ELECTRIC CHARGE AND FIELD

Mechanics (as learned in Phys 20-22) must be incomplete—how do we know?

1) In our formulation of mechanics, agreement with observation required:

- "Contact forces"

- "Internal forces"

(Fibonacci)

\( \vec{F}_{\text{table on } M} \)

(Often broken into)

\( \vec{F}_{\text{normal}} \) and

\( \vec{F}_{\text{friction}} \)

\( \vec{F}_{\text{M on table}} \)

(Usually called tension, compression, or shear)

\( \vec{F}_{\text{top on bottom}} \)

\( \vec{F}_{\text{bottom on top}} \)
MECHANICS NEEDS THESE FORCES TO MAKE $F = ma$ WORK, BUT DOESN'T EXPLAIN THEIR ORIGINS

2) "ACTION-AT-A-DISTANCE"

- FORCES (e.g., GRAVITY) ARE OBSERVED TO OCCUR BETWEEN OBJECTS WHICH ARE NOT IN CONTACT

HOW IS THIS POSSIBLE? THE PRESENCE OF MASS MUST SOMEHOW AFFECT THE SURROUNDING SPACE TO PRODUCE A FORCE

THIS IS THE PHYSICS CONCEPT OF A "FIELD":

SOURCE MASS → GRAV. FIELD → "TEST" MASS

OTHER EXAMPLES: ANCIENT GREEKS KNEW THAT AMBER, WHEN RUBBED WITH FUR, ATTRACTS STRAW

- NOT ENOUGH MASS TO EXPLAIN THIS FORCE, MUST BE SOME OTHER SOURCE OF FIELD, CALL IT "ELECTRIC CHARGE"
Source charge creates electric field affects "test" charge

More sophisticated experiments revealed:

1) Electric force can be attractive or repulsive (must be two types of charge)

\[ \vec{F}_{12} = k \frac{q_1 q_2}{r_{12}^2} \]

2) \[ \vec{F}_{12} = k \frac{q_1 q_2}{r_{12}^2} \]

(Coulomb's Law)

WHERE

\( q_1, q_2 \) are charges (measured in "Coulombs": C)

\( \vec{r}_{12} = \text{vector from } q_1 \rightarrow q_2 \)

\( k \approx 9 \times 10^9 \text{ N m}^2/\text{C}^2 = \frac{1}{4\pi \varepsilon_0} \)

Note:

<table>
<thead>
<tr>
<th>+</th>
<th>+</th>
<th>Repulsive</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td>Attractive</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
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</tr>
<tr>
<td>-</td>
<td>-</td>
<td>Repulsive</td>
</tr>
</tbody>
</table>

"Inverse-square law" similar to grav. field

In order to define \( \vec{r}_{12} \), \( q_1 + q_2 \) must be "point charges"
CHARGE DISTRIBUTIONS

- Just like mass, charge can be "spread out" in space (i.e., not a point charge): 

\[ q_1 \quad q_2 \]

- We can still use Coulomb's Law here only if we know how the charge is "distributed" throughout space.

EX: Uniform Line of Charge

Find the force on \( q_{bb} \) using calculus, we can treat \( q_A \) as a bunch of point charges in a row:

\[
\sum_{i=1}^{n} \int d\mathbf{F}_i = \int d\mathbf{F}_{tot}
\]

\[ d\mathbf{F}_i = \frac{k q_{bb} \cdot q_i}{r_{bb,i}^2} \hat{r}_{bb,i} \]

\[ d\mathbf{F}_{tot} = \int d\mathbf{F} \]
NOTE: THIS REALLY MEANS

$$F_{\text{tot},x} = \int dF_x \quad F_{\text{tot},y} = \int dF_y \quad F_{\text{tot},z} = \int dF_z$$

so

$$F_{\text{tot},x} = dF_{1B,x} + dF_{2B,x} + \ldots$$

Coulomb's Law:

$$d\vec{F}_{1B} = k \frac{(dq_1)q_B}{|\vec{r}_{1B}|^3} \vec{r}_{1B}$$

so

$$dF_{1B,x} = k \frac{(dq_1)q_B}{|\vec{r}_{1B}|^3} r_{1B,x}$$

--- HOW TO FIND \( dF_{1B,y} \) OR \( dF_{2B,x} \) OR \( dF_{2B,y} \)?
--- JUST CHANGE APPROPRIATE SUBSCRIPTS

HOW TO FIND \( dq_1 \)? WELL \( \frac{dy}{L} = \frac{dq_1}{QA} \) (BECAUSE IT'S UNIFORM)

