Generalized Coordinates

• “State” of system of \( N \) particles (Newtonian view):
  - PE, KE, Momentum, L → calculated from \( m_i, r_i, \dot{r}_i \)
  - Subscript \( i \) covers: 1) particles – \( N \)  2) dimensions – 2, 3, etc.
    \[
    PE \equiv U(r_i) = U(x_1, y_1, z_1, x_2, y_2, z_2, ...)
    \]
    \[
    KE \equiv T(\dot{r}_i) = T(\dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{x}_2, \dot{y}_2, \dot{z}_2, ...)
    \]

• Calculating \( U(r_i) \) and \( T(\dot{r}_i) \) can often be simplified
  - Using a different set of “coordinates” \((q_n)\) for the system
  - Examples: Pendulum and “Double Pendulum”
  - KE and PE easily expressed in terms of \( \theta_1 \) (and \( \theta_2 \))
  - Example: marble sliding in hemispherical bowl
  - What are some possible generalized coordinates?
  - Express \( U \) and \( T \) in terms of \( q_n \) and \( \dot{q}_n \)
Newton's 2\textsuperscript{nd} Law – Another View

- Derive $\mathbf{F}_{\text{net}} = \mathbf{ma}$ from energy conservation ($\frac{dE}{dt} = 0$)
  - Can get EOM from knowing only $U$ and $T$ (don't need forces!)

\begin{align*}
\text{Cartesian Coordinates} \\
U (r_i) & \quad T (\dot{r}_i) \\
\text{Chain Rule:} & \\
\frac{dE}{dt} &= \sum_i \left[ \frac{\partial U}{\partial r_i} \dot{r}_i + \frac{\partial T}{\partial \dot{r}_i} \ddot{r}_i \right] = 0 \\
T &= \frac{1}{2} \sum_i m_i \dot{r}_i^2 \rightarrow \frac{\partial T}{\partial \dot{r}_i} = m_i \ddot{r}_i \\
\frac{dE}{dt} &= \sum_i \left[ \frac{\partial U}{\partial r_i} + m_i \dot{r}_i \right] \dot{r}_i = 0 \\
\frac{dE}{dt} &= \sum_i \left[ \frac{\partial U}{\partial r_i} + \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{r}_i} \right) \right] \dot{r}_i = 0 \\
\text{Newton's 2\textsuperscript{nd} Law → EOM}
\end{align*}

\begin{align*}
\text{Generalized Coordinates} \\
U (q_n) & \quad T (q_n, \dot{q}_n) \\
\text{Chain Rule:} & \\
\frac{dE}{dt} &= \sum_n \left[ \frac{\partial U}{\partial q_n} \dot{q}_n + \frac{\partial T}{\partial q_n} \dot{q}_n + \frac{\partial T}{\partial \dot{q}_n} \ddot{q}_n \right] = 0 \\
T &= \sum_i \frac{1}{2} m_i \dot{r}_i^2 \rightarrow \frac{\partial T}{\partial q_n} = ? \ ? \\
\frac{dE}{dt} &= \sum_n \left[ \text{a bunch of derivatives} \right] \dot{q}_n = 0 \\
\text{“Generalized” Newton's 2\textsuperscript{nd} Law → EOM}
\end{align*}
Example: Mass-Spring System

- Spring has equilibrium length \( L \)
  - Cartesian: \((x_1, x_2)\)  Generalized: \((r, R)\)
  - Find “transformation equations” between coordinate systems
  - In each coordinate system:
    - Express \( U \) and \( T \) in terms of coordinates
    - Calculate the equations of motion \( \rightarrow \) interpret the results
Constraints

- Systems often have “forces of constraint”
  - e.g. normal force on marble in bowl
  - Mathematically described by “constraint equations”

- Cartesian constraints → often cumbersome
  - Example: Calculate $T(x, y, z, \dot{x}, \dot{y}, \dot{z})$ for marble in bowl

- Generalized coordinates → can “include” constraints
  - Keeping $T$ and $U$ in simpler forms

- System with $N$ particles and $M$ constraint equations
  - Would require $3N - M$ generalized coordinates
Constraint Example: Pendulum

Cartesian
\[ r_i = (x, y) \]

Constraint:
\[ x^2 + y^2 = L^2 \]

\[ U = mgy \]
\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]

Generalized
\[ q_n = (\theta) \]

(Constraint already included)

“transformation equations”

\[ x = L \sin \theta \]
\[ y = -L \cos \theta \]

\[ U = -mgL \cos \theta \]
\[ T = \frac{1}{2} m L^2 \dot{\theta}^2 \]

- In each coordinate system:
  - Calculate the EOM → interpret the results

- How many generalized coordinates if pendulum moves in 3-D?
Coordinate Transformations

- **Goal**: use chain rule to plug into $\frac{dE}{dt} = 0$ and construct a generalized way to get equations of motion

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2$$

Now must use chain rule some more to evaluate these derivatives

$$\frac{\partial T}{\partial q_n} = \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_n}$$

$$\frac{\partial T}{\partial \dot{q}_n} = \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_n}$$

