac{k}{h\nu} \frac{g_j}{g_k} \frac{\nu_{jk}}{c} \right) \int T_b(w) \left[ \frac{\tau_{\nu r}}{1 - e^{-\tau_{\nu r}}} \right] dw$$

or equivalently from Spitzer Eq. (3-38),

$$N(HI) = 1.823 \times 10^{13} \int T_b(w) \left[ \frac{\tau_{\nu r}}{1 - e^{-\tau_{\nu r}}} \right] dw$$