\( dq_1 = \left(\frac{QA}{L}\right)dy \quad \left(\frac{QA}{L} \equiv \lambda \right) = "\text{LINEAR CHARGE DENSITY}"\)

So

$$dF_{1B,x} = k \frac{q_B}{L} \left(\frac{QA}{L}\right)dy \quad \frac{|\vec{r}_{1B}|}{|\vec{r}_{1B}|^3} r_{1B,x} \quad \text{(FROM HERE)}$$

(PLUG IN AND INTEGRATE)
CHARGE DENSITY

- MANY POINT CHARGES CAN BE ARRANGED TO FORM:

1) A LINE OR CURVE \( \rightarrow d\mathbf{q} = \lambda(s) \, ds \)

2) A 2-D SURFACE \( \rightarrow d\mathbf{q} = \sigma(u,v) \, d\text{Area} \)

3) A 3-D OBJECT \( \rightarrow d\mathbf{q} = \rho(x,y,z) \, d\text{Vol} \)

\( \lambda, \sigma, \rho \) ARE "CHARGE DENSITIES" (NOT NECESSARILY UNIFORM)

DIRAC \( \delta \)-FUNCTION

- IMAGINE A POINT CHARGE \( q \) AT \( x = x_0 \)

- TRY TO WRITE \( \lambda(x) \) FOR THIS CHARGE:

\[
\lambda(x) = \begin{cases} 
\infty & \text{at } x = x_0 \\
0 & \text{everywhere else}
\end{cases}
\]

ALSO MUST HAVE:

\[
\int_{-\infty}^{\infty} \lambda(x) \, dx = q
\]
- This is similar to a "Dirac $\delta$-function":

$$\delta(x) = \begin{cases} \infty & \text{at } x = 0 \\ 0 & \text{everywhere else} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

- "Bridges the gap" between point charges and distributions.

<table>
<thead>
<tr>
<th>1-D</th>
<th>2-D</th>
<th>3-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(x) = q_0 \delta(x-x_0)$</td>
<td>$\sigma(x,y) = q_0 \delta(x-x_0) \delta(y-y_0)$</td>
<td>$\rho(x,y,z) = q_0 \delta(x-x_0) \delta(y-y_0) \delta(z-z_0)$</td>
</tr>
</tbody>
</table>

**Spherical Charge Distributions**

**Shell Theorem**

1. $\vec{F}_{q_1} = 0 \quad \text{(anywhere inside sphere)}$
2. $\vec{F}_{q_2}$ can be found by pretending $q$ is a point charge at sphere's center.

- **Q** source is uniform
  - i.e. $\sigma(\theta, \phi) = \text{constant} = \frac{Q}{4\pi R^2}$

- **You will prove this for homework!**

- **The moral**: "Spherically symmetric" charge distributions (i.e. $\rho(x,y,z) = \rho(|r|)$) are easy to deal with.
VECTOR FIELDS

- Mathematically, a "field" maps every point in space to a "value" at that point
- The "value" can be a number (scalar) or a group of numbers (vector, tensor, etc.)

EX: Ideal Gas in a Box

\[
\text{Temperature, Pressure, Density} \rightleftharpoons \text{"Scalar Fields"}
\]

- Typically these fields are all uniform

Now change it:

- Scalar fields are no longer uniform

- The non-uniformity is defined by a "gradient":

\[
\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]
so \( \vec{\nabla}T = \frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} + \frac{\partial T}{\partial z} \hat{k} \)

- This is defined at every point, so \( \vec{\nabla}T \) is a “vector field”

Here \( \frac{\partial T}{\partial y} \) = \( \frac{\partial T}{\partial z} \) = 0 and \( \frac{\partial T}{\partial x} < 0 \) everywhere.

\( \vec{\nabla}p \) and \( \vec{\nabla}p \) are also vector fields.

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**FIELD EQUATIONS**

- Constructed by applying physics principles to fields

**EX:** \( PV = NRT \Rightarrow P = \frac{N}{V} \) \( RT = \frac{\rho}{m} \) \( RT \)

So \( \vec{\nabla}P = \frac{R}{m} \left[ T(\vec{\nabla}p) + \rho(\vec{\nabla}T) \right] \) (Product Rule)
THE ELECTRIC FIELD

RECALL:

\[ q_{\text{source}} \rightarrow \vec{E} \text{ FIELD} \rightarrow q_{\text{test}} \]

- \( \vec{E} \) MUST MEASURE HOW "TWEAKED" SPACE IS DUE TO \( q_{\text{source}} \)
- \( q_{\text{test}} \) IS NOT NECESSARY IN DEFINING \( \vec{E} \)