- From the transformation equations:

<table>
<thead>
<tr>
<th>$r_i (q_1, q_2, \ldots)$</th>
<th>$\dot{r}_i (q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots) = \sum_n \left( \frac{\partial r_i}{\partial q_n} \right) \dot{q}_n$ (Chain rule)</th>
</tr>
</thead>
</table>

- Viewing $\dot{r}_i$ as a sum of terms, can take derivative $\rightarrow \frac{\partial \dot{r}_i}{\partial \dot{q}_n} = \frac{\partial r_i}{\partial q_n}$

- Also:

$$\frac{\partial \dot{r}_i}{\partial q_n} = \frac{\partial}{\partial q_n} \left( \sum_p \left( \frac{\partial r_i}{\partial q_p} \right) \dot{q}_p \right) = \sum_p \frac{\partial}{\partial q_p} \left( \frac{\partial r_i}{\partial q_n} \right) \dot{q}_p = \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_n} \right)$$

"Cancellation of dots"
Generalized Newton's 2\textsuperscript{nd} Law

- Plugging in:
  \[
  \frac{\partial T}{\partial q_n} = \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_n} \quad \text{(chain rule)}
  \]

  \[
  \frac{\partial T}{\partial q_n} = \sum_i m_i \dot{r}_i \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_n} \right) \quad \text{(from transformation equations)}
  \]

  \[
  \frac{\partial T}{\partial q_n} = \frac{d}{dt} \left( \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_n} \right) - \sum_i m_i \ddot{r}_i \left( \frac{\partial r_i}{\partial q_n} \right) \quad \text{(product rule)}
  \]

  \[
  \frac{\partial T}{\partial q_n} = \frac{d}{dt} \left( \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_n} \right) + \sum_i \frac{\partial U}{\partial r_i} \left( \frac{\partial r_i}{\partial q_n} \right) \quad \text{(from transformation equations)}
  \]

  \[
  \frac{\partial T}{\partial q_n} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_n} \right) + \frac{\partial U}{\partial q_n} \quad \text{(chain rule)}
  \]

This differential equation is a “factory” for equations of motion

Once \( T \) and \( U \) are expressed in generalized coordinates → just plug in
The Lagrangian Function

- Conservative forces → **U** is a function of \( q_n \) only \( \left( \frac{\partial U}{\partial \dot{q}_n} = 0 \right) \)
  - "Generalized Newton's 2\(^{nd}\) Law" can be re-written as:

\[
\frac{\partial (T - U)}{\partial q_n} - \frac{d}{dt} \left( \frac{\partial (T - U)}{\partial \dot{q}_n} \right) = 0
\]

\[
L(q_n, \dot{q}_n) \equiv T - U
\]

"Lagrangian"  

\[
\frac{\partial L}{\partial q_n} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = 0
\]

“Euler-Lagrange equations of motion” (one for each n)

- Lagrangian named after Joseph Lagrange (1700's)
  - Fundamental quantity in the field of Lagrangian Mechanics
  - Example: Show that this holds for Cartesian coordinates
Examples

• Mass-spring on table-top (top view)
  - Spring has equilibrium length $r_0$
  - Calculate EOM in polar coordinates
  - Is circular motion possible? Is it stable?
  - Find frequency of small oscillations in $r$

• Double Pendulum
  - Calculate the EOM for $\theta_1$ and $\theta_2$
  - Approximate EOM's for small $\theta_1$ and $\theta_2$
  - Does motion have \textit{consistent} frequencies?
Symmetry and Conservation Laws

- Euler-Lagrange equations of motion: \[ \frac{\partial L}{\partial q_n} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = 0 \]

  - Notice that if \( \frac{\partial L}{\partial q_n} = 0 \) \( \rightarrow \) \( \frac{\partial L}{\partial \dot{q}_n} \) is a conserved quantity

\( \frac{\partial L}{\partial \dot{q}_n} \) is called the generalized momentum in the \( q_n \) “direction”

- Common Examples:
  - Conservation of linear momentum: \( \frac{\partial L}{\partial \dot{x}_{CM}} = m \dot{x}_{CM} = \text{Constant} \)
  - Conservation of angular momentum: \( \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{Constant} \)
  - Conservation of energy: \( \frac{\partial L}{\partial t} = 0 \) (assumed previously)
Lagrangian/Hamiltonian “Revolution”

• Dynamics of a physical system
  – Can be described by energy functions $T$ and $U$ in state space
  – Mathematically → system need not be divisible into “particles”

• This opens possibilities for new “models” of matter
  – Matter distributions $\rho(q_n)$ with equations of motion
  – i.e. “generalized Newton's 2\textsuperscript{nd} Laws”
  – Idea eventually led to the development of Quantum Mechanics

• Generalized coordinates: good for describing “fields”
  – Value of field (at one point) → generalized coordinate(s)
  – Scalar/vector field $f(x,y,z)$ → “state vector” in state space
  – Points in physical space $(x,y,z) → “unit vectors” in state space