- SO \( \vec{E} \) IS LIKE \( \vec{F}_{\text{Coulomb}} \) WITHOUT \( q_{\text{test}} \):

\[
\vec{E} = \frac{\vec{F}}{q_{\text{test}}} = \frac{q_{\text{source}}}{4\pi\varepsilon_0} \frac{1}{r^2}
\]

\( r \) FROM SOURCE TO POINT P
POINT SOURCE CHARGES:

SOURCE CHARGE DISTRIBUTIONS:

LINE:  

PLANE:

DIPOLE:

- REMEMBER, WE DON'T COMPUTE \( \mathbf{F} \) UNTIL WE INTRODUCE \( q \) TEST.
“ELECTRIC FIELD EQUATIONS”

- We have discussed two fields:
  1. \( \rho(x, y, z) = “\text{charge density}” \) (scalar field)
  2. \( \vec{E}(x, y, z) = “\text{electric field}” \) (vector field)

Can we relate them using physics?

\[
\vec{E}(\vec{r}) = k \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3 r'
\]

In words: The \( \vec{E} \) field comes from all the tiny point charges in the distribution \( \rho(\vec{r}') \)

- Leads to very difficult integrals; there is often a simpler way: Gauss’ Law
**FIELD LINES**

- Notice \( \vec{E} \):  
  - diverges from \( + \)  
  - converges toward \( - \)

- Imagine a "path" along \( \vec{F} \) which starts at \( + \) and ends at \( - \)

- Why don't paths cross?  
  \( \vec{E}(x,y,z) \) is single-valued

- Field lines are nothing physical, just a visualization tool

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**ELECTRIC FLUX**

- Strong \( \vec{E} \) → tightly packed field lines

\( \vec{E}_{\text{weak}} \) → \( \vec{E}_{\text{strong}} \)

(e.g. look at point charge \( \vec{E} \) fields above)
- The "electric flux" \( \Phi_E \) measures field lines passing through a surface:

\[
\Phi_E \text{ depends on:}
\]

1. Strength of \( \vec{E} \)
2. Angle of \( \vec{E} \)
3. Area of surface

- Since \( \vec{E} \) can vary from point to point,

\[
\Phi_E = \int d\Phi_E \quad \text{(add up the flux from all points on surface)}
\]

Look at one point on surface:

\[
d\Phi_E = |\vec{E}| |d\vec{A}| \cos \theta
\]

\[
d\Phi_E = \vec{E} \cdot d\vec{A}
\]

\[
\Phi_E = \int \vec{E} \cdot d\vec{A}
\]
- Why is $\Phi_E$ useful?

Since $E$ comes from sources, $\Phi_E$ is related to charge (can get field equations).

- Put an imaginary box around a charge:

<table>
<thead>
<tr>
<th>Box Location</th>
<th>$\vec{E}$</th>
<th>$\Phi_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Around $+$</td>
<td>Points out of box</td>
<td>$\Phi_E &gt; 0$</td>
</tr>
<tr>
<td>Around $-$</td>
<td>Points into box</td>
<td>$\Phi_E &lt; 0$</td>
</tr>
<tr>
<td>In between</td>
<td>In on one side, out on other</td>
<td>$\Phi_E = 0$</td>
</tr>
</tbody>
</table>

Gauss' Law

- For any closed surface:

$$\Phi_E = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

Note: $\epsilon_0 = \frac{1}{4\pi k}$ from Coulomb's Law.
You will prove this for homework (!)

Note: Gauss' law is only useful for calculations if we have symmetry

Ex: Sphere of charge—find \( \vec{E} \) around it

\[
\Phi_E = \frac{q_{\text{enclosed}}}{\varepsilon_0}
\]

\[
\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}
\]

This means “integral over a closed surface”

Symmetry: \( \vec{E} \) must point radially outward and be equally strong everywhere on the “Gaussian sphere”

\[
\oint \vec{E} \cdot d\vec{A} = \oint |\vec{E}| \, |d\vec{A}| = E(4\pi r^2)
\]
So \[ E \left( 4\pi r^2 \right) = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \]

In vector form:

\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{r} \]

Gauss' Law has reproduced Coulomb's Law for us! (And the Shell Theorem)

**Commonly Used Symmetries**

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>( \vec{E} ) Field</th>
<th>Gaussian Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical/Point Symmetry</td>
<td>Must point radially inward or outward</td>
<td>Sphere</td>
</tr>
<tr>
<td>Cylindrical Symmetry</td>
<td>Must point toward/away from axis</td>
<td>Cylinder</td>
</tr>
<tr>
<td>Plane Symmetry</td>
<td>Must point toward/away from plane</td>
<td>Box</td>
</tr>
</tbody>
</table>